A NOTE ON FOUR TYPES OF REGULAR RELATIONS

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ABSTRACT. In this paper, we study the four different types of relations, $\mathcal{P}(X, T)$, $\mathcal{R}(X, T)$, $\mathcal{L}(X, T)$, and $\mathcal{S}(X, T)$ in a transformation $(X, T)$, and obtain some of their properties. In particular, we give a relationship between $\mathcal{R}(X, T)$ and $\mathcal{S}(X, T)$.

1. Introduction

The proximal relation were first studied by Ellis and Gottschalk in [6]. The syndetically proximal relation were introduced by Clay in [3]. In [1], Auslander defined the regular minimal sets which may be described as minimal subsets of enveloping semigroups. In [8], Shoenfeld introduced the regular homomorphisms which are defined by extending regular minimal sets to homomorphisms with minimal range. Also Yu introduced the regular relation and the syndetically regular relation (see [9], [10]).

In this paper, we study the four different types of relations in a transformation and give some of their properties.

2. Preliminaries

A transformation group $(X, T)$ will consist of a jointly continuous action of the topological group $T$ on the compact Hausdorff space $X$. The group $T$, with identity $e$, is assumed to be topologically discrete and remain fixed throughout this paper, so we may write $X$ instead of $(X, T)$. 

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A flow is said to be minimal if every point has dense orbit. Minimal flows are also referred to as minimal sets.

A homomorphism of transformation groups is a continuous, equivariant map. A one-one homomorphism of $X$ onto $X$ is called an automorphism of $X$. We denote the group of automorphisms of $X$ by $A(X)$.

The compact Hausdorff space $X$ carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X \times X$. Two points $x, x' \in X$ are said to be proximal if, given any index $\alpha$, there exists $t \in T$ such that $(xt, x't) \in \alpha$. The proximal relation in $X$, denoted by $P(X,T)$, is the set of all proximal pairs in $X$. $X$ is said to be distal if $P(X,T) = \Delta_X$, the diagonal of $X \times X$ and is said to be proximal if $P(X,T) = X \times X$.

Given a transformation group $(X,T)$, we may regard $T$ as a set of self-homeomorphisms of $X$. We define $E(X)$, the enveloping semigroup of $X$ to be the closure of $T$ in $X^X$, taken with the product topology. $E(X)$ is at once a transformation group and a sub-semigroup of $X^X$. The minimal right ideals of $E(X)$, considered as a semigroup, coincide with the minimal sets of $E(X)$. A subset $A$ of $T$ is said to be syndetic if there exists a compact subset $K$ of $T$ with $T = AK$.

Two points $x, x' \in X$ are said to be syndetically proximal if, given any index $\alpha$, there exists a syndetic subset $A$ of $T$ such that $(xt, x't) \in \alpha$ for all $t \in A$. The set of syndetically proximal pairs in $X$ is called the syndetically proximal relation and is denoted by $L(X,T)$.

Two points $x, x' \in X$ are said to be regular if there exists $h \in A(X)$ such that $(h(x), x') \in P(X,T)$. The set of regular pairs in $X$ is called the regular relation and is denoted by $R(X,T)$.

Two points $x, x' \in X$ are said to be syndetically regular if there exists $h \in A(X)$ such that $(h(x), x') \in L(X,T)$. The set of syndetically regular pairs in $X$ is called the syndetically regular relation and is denoted by $S(X,T)$.

$X$ is said to be almost periodic if, given any index $\alpha$, there exists a syndetic subset $A$ of $T$ such that $xA \subset x\alpha$ for all $x \in X$, where $x\alpha = \{ y \in X \mid (x,y) \in \alpha \}$. $X$ is said to be locally almost periodic if, given $x \in X$ and $U$ a neighborhood of $x$, there exists a neighborhood $V$ of $x$ and a syndetic subset $A$ of $T$ with $VA \subset U$.

**Remark 2.1.** If $E(X)$ contains just one minimal right ideal, then $P(X,T)$ and $R(X,T)$ are invariant equivalence relations on $X$ (see [4], [9]).
Lemma 2.2. ([2]) Suppose \((X, T)\) is locally almost periodic. Then \(P(X, T) = L(X, T)\).

Lemma 2.3. ([4]) \((X, T)\) is almost periodic iff it is locally almost periodic and distal.

3. Some results on \(P(X, T), R(X, T), L(X, T)\) and \(S(X, T)\)

The following lemma is an immediate consequence of the definitions.

Lemma 3.1. Given a transformation group \((X, T)\), the following statements are true:

1. \(L(X, T) \subset P(X, T) \subset R(X, T)\).
2. \(L(X, T) \subset S(X, T) \subset R(X, T)\).
3. \(\Delta_X \subset L(X, T)\).
4. If \(P(X, T) = L(X, T)\), then \(R(X, T) = S(X, T)\).

The next lemma leads to a useful characterization of \(L(X, T)\).

Lemma 3.2. ([5]) Given a transformation group \((X, T)\), the following statements are true:

1. \(L(X, T) = \{(x, y) \in X \times X \mid (x, y)T \subset P(X, T)\}\).
2. \(L(X, T)\) is an invariant equivalence relation on \(X\).

Lemma 3.3. Given a transformation group \((X, T)\), the following statements are true:

1. \(S(X, T) = \{(x, y) \in X \times X \mid (x, y)\overline{T} \subset R(X, T)\}\).
2. If \(E(X)\) contains just one minimal right ideal, then \(S(X, T)\) is an invariant equivalence relation on \(X\).

Proof. (1) Use lemma 3.2(1). Assume that \((x, y) \in X \times X\). Then \((x, y) \in S(X, T)\) iff there exists \(h \in A(X)\) such that \((h(x), y) \in L(X, T)\) iff there exists \(h \in A(X)\) such that \((h(x), y)T \subset P(X, T)\) iff there exists \(h \in A(X)\) such that \((h(x)p, yp) \in P(X, T)\) for all \(p \in E(X)\) iff \((x, y)p \in R(X, T)\) for all \(p \in E(X)\) iff \((x, y)\overline{T} \subset R(X, T)\). This completes the proof of (1).

(2) It follows immediately from (1) that \(S(X, T)\) is a reflexive, symmetric and invariant relation. To see that \(S(X, T)\) is transitive, assume that \((x, y) \in S(X, T)\) and \((y, z) \in S(X, T)\). Then \((x, y)\overline{T} \subset R(X, T)\) and \((y, z)\overline{T} \subset R(X, T)\) and hence \((xp, yp) \in R(X, T)\) and \((yp, zp) \in R(X, T)\).
\begin{align*}
\mathcal{R}(X,T) & \text{ for all } p \in E(X). \text{ Since } E(X) \text{ contains just one minimal right ideal, we have from Remark 2.1 that } (xp,zp) \in \mathcal{R}(X,T) \text{ for all } p \in E(X). \text{ Therefore } (x,z)^T \subset \mathcal{R}(X,T) \text{ and hence } (x,z) \in \mathcal{S}(X,T). \quad \Box \\
\text{Remark 3.4. } \mathcal{P}(X,T), \mathcal{R}(X,T), \text{ and } \mathcal{S}(X,T) \text{ are not equivalence relations on } X. \text{ However, if } E(X) \text{ contains just one minimal right ideal, then they are invariant equivalence relations on } X \text{ (see Remark 2.1 and Lemma 3.3).} \\
\text{Lemma 3.5. If } \mathcal{P}(X,T) \text{ is closed, then } \mathcal{R}(X,T) \text{ is also closed.} \\
\text{Proof. Let } (x,y) \in \mathcal{R}(X,T) \text{ and let } q \in E(X). \text{ Then there exists } h \in A(X) \text{ such that } (h(x),y) \in \mathcal{P}(X,T). \text{ Since } \mathcal{P}(X,T) \text{ is closed, we have that } (h(x),y)q \in \mathcal{P}(X,T). \text{ This implies that } (x,y) \in \mathcal{R}(X,T). \text{ Thus } \mathcal{R}(X,T) \text{ is closed.} \quad \Box \\
\text{Theorem 3.6. Let } \mathcal{P}(X,T) \text{ be closed. Then} \\
(1) \quad \mathcal{P}(X,T) = \mathcal{L}(X,T) \\
(2) \quad \mathcal{R}(X,T) = \mathcal{S}(X,T). \\
\text{Proof. To see that (1) holds, assume that } (x,y) \in \mathcal{P}(X,T). \text{ Since } \mathcal{P}(X,T) \text{ is closed, it follows that } (x,y)^T \subset \mathcal{P}(X,T). \text{ By Lemma 3.2(1), it follows that } \mathcal{P}(X,T) \subset \mathcal{L}(X,T) \text{ and therefore } \mathcal{P}(X,T) = \mathcal{L}(X,T). \text{ The proof of (2) is exactly analogous to that of (1) by Lemma 3.5.} \quad \Box \\
\text{Ellis’ result [4, Lemma 5.17] is a corollary to the above theorem.} \\
\text{Corollary 3.7. Let } \mathcal{P}(X,T) \text{ be closed. Then it is an invariant equivalence relation on } X. \\
\text{Remark 3.8. Let } (X,T) \text{ is distal. Since } \mathcal{P}(X,T) = \triangle_X, \text{ it follows that } \mathcal{L}(X,T) = \mathcal{P}(X,T) \text{ and therefore } \mathcal{P}(X,T) \text{ is a closed invariant equivalence relation on } X \text{ (see [4, Lemma 5.12]).} \\
\text{We can prove Ellis’ result [4, Lemma 5.27] as follows:} \\
\text{Theorem 3.9. Suppose } (X,T) \text{ is locally almost periodic. Then the following statements are true:} \\
(1) \quad \mathcal{L}(X,T) = \mathcal{P}(X,T) \subset \mathcal{R}(X,T) = \mathcal{S}(X,T). \\
(2) \quad \mathcal{P}(X,T) \text{ and } \mathcal{R}(X,T) \text{ are closed invariant equivalence relations on } X. \\
\text{Proof. (1) This follows from Lemma 2.2 and Lemma 3.1(4).}
The fact that $\mathcal{P}(X,T)$ is an invariant equivalence relation on $X$ follows from (1) and Lemma 3.2(2). Since $\mathcal{P}(X,T)$ is transitive, it follows from [4, Proposition 5.16] that $E(X)$ contains just one minimal right ideal and therefore $\mathcal{R}(X,T)$ is an invariant equivalence relation on $X$ by Remark 2.1. The closed property of $\mathcal{P}(X,T)$ follows from [4, Proposition 5.26]. The closed property of $\mathcal{R}(X,T)$ follows from Lemma 3.5.

The proof of the following corollary follows immediately from Lemma 2.3.

**Corollary 3.10.** Suppose $(X,T)$ is almost periodic. Then the following statements are true:

1. $\mathcal{L}(X,T) = \mathcal{P}(X,T) \subset \mathcal{R}(X,T) = \mathcal{S}(X,T)$.
2. $\mathcal{P}(X,T)$ and $\mathcal{R}(X,T)$ are closed invariant equivalence relations on $X$.

**Theorem 3.11.** Suppose $A(X) = \{1_X\}$, where $\{1_X\}$ is the identity homomorphism of $X$. Then $\mathcal{L}(X,T) = \mathcal{S}(X,T) \subset \mathcal{P}(X,T) = \mathcal{R}(X,T)$.

**Proof.** Let $(x,y) \in \mathcal{S}(X,T)$. Then $(x,y)\overline{T} \subset \mathcal{R}(X,T)$ by Lemma 3.3(1). Since $A(X) = \{1_X\}$, it follows that $\mathcal{P}(X,T) = \mathcal{R}(X,T)$ and hence $(x,y) \in \mathcal{L}(X,T)$ by Lemma 3.2(1). Therefore $\mathcal{S}(X,T) = \mathcal{L}(X,T)$.

**Corollary 3.12.** Suppose $(X,T)$ is minimal and proximal. Then $\mathcal{L}(X,T) = \mathcal{S}(X,T) \subset \mathcal{P}(X,T) = \mathcal{R}(X,T)$.

**Proof.** The proof uses [7, (8) of Section 1] to show that if $(X,T)$ is minimal and proximal, then the only homomorphism $(X,T) \to (X,T)$ is the identity.

**Lemma 3.13.** Let $h \in A(X)$ and let $\bar{h} : X \times X \to X \times X$ be the map induced by $h$. Then the following statements are true:

1. $\bar{h}\mathcal{P}(X,T) \subset \mathcal{P}(X,T)$.
2. $\bar{h}\mathcal{R}(X,T) \subset \mathcal{R}(X,T)$.
3. $\bar{h}\mathcal{L}(X,T) \subset \mathcal{L}(X,T)$.
4. $\bar{h}\mathcal{S}(X,T) \subset \mathcal{S}(X,T)$.

**Proof.** The proof of (1) is analogous to that of [4, Proposition 5.22]. Let $(x,y) \in \mathcal{R}(X,T)$. Then there exists $\psi \in A(X)$ with $\psi(x), y \in \mathcal{P}(X,T)$.
$P(X, T)$. By (1) $(h \circ \psi(x), h(y)) = (h \circ \psi \circ h^{-1}(h(x)), h(y)) \in P(X, T)$. Since $h \circ \psi \circ h^{-1} \in A(X)$, it follows that $(h(x), h(y)) = \tilde{h}(x, y) \in R(X, T)$. Now let $(x, y) \in L(X, T)$. Then $(x, y) \overline{T} \subset P(X, T)$ by Lemma 3.2(1), which means that $(x, y)p \in P(X, T)$ for all $p \in E(X)$. By (1) $\tilde{h}(x, y)p \in P(X, T)$ for all $p \in E(X)$. Therefore $(h(x), h(y)) \overline{T} \subset P(X, T)$ and hence $(h(x), h(y)) \in L(X, T)$. This proves that $\tilde{h}L(X, T) \subset L(X, T)$.

The proof of (4) is analogous to that of (3).

\[\square\]

**Theorem 3.14.** Let $h \in A(X)$ and let $\tilde{h} : X \times X \rightarrow X \times X$ be the map induced by $h$. Then the following statements are true:

1. If $(X, T)$ is minimal, then $\tilde{h}P(X, T) = P(X, T)$.
2. If $(X, T)$ is minimal and $A(X)$ is abelian, then $\tilde{h}R(X, T) = R(X, T)$.

**Proof.** If $(X, T)$ is minimal, then it is pointwise almost periodic. Thus (1) follows from [4, Proposition 5.22]. To see (2), let $(y_1, y_2) \in R(X, T)$. Then there exists $\psi \in A(X)$ with $(\psi(y_1), y_2) \in P(X, T)$. By (1) there exists $(x_1, x_2) \in P(X, T)$ such that $\tilde{h}(x_1, x_2) = (\psi(y_1), y_2)$. Therefore we have that $(\psi^{-1}(h(x_1)), h(x_2)) = (y_1, y_2)$ and $\psi^{-1} \in A(X)$. Since $A(X)$ is abelian, it follows that $(h(\psi^{-1}(x_1)), h(x_2)) = \tilde{h}(\psi^{-1}(x_1), x_2) = (y_1, y_2)$, which proves that $\tilde{h}R(X, T) = R(X, T)$.

\[\square\]

For each $h \in A(X)$, we define the subsets $S_h(X)$ and $R_h(X)$ of $X \times X$ as follows:

$$S_h(X) = \{(x, x') \in X \times X \mid (h(x), x') \in L(X, T)\} \quad R_h(X) = \{(x, x') \in X \times X \mid (h(x), x') \in P(X, T)\}.$$  

Note that $S_{1_X}(X) = L(X, T)$ and $R_{1_X}(X) = P(X, T)$.

If $V$ and $H$ are relations in $X$, then $V \circ H$ is the relation in $X$ defined by as follows:

$$(x, y) \in V \circ H \text{ if and only if for some element } z, (x, z) \in H \text{ and } (z, y) \in V.$$  

**Lemma 3.15.** Let $(X, T)$ be a transformation group and let $h \in A(X)$. Then $S_h(X) \neq \emptyset$ and $R_h(X) \neq \emptyset$.

**Proof.** Let $h, k \in A(X)$ and let $x' = h(x)$. Then $(h(x), x') \in \Delta_X \subset L(X, T) \subset P(X, T)$ by Lemma 3.1. Therefore $(x, x') \in S_h(X)$ and $(x, x') \in R_h(X)$. This proves that $S_h(X) \neq \emptyset$ and $R_h(X) \neq \emptyset$.

\[\square\]
THEOREM 3.16. Suppose that \((X,T)\) is a transformation group and that \(E(X)\) contains just one minimal right ideal. Then \(R_h(X) \circ R_k(X) = R_{hk}(X)\) for all \(h,k \in A(X)\).

Proof. Let \(h,k \in A(X)\) and \((x,y) \in R_h(X) \circ R_k(X)\). Then there exists \(z \in X\) such that \((x,z) \in R_k(X)\) and \((z,y) \in R_h(X)\). Hence \((k(x),z) \in \mathcal{P}(X,T)\) and \((h(z),y) \in \mathcal{P}(X,T)\). Therefore by Theorem 3.13(1) \((h(k(x)),h(z)) \in \mathcal{P}(X,T)\). Since \(E(X)\) contains just one minimal right ideal, it follows from Remark 2.1 that \(\mathcal{P}(X,T)\) is transitive and therefore \((h(k(x)),y) \in \mathcal{P}(X,T)\). Since \(h \circ k \in A(X)\), we have that \((x,y) \in R_{hk}(X)\).

Let \((x,y) \in R_{hk}(X)\). By Theorem 3.13(1), \((h(k(x)),y) \in \mathcal{P}(X,T)\) shows that \((k(x),h^{-1}(y)) \in \mathcal{P}(X,T)\). Now let \(h^{-1}(y) = z\). Then \((k(x),z) \in \mathcal{P}(X,T)\) and \(h(z) = y\). Since \((y,y) \in \mathcal{P}(X,T)\), it follows that \((h(z),y) \in \mathcal{P}(X,T)\). Hence \((x,z) \in R_k(X)\) and \((z,y) \in R_h(X)\). Thus \((x,y) \in R_{hk}(X)\). \(\square\)

The next corollary states that if \(E(X)\) contains just one minimal right ideal, then \(\{R_h(X) \mid h \in A(X)\}, \circ\) forms a group.

COROLLARY 3.17. Suppose that \((X,T)\) is a transformation group and that \(E(X)\) contains just one minimal right ideal. For arbitrary \(h,k,r \in A(X)\), the following properties hold:

1. \(R_h(X) \circ R_k(X) \circ R_r(X) = R_h(X) \circ (R_k(X) \circ R_r(X))\).
2. There exists \(1_X \in A(X)\) such that \(\mathcal{P}(X,T) \circ R_h(X) = R_h(X) \circ \mathcal{P}(X,T) = R_h(X)\).
3. For each \(h \in A(X)\) there exists \(h^{-1} \in A(X)\) such that \(R_h(X) \circ R_h^{-1}(X) = R_h^{-1}(X) \circ R_h(X) = \mathcal{P}(X,T)\).

Proof. This follows from Lemma 3.15, Theorem 3.16, and the fact that \(A(X)\) is a group. \(\square\)

COROLLARY 3.18. Let \((X,T)\) be a transformation group. Then the following statements are true:

1. \(S_h(X) \circ S_k(X) = S_{hk}(X)\) for all \(h,k \in A(X)\).
2. \((S_h(X) \circ S_k(X)) \circ S_r(X) = S_h(X) \circ (S_k(X) \circ S_r(X))\) for all \(h,k,r \in A(X)\).
3. \(\mathcal{L}(X,T) \circ S_h(X) = S_h(X) \circ \mathcal{L}(X,T) = S_h(X)\) for all \(h \in A(X)\).
4. \((S_h(X))^{-1} = S_{h^{-1}}(X)\) for all \(h \in A(X)\).
Proof. The proof of (1) is analogous to that of Theorem 3.16. Note that $S_h(X) \neq \emptyset$ and $hL(X, T) \subset L(X, T)$ for all $h \in A(X)$, and $L(X, T)$ is an invariant equivalence relation on $X$. □

Remark 3.19. (1) The collection $\{S_h(X) \mid h \in A(X)\}$, $\circ$ is a group by Corollary 3.18.

(2) Suppose $(X, T)$ is distal. The collection $\{R_h(X) \mid h \in A(X)\}$, $\circ$ forms a group because $(X, T)$ is distal iff $E(X)$ is a minimal right ideal (see [4, Proposition 5.3]).

References


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