ALMOST SPLITTING SETS $S$ OF AN INTEGRAL DOMAIN $D$ SUCH THAT $D_S$ IS A PID

GYU WHAN CHANG

Abstract. Let $D$ be an integral domain, $S$ be a multiplicative subset of $D$ such that $D_S$ is a PID, and $D[X]$ be the polynomial ring over $D$. We show that $S$ is an almost splitting set in $D$ if and only if every nonzero prime ideal of $D$ disjoint from $S$ contains a primary element. We use this result to give a simple proof of the known result that $D$ is a UMT-domain and $Cl(D[X])$ is torsion if and only if each upper to zero in $D[X]$ contains a primary element.

1. Introduction

Let $D$ be an integral domain with quotient field $K$, $D^* = D \setminus \{0\}$, $S$ be a multiplicative subset of $D$, $X$ be an indeterminate over $D$, and $D[X]$ be the polynomial ring over $D$. For a polynomial $h \in K[X]$, we denote by $c(h)$ the fractional ideal of $D$ generated by the coefficients of $h$.

As in [12], we say that $D$ is an almost GCD-domain (AGCD-domain) if for each $0 \neq a, b \in D$, there is an integer $n \geq 1$ such that $a^nD \cap b^nD$ is principal. Clearly, GCD-domains are AGCD-domains, but not vice versa (for example, if $F$ is a field of characteristic 2, then $F[X^2, X^3]$ is an AGCD-domains but not a GCD-domain (cf. [6, Lemma 3.2])). An upper to zero in $D[X]$ is a nonzero prime ideal $Q$ of $D[X]$ with $Q \cap D = (0)$,
while $D$ is called a UMT-domain if each upper to zero in $D[X]$ is a maximal $t$-ideal of $D[X]$. (Definitions related to the $t$-operation will be reviewed in the sequel.) $D$ is a Prüfer $v$-multiplication domain (PvMD) if each nonzero finitely generated ideal of $D$ is $t$-invertible. It is known that AGCD-domains are UMT-domains with torsion class group [4, Lemma 3.1], and $D$ is a PvMD if and only if $D$ is an integrally closed UMT-domain [10, Proposition 3.2]; so $D$ is an integrally closed AGCD-domain if and only if $D$ is a PvMD with torsion class group. We say that a multiplicative subset $S$ of $D$ is an almost splitting set of $D$ if for each $0 \neq d \in D$, there is an integer $n \geq 1$ such that $d^n = sa$ for some $s \in S$ and $a \in N(S)$, where $N(S) = \{0 \neq x \in D | (x, s)'_t = D$ for all $s' \in S\}$. It is known that $D^*$ is an almost splitting set of $D[X]$ if and only if $D$ is a UMT-domain and $Cl(D[X])$ is torsion [4, Theorem 2.4]; which also implies that if $D$ is integrally closed, then $D^*$ is an almost splitting set of $D[X]$ if and only if $D$ is an AGCD-domain.

In this paper, we show that if $D_S$ is a principal ideal domain (PID), then $S$ is an almost splitting set of $D$ if and only if each nonzero prime ideal of $D$ disjoint from $S$ contains a primary element. (A nonzero element $a \in D$ is said to be primary if $aD$ is a primary ideal.) We use this result to recover [4, Theorem 2.4] that $D^*$ is an almost splitting set of $D[X]$ if and only if $D$ is a UMT-domain and $Cl(D[X])$ is torsion, if and only if each upper to zero in $D[X]$ contains a primary element. We also show that $D[X]$ is an AGCD-domain if and only if $D[X]_{N_v}$ is an AGCD-domain and $D^*$ is an almost splitting set of $D[X]$, where $N_v = \{f \in D[X] | c(f)_v = D\}$.

We first review some definitions related to the $v$- and $t$-operations. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of $D$. For each $I \in \mathbf{F}(D)$, let $I^{-1} = \{x \in K | xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup \{J_v | J \subseteq I$ and $J$ is a nonzero finitely generated fractional ideal of $D\}$. Clearly, if $I$ is finitely generated, then $I_v = I_t$. An $I \in \mathbf{F}(D)$ is called a $t$-ideal if $I_t = I$, and an integral ideal is a maximal $t$-ideal if it is maximal among proper integral $t$-ideals. Let $t$-$\text{Max}(D)$ be the set of maximal $t$-ideals of $D$. It is well known that $t$-$\text{Max}(D) \neq \emptyset$ if $D$ is not a field; a prime ideal minimal
over a $t$-ideal is a $t$-ideal (hence an upper to zero in $D[X]$ is a $t$-ideal); each proper integral $t$-ideal is contained in a maximal $t$-ideal; and each maximal $t$-ideal is a prime ideal.

We say that an $I \in \mathbf{F}(D)$ is $t$-invertible if $(II^{-1})_t = D$; equivalently, if $II^{-1} \not\subset P$ for all $P \in t\text{-Max}(D)$. Let $T(D)$ be the group of $t$-invertible fractional $t$-ideals of $D$ under the $t$-multiplication $A \ast B = (AB)_t$, and let $\text{Prin}(D)$ be its subgroup of principal fractional ideals. The ($t$-)class group of $D$ is an abelian group $Cl(D) = T(D)/\text{Prin}(D)$. It is well known that $D$ is a GCD-domain if and only if $D$ is a PrMD and $Cl(D) = 0$ [5, Corollary 1.5]. The readers can refer to [9] for any undefined notation or terminology.

2. Results

Let $D$ be an integral domain with quotient field $K$, $D^* = D \setminus \{0\}$, $X$ be an indeterminate over $D$, and $D[X]$ be the polynomial ring over $D$.

We begin this section with a nice characterization of almost splitting sets, which appears in [2, Proposition 2.7].

**Lemma 1.** Let $S$ be a multiplicative subset of $D$. Then $S$ is an almost splitting set of $D$ if and only if, for each $0 \neq d \in D$, there is a positive integer $n = n(d)$ such that $d^nD \cap D$ is principal.

As in [1], we say that a multiplicative subset $S$ of $D$ is a $t$-splitting set if each $0 \neq d \in D$, we have $dD = (AB)_t$ for some integral ideals $A, B$ of $D$, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. An almost splitting set is $t$-splitting [6, Proposition 2.3], and if $Cl(D)$ is torsion, a $t$-splitting set is almost splitting [6, Corollary 2.4]. It is known that if $D_S$ is a PID, then $S$ is a $t$-splitting set of $D$ if and only if each nonzero prime ideal of $D$ disjoint from $S$ is $t$-invertible [7, Theorem 2.8], which was used to show that $D^*$ is a $t$-splitting set in $D[X]$ if and only if $D$ is a UMT-domain [7, Corollary 2.9]. Our next result, which is the main result of this paper, is an almost splitting set analog of [7, Theorem 2.8].
THEOREM 2. Let $S$ be a multiplicative subset of $D$ such that $DS$ is a PID. Then $S$ is an almost splitting set in $D$ if and only if every nonzero prime ideal of $D$ disjoint from $S$ contains a primary element.

Proof. ($\Rightarrow$) Assume that $S$ is an almost splitting set of $D$, and let $P$ be a nonzero prime ideal of $D$ disjoint from $S$. Then $PD_S = pD_S$ for some $p \in P$, because $DS$ is a PID. By Lemma 1, there is a positive integer $n$ such that $P = PD_S \cap D \supseteq P^nD_S \cap D = p^nD_S \cap D = qD$ for some $q \in D$. Note that $q$ is a primary element, because $p^nD_S$ is primary. Thus, $P$ contains a primary element $q$.

($\Leftarrow$) Let $0 \neq d \in D$. Then since $DS$ is a PID, we have $dD_S = p_1^{e_1} \cdots p_k^{e_k}D_S$ for some $p_i \in D$ and positive integers $e_i$ such that $p_i$’s are distinct prime elements in $D_S$. Let $P_i$ be the prime ideal of $D$ such that $P_iD_S = p_iD_S$. Since $p_iD_S$ is minimal over $dD_S$, $P_i$ is minimal over $dD$. Moreover, $P_i \cap S = \emptyset$, and so $P_i$ contains a primary element $q_i$. Since $P_i D_S = p_iD_S$, there is a positive integer $n_i$ for which $q_iD_S = p_i^nD_S$. Let $n = n_1 \cdots n_k$ and $m_i = \frac{n}{n_i} e_i$. Then $d^nD_S = (p_1^{n_1} \cdots p_k^{n_k})D_S = (q_1^{m_1}D_S) \cap \cdots \cap (q_k^{m_k}D_S) = (q_1^{m_1}D_S) \cap \cdots \cap (q_k^{m_k}D_S)$, whence

$$d^nD_S \cap D = ((q_1^{m_1}D_S) \cap \cdots \cap (q_k^{m_k}D_S)) \cap D = (q_1^{m_1}D_S \cap D) \cap \cdots \cap (q_k^{m_k}D_S \cap D) = (q_1^{m_1}D) \cap \cdots \cap (q_k^{m_k}D) = (q_1^{m_1} \cdots q_k^{m_k})D,$$

where the last equality follows from the fact that each $q_i^{m_i}$ is a primary element, so [3, Corollary 2] applies. Therefore, $S$ is an almost splitting set by Lemma 1. \(\square\)

Let $N_v = \{f \in D[X] \mid c(f)_v = D\}$ and $N(D^*) = \{f \in D[X] \mid f \neq 0$ and $(f,d)_v = D[X]$ for all $d \in D^*\}$. Obviously, $N_v = N(D^*)$, and thus $Cl(D[X]_{N(D^*)}) = 0$ [11, Theorems 2.4 and 2.14]. The next result is already known, but we use Theorem 2 to give another simple proof.

COROLLARY 3. ([4, Theorem 2.4]) The following statements are equivalent.

1. $D^*$ is an almost splitting set in $D[X]$. 

(2) $D$ is a UMT-domain and $Cl(D[X])$ is torsion.

(3) Each upper to zero in $D[X]$ contains a primary element.

Proof. (1) $\Rightarrow$ (2) Suppose that $D^*$ is an almost splitting set in $D[X]$. Then $Cl(D[X]_{D^*}) = Cl((D[X])_{N(D^*)}) = 0$, and thus $Cl(D[X])$ is torsion [4, Lemma 2.3]. Also, if $Q$ is an upper to zero in $D[X]$, then $Q \cap D^* = \emptyset$, and hence $Q$ contains a primary element by Theorem 2. For $g \in D[X] \setminus Q$, if $u \in (g, f)^{-1}$, then $uf \cdot g = ug \cdot f \in fD[X]$, and since $g \notin Q$, we have $uf \in fD[X]$. Hence, $u \in D[X]$, which means that $(f, g)^{-1} = (f, g)_0 = D[X]$. Thus, $Q$ is a maximal $t$-ideal.

(2) $\Rightarrow$ (3) Assume that $D$ is a UMT-domain and $Cl(D[X])$ is torsion, and let $Q$ be an upper to zero in $D[X]$. Then $Q$ is a maximal $t$-ideal of $D[X]$, and hence $Q$ is $t$-invertible [10, Theorem 1.4]. Also, since $Cl(D[X])$ is torsion, there is an integer $n \geq 1$ such that $(Q^n)_t = fD[X]$ for some $f \in D[X]$. If $g, h \in D[X]$ such that $gh \in fD[X]$ and $g \notin Q$, then $(Q^n, g)_t = D[X]$, because $Q$ is a maximal $t$-ideal. Hence $Q \supseteq h(Q^n, g)_t = hD[X] \ni h$. Thus, $f$ is a primary element such that $f \in Q$.

(3) $\Rightarrow$ (1) This is an immediate consequence of Theorem 2, because $D[X]_{D^*}$ is a PID and each nonzero prime ideal of $D[X]$ disjoint from $D^*$ is an upper to zero in $D[X]$.

It is known that $D[X]$ is an AGCD-domain if and only if $D$ is an AGCD-domain and $\bar{D}[X]$ is a root extension of $D[X]$, where $\bar{D}$ is the integral closure of $D$ [2, Theorem 3.4]. (Let $A \subseteq B$ be an extension of integral domains. Then $B$ is said to be a root extension of $A$ if for each $b \in B$, $b^n \in A$ for some integer $n \geq 1$.) We next give another characterization of $D[X]$ being an AGCD-domain.

**Corollary 4.** $D[X]$ is an AGCD-domain if and only if $D[X]_{N_v}$ is an AGCD-domain and $D^*$ is an almost splitting set of $D[X]$.

Proof. Assume that $D[X]$ is an AGCD-domain. Then $D[X]_{N_v}$ is an AGCD-domain [6, Corollary 2.12], and since an AGCD-domain is a UMT-domain with torsion class group, $D^*$ is an almost splitting set.
of $D[X]$ by Corollary 3. Conversely, assume that $D[X]_{N_v}$ is an AGCD-domain and $D^*$ is an almost splitting set of $D[X]$. Note that $N(D^*) = N_v$ and $D[X]_{D^*} = K[X]$ is a PID (hence an AGCD-domain). Thus, $D[X]$ is an AGCD-domain \cite[Corollary 2.12]{6}.

\begin{corollary}
If $D$ is integrally closed, the following statements are equivalent.

(1) $D^*$ is an almost splitting set in $D[X]$.
(2) $D$ is an AGCD-domain.
(3) $D$ is a PvMD and $Cl(D)$ is torsion.
(4) $D[X]$ is an AGCD-domain.
(5) Each upper to zero in $D[X]$ contains a primary element.
\end{corollary}

\begin{proof}
(1) $\iff$ (2) \cite[Proposition 2.6]{6}. (1) $\iff$ (3) If $D$ is integrally closed, then $Cl(D[X]) = Cl(D)$ \cite[Theorem 3.6]{8}, and $D$ is a UMT-domain if and only if $D$ is a PvMD \cite[Proposition 3.2]{10}. Thus, the result follows from Corollary 3. (3) $\Rightarrow$ (4) This follows, because $D[X]$ is a PvMD and $Cl(D[X]) = Cl(D)$. (4) $\Rightarrow$ (1) Corollary 4. (1) $\iff$ (5) Corollary 3.
\end{proof}

We end this paper with an example of non-integrally closed AGCD-domain. Let $S$ be a multiplicative subset of $D$, and let $R = D + XD_S[X]$. It is known that $R$ is an AGCD-domain if and only if $D$ is an AGCD-domain and $D_S[X]$ is a root extension of $D_S[X]$ \cite[Theorems 3.4 and 3.12]{2}. Clearly, $D^*$ is an almost splitting set of $D$. Thus, $D + XK[X]$ is an AGCD-domain if and only if $D$ is an AGCD-domain. For example, let $F$ be a field of characteristic $> 0$, $Z$ be an indeterminate over $F$, and $D = F[Z^2, Z^3]$. Then $D + XK[X]$ is a non-integrally closed AGCD-domain.

\begin{acknowledgement}
The author would like to thank the referees for several helpful comments.
\end{acknowledgement}
References


Department of Mathematics
University of Incheon
Incheon 402-749, Korea
E-mail: whan@incheon.ac.kr