STABILITY OF THE JENSEN TYPE FUNCTIONAL EQUATION IN BANACH ALGEBRAS: A FIXED POINT APPROACH

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Abstract. Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras and of derivations on Banach algebras for the following Jensen type functional equation:

\[ f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x). \]

1. Introduction and preliminaries

The stability problem of functional equations was originated from a question of Ulam [30] concerning the stability of group homomorphisms: Let \((G_1, \ast)\) be a group and let \((G_2, \circ, d)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \(\varepsilon > 0\), does there exist a \(\delta(\varepsilon) > 0\) such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality

\[ d(h(x \ast y), h(x) \circ h(y)) < \delta \]

for all \(x, y \in G_1\), then there is a homomorphism \(H : G_1 \to G_2\) with

\[ d(h(x), H(x)) < \varepsilon \]
for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x \ast y) = H(x) \circ H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let $X$ and $Y$ be Banach spaces. Assume that $f: X \to Y$ satisfies

$$
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon
$$

for all $x, y \in X$ and some $\varepsilon \geq 0$. Then there exists a unique additive mapping $T: X \to Y$ such that

$$
\|f(x) - T(x)\| \leq \varepsilon
$$

for all $x \in X$.

Th.M. Rassias [20] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded.

**Theorem 1.1.** (Th.M. Rassias). Let $f: E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)
$$

for all $x, y \in E$, where $\varepsilon$ and $p$ are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$
L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
$$

exists for all $x \in E$ and $L: E \to E'$ is the unique additive mapping which satisfies

$$
\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p
$$

for all $x \in E$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is $\mathbb{R}$-linear.

The above inequality (1.1) that was introduced for the first time by Th.M. Rassias [20] for the proof of the stability of the linear mapping between Banach spaces has provided a lot of influence in the development of what is now known as generalized Hyers-Ulam stability or as Hyers-Ulam-Rassias stability of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability
of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [7] extended the Hyers-Ulam stability by proving the following theorem in the spirit of Th.M. Rassias’ approach.

**Theorem 1.2.** [7] Let $f : E \to E'$ be a mapping for which there exists a function $\varphi : E \times E' \to [0, \infty)$ such that

$$
\varphi(x, y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(2^jx, 2^jy) < \infty,
$$

$$
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)
$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$
\|f(x) - T(x)\| \leq \frac{1}{2}\varphi(x, x)
$$

for all $x \in E$.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 2, 4, 5, 10, 11, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29]).

We recall the following theorem by Diaz and Margolis. The reader is referred to the book of D.H. Hyers, G. Isac and Th.M. Rassias [9] for an extensive account of fixed point theory with several applications.

**Theorem 1.3.** [6] Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$
d(J^n x, J^{n+1} x) = \infty
$$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
2. the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

This paper is organized as follows: In Section 2, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the Jensen type functional equation.
In Section 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the Jensen type functional equation.

In 1996, G. Isac and Th.M. Rassias [12] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Throughout this paper, assume that $A$ is a real Banach algebra with norm $\| \cdot \|_A$ and that $B$ is a real Banach algebra with norm $\| \cdot \|_B$.

2. Stability of homomorphisms in Banach algebras

For a given mapping $f : A \to B$, we define

$$Df(x, y) := f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - f(x)$$

for all $x, y \in A$.

Note that an $\mathbb{R}$-linear mapping $H : A \to B$ is called a homomorphism in Banach algebras if $H$ satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$.

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the functional equation $Df(x, y) = 0$.

**Theorem 2.1.** Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that

$$\|Df(x, y)\|_B \leq \varphi(x, y),$$
$$\|f(xy) - f(x)f(y)\|_B \leq \varphi(x, y)$$

for all $x, y \in A$. If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ and if there exists an $L < 1$ such that $\varphi(x, y) \leq 2L\varphi(x, y)\sqrt{\frac{1}{2}}$ for all $x, y \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{L}{1 - L}\varphi(x, 0)$$

for all $x \in A$. 
Proof. Consider the set
\[ X := \{ g : A \to B \} \]
and introduce the \textit{generalized metric} on \( X \):
\[ d(g, h) = \inf \{ C \in \mathbb{R}^+ : \| g(x) - h(x) \|_B \leq C\varphi(x, 0) \text{ for all } x \in A \}. \]
It is easy to show that \((X, d)\) is complete.

Now we consider the linear mapping \( J : X \to X \) such that
\[ Jg(x) := \frac{1}{2} g(2x) \]
for all \( x \in A \).

By Theorem 3.1 of [3],
\[ d(Jg, Jh) \leq Ld(g, h) \]
for all \( g, h \in X \).

Letting \( y = 0 \) in (2.1), we get
\[ 2f\left( \frac{x}{2} \right) - f(x) \leq \varphi(x, 0) \]
for all \( x \in A \). So
\[ \left\| f(x) - \frac{1}{2} f(2x) \right\|_B \leq \frac{1}{2} \varphi(2x, 0) \leq L\varphi(x, 0) \]
for all \( x \in A \). Hence \( d(f, Jf) \leq L \).

By Theorem 1.3, there exists a mapping \( H : A \to B \) such that
(1) \( H \) is a fixed point of \( J \), i.e.,
\[ H(2x) = 2H(x) \]
for all \( x \in A \). The mapping \( H \) is a unique fixed point of \( J \) in the set
\[ Y = \{ g \in X : d(f, g) < \infty \}. \]
This implies that \( H \) is a unique mapping satisfying (2.4) such that there exists \( C \in (0, \infty) \) satisfying
\[ \| H(x) - f(x) \|_B \leq C\varphi(x, 0) \]
for all \( x \in A \).

(2) \( d(J^n f, H) \to 0 \) as \( n \to \infty \). This implies the equality
\[ \lim_{n \to \infty} \frac{f(2^nx)}{2^n} = H(x) \]
for all \( x \in A \).
(3) \(d(f, H) \leq \frac{1}{1-L}d(f, Jf)\), which implies the inequality
\[
d(f, H) \leq \frac{L}{1-L}.
\]
This implies that the inequality (2.3) holds.

One can easily show that
\[
\lim_{j \to \infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) = 0
\]
for all \(x, y \in A\). It follows from (2.1), (2.5) and (2.6) that
\[
\left\| H\left(\frac{x+y}{2}\right) + H\left(\frac{x-y}{2}\right) - H(x) \right\|_B
\leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0
\]
for all \(x, y \in A\). So
\[
H\left(\frac{x+y}{2}\right) + H\left(\frac{x-y}{2}\right) = H(x)
\]
for all \(x, y \in A\). Letting \(z = \frac{x+y}{2}\) and \(w = \frac{x-y}{2}\) in the above equation, we get
\[
H(z) + H(w) = H(z + w)
\]
for all \(z, w \in A\). So the mapping \(H : A \to B\) is Cauchy additive, i.e.,
\[
H(z + w) = H(z) + H(w)
\]
for all \(z, w \in A\).

By the same reasoning as in the proof of Theorem 1.1 [20], one can show that the mapping \(H : A \to B\) is \(\mathbb{R}\)-linear.

It follows from (2.2) that
\[
\|H(xy) - H(x)H(y)\|_B = \lim_{n \to \infty} \frac{1}{4^n} \|f(4^n xy) - f(2^n x)f(2^n y)\|_B
\leq \lim_{n \to \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0
\]
for all \(x, y \in A\). So
\[
H(xy) = H(x)H(y)
\]
for all \(x, y \in A\).

Thus \(H : A \to B\) is a homomorphism satisfying (2.3), as desired. \(\Box\)
Corollary 2.2. Let $r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that

\begin{align}
\|Df(x, y)\|_B & \leq \theta(\|x\|_A^r + \|y\|_A^r), \\
\|f(xy) - f(x)f(y)\|_B & \leq \theta(\|x\|_A^r + \|y\|_A^r)
\end{align}

for all $x, y \in A$. If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique homomorphism $H : A \to B$ such that

$$
\|f(x) - H(x)\|_B \leq \frac{2^r \theta}{2 - 2^r} \|x\|_A^r
$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$
\varphi(x, y) := \theta(\|x\|_A^r + \|y\|_A^r)
$$

for all $x, y \in A$. Then we can choose $L = 2^{r-1}$ and we get the desired result. \hfill \square

Theorem 2.3. Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying (2.1) and (2.2). If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ and if there exists an $L < 1$ such that $\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2^r} \varphi(x, y)$ for all $x, y \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$
\|f(x) - H(x)\|_B \leq \frac{1}{1 - L} \varphi(x, 0)
$$

for all $x \in A$.

Proof. Consider the complete generalized metric space $(X, d)$ given in the proof of Theorem 2.1.

Now we consider the linear mapping $J : X \to X$ such that

$$
Jg(x) := 2g\left(\frac{x}{2}\right)
$$

for all $x \in A$.

By Theorem 3.1 of [3],

$$
d(Jg, Jh) \leq Ld(g, h)
$$

for all $g, h \in X$.

Letting $y = 0$ in (2.1), we get

$$
\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_B \leq \varphi(x, 0)
$$
for all $x \in A$. So

$$\left\| f(x) - 2f \left( \frac{x}{2} \right) \right\|_B \leq \varphi(x, 0) \leq \frac{L}{2} \varphi(2x, 0)$$

for all $x \in A$. Hence $d(f, Jf) \leq 1$.

By Theorem 1.3, there exists a mapping $H : A \to B$ such that

1. $H$ is a fixed point of $J$. This implies that $H$ is a unique mapping satisfying (2.4) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C \varphi(x, 0)$$

for all $x \in A$.

2. $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right) = H(x)$$

for all $x \in A$.

3. $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{1-L}.$$ 

This implies that the inequality (2.9) holds.

One can easily show that

$$\lim_{j \to \infty} 4^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) = 0$$

for all $x, y \in A$. By (2.1), we see that

$$\left\| H \left( \frac{x + y}{2} \right) + H \left( \frac{x - y}{2} \right) - H(x) \right\|_B$$

$$= \lim_{n \to \infty} 2^n \left\| f \left( \frac{x + y}{2^{n+1}} \right) + f \left( \frac{x - y}{2^{n+1}} \right) - f \left( \frac{x}{2^n} \right) \right\|_B$$

$$\leq \lim_{n \to \infty} 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \leq \lim_{n \to \infty} 4^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0$$

for all $x, y \in A$.

By the proof of Theorem 2.1, the mapping $H : A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem 1.1 [20], one can show that the mapping $H : A \to B$ is $\mathbb{R}$-linear.
It follows from (2.2) that
\[
\|H(xy) - H(x)H(y)\|_B = \lim_{n \to \infty} 4^n \left\| f \left( \frac{xy}{4^n} \right) - f \left( \frac{x}{2^n} \right) f \left( \frac{y}{2^n} \right) \right\|_B
\]
\[
\leq \lim_{n \to \infty} 4^n \phi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0
\]
for all \(x, y \in A\). So
\[
H(xy) = H(x)H(y)
\]
for all \(x, y \in A\).

Thus \(H : A \to B\) is a homomorphism satisfying (2.9), as desired.

**Corollary 2.4.** Let \(r > 2\) and \(\theta\) be nonnegative real numbers, and let \(f : A \to B\) be a mapping satisfying (2.7) and (2.8). If for each \(x \in A\) the mapping \(f(tx)\) is continuous in \(t \in \mathbb{R}\), then there exists a unique homomorphism \(H : A \to B\) such that
\[
\|f(x) - H(x)\|_B \leq \frac{2^r \theta}{2^r - 4} \|x\|_A^r
\]
for all \(x \in A\).

**Proof.** The proof follows from Theorem 2.3 by taking
\[
\phi(x, y) := \theta(\|x\|_A^r + \|y\|_A^r)
\]
for all \(x, y \in A\). Then we can choose \(L = 2^{2-r}\) and we get the desired result.

**3. Stability of derivations on Banach algebras**

Note that an \(\mathbb{R}\)-linear mapping \(\delta : A \to A\) is called a derivation on \(A\) if \(\delta\) satisfies \(\delta(xy) = \delta(x)y + x\delta(y)\) for all \(x, y \in A\).

We prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the functional equation \(Df(x, y) = 0\).

**Theorem 3.1.** Let \(f : A \to A\) be a mapping for which there exists a function \(\phi : A^2 \to [0, \infty)\) such that
\[
\|Df(x, y)\|_A \leq \phi(x, y),
\]
\[
\|f(xy) - f(x)y - xf(y)\|_A \leq \phi(x, y)
\]
for all \( x, y \in A \). If there exists an \( L < 1 \) such that \( \varphi(x, y) \leq 2L\varphi(\frac{x}{2}, \frac{y}{2}) \) for all \( x, y \in A \). If for each \( x \in A \) the mapping \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then there exists a unique derivation \( \delta : A \to A \) such that

\[
\| f(x) - \delta(x) \|_A \leq \frac{L}{1 - L}\varphi(x, 0)
\]

for all \( x \in A \).

\textbf{Proof.} By the same reasoning as the proof of Theorem 2.1, there exists a unique \( \mathbb{R} \)-linear mapping \( \delta : A \to A \) satisfying (3.3). The mapping \( \delta : A \to A \) is given by

\[
\delta(x) = \lim_{n \to \infty} f \left( \frac{2^n x}{2^n} \right)
\]

for all \( x \in A \).

It follows from (3.2) that

\[
\| \delta(xy) - \delta(x)y - x\delta(y) \|_A = \lim_{n \to \infty} \frac{1}{4^n} \| f(4^n xy) - f(2^n x) \cdot 2^n y - 2^n x f(2^n y) \|_A \\
\leq \lim_{n \to \infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0
\]

for all \( x, y \in A \). So

\[
\delta(xy) = \delta(x)y + x\delta(y)
\]

for all \( x, y \in A \). Thus \( \delta : A \to A \) is a derivation satisfying (3.3). \( \square \)

\textbf{Corollary 3.2.} Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping such that

\[
\| Df(x, y) \|_A \leq \theta(\| x \|_A^r + \| y \|_A^r),
\]

\[
\| f(xy) - f(x)y - xf(y) \|_A \leq \theta(\| x \|_A^r + \| y \|_A^r)
\]

for all \( x, y \in A \). If for each \( x \in A \) the mapping \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then there exists a unique derivation \( \delta : A \to A \) such that

\[
\| f(x) - \delta(x) \|_A \leq \frac{2r\theta}{2 - 2r} \| x \|_A^r
\]

for all \( x \in A \).

\textbf{Proof.} The proof follows from Theorem 3.1 by taking

\[
\varphi(x, y) := \theta(\| x \|_A^r + \| y \|_A^r)
\]

for all \( x, y \in A \). Then we can choose \( L = 2^{r-1} \) and we get the desired result. \( \square \)
Theorem 3.3. Let \( f : A \rightarrow A \) be a mapping for which there exists a function \( \varphi : A^2 \rightarrow [0, \infty) \) satisfying (3.1) and (3.2). If there exists an \( L < 1 \) such that \( \varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{4}\varphi(x, y) \) for all \( x, y \in A \). If for each \( x \in A \) the mapping \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then there exists a unique derivation \( \delta : A \rightarrow A \) such that
\[
\|f(x) - \delta(x)\|_A \leq \frac{1}{1 - L}\varphi(x, 0) \tag{3.6}
\]
for all \( x \in A \).

Proof. By the same reasoning as the proof of Theorem 2.3, there exists a unique \( \mathbb{R} \)-linear mapping \( \delta : A \rightarrow A \) satisfying (3.6). The mapping \( \delta : A \rightarrow A \) is given by
\[
\delta(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)
\]
for all \( x \in A \).

It follows from (3.2) that
\[
\|\delta(xy) - \delta(x)y - x\delta(y)\|_A = \lim_{n \rightarrow \infty} 4^n \left\| f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right) \cdot \frac{y}{2^n} - x f\left(\frac{y}{2^n}\right) \right\|_A \\
\leq \lim_{n \rightarrow \infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\]
for all \( x, y \in A \). So
\[
\delta(xy) = \delta(x)y + x\delta(y)
\]
for all \( x, y \in A \). Thus \( \delta : A \rightarrow A \) is a derivation satisfying (3.6). \( \Box \)

Corollary 3.4. Let \( r > 2 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \rightarrow A \) be a mapping satisfying (3.4) and (3.5). If for each \( x \in A \) the mapping \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then there exists a unique derivation \( \delta : A \rightarrow A \) such that
\[
\|f(x) - \delta(x)\|_A \leq \frac{2^r \theta}{2^r - 4}\|x\|_A^r
\]
for all \( x \in A \).

Proof. The proof follows from Theorem 3.3 by taking
\[
\varphi(x, y) := \theta(\|x\|_A^r + \|y\|_A^r)
\]
for all \( x, y \in A \). Then we can choose \( L = 2^{2-r} \) and we get the desired result. \( \Box \)
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