STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR SYSTEMS OF VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS IN \(q\)-UNIFORMLY SMOOTH BANACH SPACES

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Abstract. In this paper, we introduce a new iterative scheme to investigate the problem of finding a common element of nonexpansive mappings and the set of solutions of generalized variational inequalities for a \(k\)-strict pseudo-contraction by relaxed extra-gradient methods. Strong convergence theorems are established in \(q\)-uniformly smooth Banach spaces.

1. Introduction

Throughout this paper, we assume that \(E\) is a real Banach space and \(E^*\) the dual space of \(E\). Let \(C\) be a subset of \(E\) and \(T\) be a self mapping of \(C\). Denote by \(Fix(T)\) the set of fixed points of \(T\), that is, \(Fix(T) = \{x \in C : Tx = x\}\). When \(\{x_n\}\) is a sequence in \(E\), \(x_n \to x(x_n \rightharpoonup x)\) will denote strong(weak) convergence of the sequence \(\{x_n\}\) to \(x\).

Let \(q > 1\) be a real number. The duality mapping \(J_q : E \to 2^{E^*}\) is defined by

\[ J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \quad \|f\| = \|x\|^{q-1}\}, \quad \forall x \in E. \]

In particular, \(J = J_2\) is called the normalized duality mapping and \(J_q(x) = \|x\|^{q-2}J_2(x)\) for \(x \neq 0\). If \(E\) is a Hilbert space, then \(J = I\), where \(I\) is the identity mapping.

2010 Mathematics Subject Classification: 41A65, 47J20, 47H09.
Key words and phrases: Strong convergence, \(k\)-strict pseudo-contraction, \(q\)-uniformly smooth Banach space, variational inequality.
Recall that a mapping $T$ is said to be nonexpansive if
\[ \|Tx -Ty\| \leq \|x - y\| \]
for all $x, y \in C$. A mapping $T$ is called a pseudo-contraction if there exists some $j_q(x - y) \in J_q(x - y)$ such that
\[ \langle Tx -Ty, j_q(x - y) \rangle \leq \|x - y\|^q \]
for all $x, y \in C$. $T$ is said to be a $k$-strict pseudo-contraction in the terminology of Browder and Petryshyn [1] if there exists a constant $k > 0$ such that
\[ \langle Tx -Ty, j_q(x - y) \rangle \leq \|x - y\|^q - k\|(I - T)x - (I - T)y\| \]
for every $x, y \in C$ and for some $j_q(x - y) \in J_q(x - y)$.

**Remark 1.1.** From (1.1) we can prove that if $T$ is $k$-strict pseudo-contraction, then $T$ is Lipschitz continuous with the Lipschitz constant $L = 1 + \frac{k}{q}$. A Banach space $E$ is called uniformly convex if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. It is known that a uniformly convex Banach space is reflexive and strictly convex. Let $S(E) = \{x \in E : \|x\| = 1\}$. $E$ is said to be Gâteaux differentiable if the limit
\[ \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \]
exists for each $x, y \in S(E)$. In this case, $E$ is called smooth. Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of $E$ defined by
\[ \rho_E(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \geq t \right\}. \]

A Banach space $E$ is said to be uniformly smooth if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$. Let $q > 1$. A Banach space $E$ is said to be $q$-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho(t) \leq ct^q$. Recall that construction of fixed points for nonexpansive mappings and $\lambda$-strict pseudo-contractions via the Mann’s iterative algorithm has been extensively investigated by many authors (see [3,6,7,8]). The Mann iteration is extensively and successfully used to approximate fixed points of nonexpansive mappings.

However, iterative methods for strict pseudo-constractions are far less developed than for nonexpansive mappings. On the other hand, strict pseudo-constructions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [11]). Therefore it is
interesting to develop the theory of iterative methods for strict pseudo-contractions. In 1967, Halpen [4] introduced the following explicit iteration scheme for a nonexpansive mapping $T$ which was referred to Halpern iteration: for $u, x_0 \in K$, $\alpha_n \in [0, 1]$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n.$$ 

Recently, Zhou [17] obtained strong convergence theorem for the following iterative sequence in a 2-uniformly smooth Banach space $E$: for $u, x_0 \in E$ and a $\lambda$-strict pseudo-contraction $T$,

$$x_{n+1} = \beta_n u + \gamma_n x_n + (1 - \beta_n - \gamma_n)[\alpha_nTx_n + (1 - \alpha_n)x_n],$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$ satisfy

(i) $a \leq \alpha_n \leq \frac{1}{\lambda^2}$ for some $a > 0$ and for all $n \geq 0$;

(ii) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $\lim_{n \to \infty} |\alpha_{n+1} - \alpha_n| = 0$;

(iv) $0 < \lim \inf_{n \to \infty} \gamma_n \leq \lim \sup_{n \to \infty} \gamma_n < 1$.

Very recently, Zhang and Shu [16] extended Zhou’s results to $q$-uniformly smooth Banach space.

Motivated and inspired by the above works, in this paper, we consider the problem of convergence of an iterative algorithm for a system of generalized variational inequalities involving strictly pseudo-contractions and a nonexpansive mapping. We prove the strong convergence of proposed iterative scheme in uniformly convex and $q$-uniformly smooth Banach spaces.

2. Preliminaries

Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $E^*$ the dual space of $E$.

**Definition 2.1.** Let $E$ be a real Banach space, $C$ a nonempty closed and convex subset of $E$ and $K$ a nonempty subset of $C$. Let $Q$ be a mapping of $C$ into $K$. Then $Q$ is said to be:

1. sunny if for each $x \in C$ and $t \in [0, 1]$ we have
   $$Q(tx + (1-t)x) = Qx;$$

2. a retraction of $C$ onto $K$ if
   $$Qx = x, \quad \forall x \in K;$$
(3) a sunny nonexpansive retraction if $Q$ is sunny nonexpansive and a retraction onto $K$.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

**Proposition 2.1.** ([9]) Let $E$ be a smooth Banach space and let $K$ be a nonempty subset of $E$. Let $Q : E \to K$ be a retraction and let $J$ be the normalized duality mapping on $E$. Then the following are equivalent:

(a) $Q$ is sunny and nonexpansive;
(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy) \rangle$, $\forall x, y \in E$;
(c) $\langle x - Qx, J(y - Qx) \rangle \leq 0$, $\forall x \in E, y \in K$.

**Proposition 2.2.** ([5]) Let $K$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$ and $S$ a nonexpansive mapping of $C$ into itself with $\text{Fix}(S) \neq \emptyset$. Then the set $\text{Fix}(S)$ is a sunny nonexpansive retract of $C$. Reich [10], in 1980, proved the following behavior for nonexpansive mappings.

**Proposition 2.3.** Let $E$ be a real uniformly smooth Banach space and $C$ a nonempty closed convex subset of $E$. Let $T : C \to C$ be a nonexpansive mapping with a fixed point and let $z \in C$. For each $t \in (0, 1)$, let $z_t$ be the unique solution of the equation $x = tz + (1 - t)Tx$. Then $\{z_t\}$ converges to a fixed point of $T$ as $t \to 0$ and

$$Qz = s - \lim_{t \to 0} z_t$$

defines the unique sunny nonexpansive retraction from $C$ onto $\text{Fix}(T)$, that is, $Q$ satisfies the property:

$$\langle u - Qu, J(y - Qu) \rangle \leq 0, \quad \forall u \in C, y \in \text{Fix}(T).$$

Motivated by Wang and Chen [13], we consider the following general system of variational inequalities in a uniformly smooth Banach space $E$. Let $S : C \to C$ be a $k$-strict pseudo-contraction. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases}
(\lambda(I - S)y^* + x^* - y^*, J(x - x^*)) \geq 0, & \forall x \in C, \\
(\mu(I - S)x^* + y^* - x^*, J(x - x^*)) \geq 0, & \forall x \in C.
\end{cases}$$

(2.1)

In order to prove our main results, we need the following lemmas.
Lemma 2.1. ([14]) Let $E$ be a real $q$-uniformly smooth Banach space. Then there exists a constant $c_q > 0$ such that
\[ \|x + y\|^q \leq \|x\|^q + q \langle y, j_q(x) \rangle + c_q\|y\|^q \]
for all $x, y \in E$.

Lemma 2.2. ([12]) Let $\{z_n\}$ and $\{w_n\}$ be two bounded sequences in a Banach space $E$ such that
\[ z_{n+1} = (1 - \gamma_n)z_n + \gamma_n w_n, \quad n \geq 1, \]
where $\{\gamma_n\}$ satisfies condition: $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1$. If $\limsup_{n \to \infty} (\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|) \leq 0$, then $w_n - z_n \to 0$ as $n \to \infty$.

Lemma 2.3. ([2]) Let $C$ be a nonempty closed convex subset of a real Banach space $E$. Let $T_1$ and $T_2$ be nonexpansive mappings from $C$ into itself with a common fixed point. Define a mapping $T : C \to C$ by
\[ Tx = \delta T_1 x + (1 - \delta)T_2 x, \quad \forall x \in C, \]
where $\delta$ is a constant in $(0, 1)$. Then $T$ is nonexpansive and $\text{Fix}(T) = \text{Fix}(T_1) \cap \text{Fix}(T_2)$.

Lemma 2.4. ([15]) Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that
\[ \alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \]
where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that
(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\limsup_{n \to \infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n \to \infty} \alpha_n = 0$.

Lemma 2.5. For given $(x^*, y^*) \in C \times C$, where $y^* = Q_C(x^* - \mu(I - S)x^*)$, $(x^*, y^*)$ is a solution of problem (2.1) if and only if $x^*$ is a fixed point of the mapping $D : C \to C$ defined by
\[ D(x) = Q_C[Q_C(x - \mu(I - S)x) - \lambda(I - S)Q_C(x - \mu(I - S)x)], \quad \forall x \in C, \]
where $\lambda, \mu > 0$ are constants and $Q_C$ is a sunny nonexpansive retraction from $E$ onto $C$.

Proof. Observe that
\[
\begin{cases}
\langle \lambda(I - S)y^* + x^* - y^*, J(x - x^*) \rangle \geq 0, & \forall x \in C, \\
\langle \mu(I - S)x^* + y^* - x^*, J(x - x^*) \rangle \geq 0, & \forall x \in C.
\end{cases}
\]
Now, we consider the following main result of this paper.

**Theorem 3.1.** Let $C$ be a nonempty closed convex subset of a uniformly convex and $q$-uniformly smooth Banach space $E$ and $Q_C$ a sunny nonexpansive retraction from $E$ onto $C$. Let $S : C \to C$ be a $k$-strict pseudo-contraction such that $\text{Fix}(S) \neq \emptyset$ and $T : C \to C$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Assume that $F = \text{Fix}(T) \cap \text{Fix}(D) \neq \emptyset$, where $D$ is defined as Lemma 2.5. Let a sequence $\{x_n\}$ be generated by

$$
\begin{align*}
&x_1 = u \in C, \\
y_n = Q_C(x_n - \mu(I - S)x_n), \\
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta Tx_n + (1 - \delta)Q_C(y_n - \lambda(I - S)y_n)], \\
&n \geq 1.
\end{align*}
$$

where $\delta \in (0, 1)$, $\lambda, \mu \in (0, \min\{1, (\frac{q_k}{c_q})^{\frac{1}{q-1}}\})$ and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that

(H1) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$,

(H2) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(H3) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

Then $\{x_n\}$ defined by (3.1) converges strongly to $\bar{x} = Q_F u$ and $(\bar{x}, \bar{y})$, where $\bar{y} = Q_C(\bar{x} - \mu(I - S)\bar{x})$ and $Q_F$ is the unique sunny nonexpansive retraction from $C$ onto $F$, is a solution of the problem (2.1).

**Proof.** We divide our proofs into several steps as follows.

(Step 1.) First, we show that $F$ is closed and convex.

It is well known that $\text{Fix}(T)$ is closed and convex. Next, we show that $\text{Fix}(D)$ is closed and convex. For any $\lambda, \mu \in (0, M]$, $M = \min\{1, (\frac{q_k}{c_q})^{\frac{1}{q-1}}\}$,
we have that the mappings \( I - \mu(I - S) \) and \( I - \lambda(I - S) \) are nonexpansive mappings. Indeed, from Lemma 2.1, we have for all \( x, y \in C \),

\[
\|(I - \lambda(I - S))x - (I - \lambda(I - S))y\|_q^q \\
= \|x - y - \lambda(x - y - (Sx - Sy))\|_q^q \\
\leq \|x - y\|_q^q - q\lambda\|x - y - (Sx - Sy)\| + c_\theta \lambda\|x - y - (Sx - Sy)\|_q^q \\
\leq \|x - y\|_q^q - q\lambda\|x - y\|_q^q + q\lambda\|x - y\|_q^q - k\|\|x - (I - S)\|_q^q \\
+ c_\theta \lambda\|x - y - (Sx - Sy)\|_q^q \\
= \|x - y\|_q^q - \lambda(qk - c_\theta \lambda q^2 - 1)\|x - y - (Sx - Sy)\|_q^q \\
\leq \|x - y\|_q^q,
\]

which shows that \( I - \lambda(I - S) \) is a nonexpansive mapping. So is \( I - \mu(I - S) \). By Lemma 2.5, we can see that

\[
D = Q C(Q C(I - \mu(I - S)) - \lambda(I - S)Q C(I - \mu(I - S))) \\
= Q C(I - \lambda(I - S))Q C(I - \mu(I - S))
\]

is nonexpansive. Thus, \( F = Fix(T) \cap Fix(D) \) is closed and convex.

(Step 2.) The sequences \( \{x_n\} \) is bounded. For \( x^* \in F = Fix(T) \cap Fix(D) \), we have that

\[
x^* = Q C(Q C(x^* - \mu(I - S)x^*) - \lambda(I - S)Q C(x^* - \mu(I - S)x^*)).
\]

Set \( y^* = Q C(x^* - \mu(I - S)x^*). \) We obtain \( x^* = Q C(y^* - \lambda(I - S)y^*) \). Since \( y_n = Q C(x_n - \mu(I - S)x_n) \), we have

\[
\|y_n - y^*\| = \|Q C(x_n - \mu(I - S)x_n) - Q C(x^* - \lambda(I - S)x^*)\| \\
\leq \|x_n - x^*\|.
\]

For the sake of simplicity, let \( u_n = \delta Tx_n + (1 - \delta)Q C(y_n - \lambda(I - S)y_n) \) for each \( n \geq 1 \). By (3.2), we have

\[
\|u_n - x^*\| = \|\delta Tx_n + (1 - \delta)Q C(y_n - \lambda(I - S)y_n) - x^*\| \\
\leq \|\delta Tx_n - x^*\| \\
\leq \|x_n - x^*\| + (1 - \delta)\|y_n - y^*\| \\
\leq \|x_n - x^*\| + (1 - \delta)\|x_n - x^*\| \\
= \|x_n - x^*\|.
\]
Then we have
\[
\|x_{n+1} - x^*\| = \|\alpha_n u + \beta_n x_n + \gamma_n u_n - x^*\|
\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|u_n - x^*\|
\leq \alpha_n \|u - x^*\| + (1 - \alpha_n) \|x_n - x^*\|
\leq \max\{\|u - x^*\|, \|x_n - x^*\}\}.
\]

By induction, we get
\[
\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\}\}.
\]

Thus, \(\{x_n\}\) is bounded, so are \(\{y_n\}\) and \(\{u_n\}\).

(Step 3.) \(x_{n+1} - x_n \to 0\) as \(n \to \infty\). We now define \(w_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}\).

Set \(M_1 = \|u\| + \sup\{\|u_n\|\}\). By using (3.1), we get
\[
\|w_{n+1} - w_n\| = \left\| \frac{\alpha_{n+1} u + \gamma_{n+1} u_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n u_n}{1 - \beta_n} \right\|
\]
\[
(3.4)
\leq \left\| \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) u + \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) \right\| u_n
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \left( \|u\| + \|u_n\| \right) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|u_{n+1} - u_n\|
\leq M_1 \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) + \|u_{n+1} - u_n\|
\]
and
\[
\|u_{n+1} - u_n\| = \|\delta T x_{n+1} + (1 - \delta) Q C (y_{n+1} - \lambda (I - S) y_{n+1})
- (\delta T x_n + (1 - \delta) Q C (y_n - \lambda (I - S) y_n))\|
\leq \delta \|T x_{n+1} - T x_n\|
+ (1 - \delta) \|Q C (y_{n+1} - \lambda (I - S) y_{n+1)) - Q C (y_n - \lambda (I - S) y_n)\|
\leq \delta \|x_{n+1} - x_n\| + (1 - \delta) \|y_{n+1} - y_n\|
\leq \delta \|x_{n+1} - x_n\| + (1 - \delta) \|x_{n+1} - x_n\|
= \|x_{n+1} - x_n\|.
\]

Substituting (3.5) into (3.4) yields
\[
\|w_{n+1} - w_n\| \leq M_1 \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) + \|x_{n+1} - x_n\|.
\]

By the assumptions on \(\{\alpha_n\}\) and \(\{\beta_n\}\), we get
\[
\limsup_{n \to \infty} \|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \leq 0.
\]
By using Lemma 2.2, we conclude that \( w_n - x_n \to 0 \) as \( n \to \infty \). Noting that 
\[
x_{n+1} - x_n = (1 - \beta_n)(w_n - x_n),
\]
we get \( x_{n+1} - x_n \to 0 \) as \( n \to \infty \).

(Step 4.) There exists a continuous path \( \{x_t\} \) such that \( x_t \to \overline{x} \) as \( t \to 0 \), where \( \overline{x} = Q_F u \) and \( Q_F : C \to F \) is the unique sunny nonexpansive retraction from \( C \) onto \( F \). Define a mapping \( T_\delta : C \to C \) by
\[
T_\delta x = \delta T x + (1 - \delta)Q_C(I - \lambda(I - S))Q_C(I - \mu(I - S))x, \quad \forall x \in C.
\]
Then \( T_\delta \) is nonexpansive and
\[
\text{Fix}(T_\delta) = \text{Fix}(T) \cap \text{Fix}(Q_C(I - \lambda(I - S))Q_C(I - \mu(I - S)))
\]
\[
= \text{Fix}(T) \cap \text{Fix}(D)
\]
\[
= F
\]
by Lemma 2.3. For \( t \in (0, 1) \) we define a contraction via
\[
T^t_\delta x = tu + (1 - t)T_\delta x, \quad \forall x \in C.
\]
Then, the Banach contraction mapping principle ensures that there exists a unique path \( x_t \in C \) such that
\[
x_t = tu + (1 - t)T_\delta x_t
\]
for every \( t \in (0, 1) \). By Proposition 2.3, we know that \( x_t \to \overline{x} \in \text{Fix}(T_\delta) \) as \( t \to \infty \). Further, if we define \( Q_{\text{Fix}(T_\delta)}u = \overline{x} \), then \( Q_{\text{Fix}(T_\delta)} : C \to \text{Fix}(T_\delta) \) is a unique sunny nonexpansive retraction from \( C \) onto \( \text{Fix}(T_\delta) \). Noting that \( \text{Fix}(T_\delta) = F \), we see that \( Q_F : C \to F \) is indeed the unique sunny nonexpansive retraction from \( C \) onto \( F \).

(Step 5.) \( \limsup_{n \to \infty} \langle u - \overline{x}, J(x_n - \overline{x}) \rangle \leq 0 \), where \( \overline{x} = Q_F u \).

We note that
\[
\|x_n - T_\delta x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_\delta x_n\|
\]
\[
\leq \|x_n - x_{n+1}\| + \alpha_n\|u - T_\delta x_n\| + \beta_n\|x_n - T_\delta x_n\|.
\]
This implies that
\[
(1 - \beta_n)\|x_n - T_\delta x_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|u - T_\delta x_n\|.
\]
It follows from conditions (H2), (H3) and Step 3 that \( x_n - T_\delta x_n \to 0 \) as \( n \to \infty \). Since
\[
x_t - x_n = tu + (1 - t)T_\delta x_t - x_n
\]
\[
= (1 - t)(T_\delta x_t - x_n) + t(u - x_n),
\]
then
\[
\|x_t - x_n\|^2 = (1 - t)\langle T_\delta x_t - x_n, J(x_t - x_n) \rangle + t\langle u - x_n, J(x_t - x_n) \rangle
\]
\[
= (1 - t)[\langle T_\delta x_t - T_\delta x_n, J(x_t - x_n) \rangle + \langle T_\delta x_n - x_n, J(x_t - x_n) \rangle]
\]
\[
+ t\langle u - x_t, J(x_t - x_n) \rangle + t\langle x_t - x_n, J(x_t - x_n) \rangle
\]
\[
\leq (1 - t)(\|x_t - x_n\|^2 + \|T_\delta x_n - x_n\|\|x_t - x_n\|)
\]
\[
+ t\langle u - x_t, J(x_t - x_n) \rangle + t\|x_t - x_n\|^2
\]
\[
= \|x_t - x_n\|^2 + \|T_\delta x_n - x_n\|\|x_t - x_n\| + t\langle u - x_t, J(x_t - x_n) \rangle.
\]
It turns out that
\[
\langle x_t - u, J(x_t - x_n) \rangle \leq \frac{1}{t}\|T_\delta x_n - x_n\|\|x_t - x_n\|, \forall t \in (0, 1).
\]
By the above inequality, we have
\[
\limsup_{n \to \infty}(x_t - u, J(x_t - x_n)) \leq 0.
\]
Since \(J\) is strong to weak* uniformly continuous on bounded subset of \(E\), we see that
\[
|\langle u - \varpi, J(x_n - \varpi) \rangle - \langle x_t - u, J(x_t - x_n) \rangle|
\]
\[
\leq |\langle u - \varpi, J(x_n - \varpi) \rangle - \langle u - \varpi, J(x_n - x_t) \rangle|
\]
\[
+ |\langle u - \varpi, J(x_n - x_t) \rangle - \langle x_t - u, J(x_t - x_n) \rangle|
\]
\[
= |\langle u - \varpi, J(x_n - \varpi) - J(x_n - x_t) \rangle| + |\langle x_t - \varpi, J(x_n - x_t) \rangle|
\]
\[
\leq \|u - \varpi\|\|J(x_n - \varpi) - J(x_n - x_t)\| + \|x_t - \varpi\|\|x_n - x_t\|
\]
\[
\to 0 \quad \text{as} \quad t \to 0.
\]
For any \(\varepsilon > 0\), there exists \(\delta > 0\) such that for every \(t \in (0, \delta)\)
\[
\langle u - \varpi, J(x_n - \varpi) \rangle \leq \langle x_t - u, J(x_t - x_n) \rangle + \varepsilon.
\]
Therefore
\[
\limsup_{n \to \infty}\langle u - \varpi, J(x_n - \varpi) \rangle \leq \limsup_{n \to \infty}\langle x_t - u, J(x_t - x_n) \rangle + \varepsilon.
\]
This implies that
\[
\limsup_{n \to \infty}(u - \varpi, J(x_n - \varpi)) \leq 0.
\]
Strong convergence of an iterative algorithm

(Step 6.) $x_n \to \overline{x} \in Q_F u$ as $n \to \infty$. By using (3.3) we have

$$
\|x_{n+1} - \overline{x}\|^2 = \langle \alpha_n u + \beta_n x_n + \gamma_n u_n - \overline{x}, J(x_{n+1} - \overline{x}) \rangle
$$

$$
= \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x}) \rangle + \beta_n \langle x_n - \overline{x}, J(x_{n+1} - \overline{x}) \rangle
$$

$$
+ \gamma_n \langle u_n - \overline{x}, J(x_{n+1} - \overline{x}) \rangle
$$

$$
\leq \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x}) \rangle + \beta_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|
$$

$$
+ \gamma_n \|u_n - \overline{x}\| \|x_{n+1} - \overline{x}\|
$$

$$
\leq \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x}) \rangle + \beta_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|
$$

$$
+ \gamma_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|
$$

$$
\leq \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x}) \rangle + (1 - \alpha_n) \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|
$$

$$
\leq \alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x}) \rangle + \frac{1 - \alpha_n}{2} (\|x_n - \overline{x}\|^2 + \|x_{n+1} - \overline{x}\|^2),
$$

which implies that

$$
\|x_{n+1} - \overline{x}\|^2 \leq (1 - \alpha_n) \|x_n - \overline{x}\|^2 + 2\alpha_n \langle u - \overline{x}, J(x_{n+1} - \overline{x}) \rangle
$$

and hence $x_n \to \overline{x}$ as $n \to \infty$ by virtue of Lemma 2.4. This completes the proof. \qed

**Remark 3.1.** Since $L^p(1 < p \leq 2)$ is uniformly convex and $p$-uniformly smooth, we see that Theorem 3.1 is applicable to $L^p$ for $1 < p \leq 2$.

## 4. Applications

In real Hilbert spaces, Lemma 2.3 is reduced to the following.

**Lemma 4.1.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. For given $(\overline{x}, \overline{y}) \in C \times C$, where $\overline{y} = P_C(\overline{x} - \mu(I - S)\overline{x})$, $(\overline{x}, \overline{y})$ is a solution of the following problem:

$$
\begin{align*}
\langle \lambda(I - S)\overline{y} + \overline{x} - \overline{y}, x - \overline{x} \rangle \geq 0, \quad &\forall x \in C, \\
\langle \mu(I - S)\overline{x} + \overline{y} - \overline{x}, x - \overline{x} \rangle \geq 0, \quad &\forall x \in C,
\end{align*}
$$

if and only if $\overline{x}$ is a fixed point of the mapping $\overline{D} : C \to C$ defined by

$$
\overline{D}(x) = P_C[P_C(x - \mu(I - S)x) - \lambda(I - S)P_C(x - \mu(I - S)x)],
$$

where $P_C$ is a metric projection $H$ onto $C$. Utilizing Theorem 3.1 we can obtain the following results.
Theorem 4.1. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $S : C \to C$ be a $k$-strict pseudo-contraction such that $\text{Fix}(S) \neq \emptyset$ and $T : C \to C$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Assume that $F = \text{Fix}(T) \cap \text{Fix}(D) \neq \emptyset$, where $D$ is defined as Lemma 4.1. Let a sequence $\{x_n\}$ be generated by

\begin{align}
\begin{cases}
x_1 = u \in C, \\
y_n = P_C(x_n - \mu(I-S)x_n), \\\nx_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \delta T x_n + (1 - \delta) P_C(y_n - \lambda(I-S)y_n), \quad n \geq 1,
\end{cases}
\end{align}

where $\delta \in (0, 1)$, $\lambda, \mu \in (0, 2k)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that

\begin{enumerate}
    \item [(H1)] $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1,$
    \item [(H2)] $\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$
    \item [(H3)] $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1.$
\end{enumerate}

Then $\{x_n\}$ defined by (4.2) converges strongly to $\bar{x} = P_F u$ and $(\bar{x}, \bar{y})$ is a solution of problem (4.1), where $\bar{y} = P_C(\bar{x} - \mu(I-S)\bar{x})$.

Theorem 4.2. Let $C$ be a nonempty closed convex subset of $H$. Let $T, S : C \to C$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$ and $\text{Fix}(S) \neq \emptyset$. Assume that $F = \text{Fix}(T) \cap \text{Fix}(D) \neq \emptyset$, where $D$ is defined as Lemma 4.1. Let the sequence $\{x_n\}$ generated by (4.2) such that the conditions (H1), (H2), (H3) hold. Then $\{x_n\}$ converges strongly to $\bar{x} = P_F u$ and $(\bar{x}, \bar{y})$ is a solution of problem (4.1), where $\bar{y} = P_C(\bar{x} - \mu(I-S)\bar{x})$.

References

Strong convergence of an iterative algorithm


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