HARMONIC MAPPINGS WITH ANALYTIC FUNCTIONS

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Abstract. In this paper, we study harmonic, orientation-preserving, univalent mappings defined on \( \Delta = \{ z : |z| > 1 \} \) that have real coefficients or starlike analytic functions and obtain some coefficients bounds.

1. Introduction

A continuous function \( f = u + iv \) defined in a domain \( D \subseteq \mathbb{C} \) is harmonic in \( D \) if \( u \) and \( v \) are real harmonic in \( D \). We consider complex-valued, harmonic, orientation-preserving, univalent mappings \( f \) defined on \( \Delta = \{ z : |z| > 1 \} \), that are normalized at infinity by \( f(\infty) = \infty \). Such functions admit the representation

\[
f(z) = h(z) + g(z) + A \log |z|,
\]

where

\[
h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}
\]

are analytic in \( \Delta \) and \( 0 \leq |\beta| < |\alpha| \). In addition, \( a = \overline{f'}/f_z \) is analytic.
and satisfies $|a(z)| < 1$. Also one can easily show that $|A|/2 < |\alpha| + |\beta|$ by using the bound $|s_1| \leq 1 - |s_0|^2$ for analytic function $a = s_0 + s_1z^{-1} + \cdots$ in $\Delta$ that are bounded by one. By applying an affine post-mapping to $f$ we may normalize $f$ so that $\alpha = 1, \beta = 0, \text{and } a_0 = 0$. Therefore let $\Sigma$ be the set of all harmonic, orientation-preserving, univalent mappings

$$f(z) = h(z) + \overline{g(z)} + A \log |z|$$

of $\Delta$, where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in $\Delta$ and $A \in \mathbb{C}$. Hengartner and Schober[2] used the representation (1.1) to obtain coefficient bounds and distortion theorems. Some coefficients bounds for $f \in \Sigma$ also obtained by Jun[3].

In this article, we continue to investigate harmonic, orientation-preserving, univalent mappings $f$ in $\Sigma$ to get coefficients bounds for $f$ with some restrictions. In next section we consider univalent harmonic mappings $f \in \Sigma$ with real $A$ which have real coefficients and obtain estimates $|b_n - a_n| \leq n|1 + b_1 - a_1|$ for $n \geq 2$.

Also $f \in \Sigma$ with starlike analytic functions $h + g$ will be considered in section 3.

2. Harmonic mappings with real coefficients

**Theorem 2.1.** If $f \in \Sigma$ with real $A$ has real coefficients, then

$$|b_n - a_n| \leq n|1 + b_1 - a_1| \quad \text{for} \quad n \geq 2.$$

**Proof.** For $z = re^{i\theta}, \ r > 1$,

$$\text{Im}\{f(re^{i\theta})\} = \sum_{k=1}^{\infty} c_k \sin k\theta$$

(2.1)
where $c_1 = r + (b_1 - a_1)r^{-1}$ and $c_k = (b_k - a_k)r^{-k}$ for $k \geq 2$. Multiply $\sin n\theta$ to (2.1) and integrate from 0 to $\pi$, then we have

$$
\frac{2}{\pi} \int_0^\pi \text{Im}\{f(re^{i\theta})\} \sin n\theta \ d\theta
= \frac{2}{\pi} \int_0^\pi \left( \sum_{k=1}^\infty c_k \sin k\theta \right) \sin n\theta \ d\theta = \frac{2}{\pi} \int_0^\pi c_n \sin^2 n\theta \ d\theta
= c_n.
$$

From the relationship

$$
|\sin(n+1)\theta| = |\sin n\theta \cos \theta + \cos n\theta \sin \theta| \leq |\sin n\theta| + |\sin \theta|,
$$

we can show that $|\sin n\theta| \leq n|\sin \theta|$ by the mathematical induction. Thus we have

$$
|c_n| = \left| \frac{2}{\pi} \int_0^\pi \text{Im}\{f(re^{i\theta})\} \sin n\theta \ d\theta \right|
\leq \frac{2}{\pi} \int_0^\pi \left| \text{Im}\{f(re^{i\theta})\}\right| \sin n\theta \ d\theta
\leq \frac{2n}{\pi} \int_0^\pi \left| \text{Im}\{f(re^{i\theta})\}\right| \sin \theta \ d\theta.
$$

The univalence of $f$ implies that $0 \neq f(re^{i\theta}) - f(re^{-i\theta})$ since $re^{i\theta} \neq re^{-i\theta}$ for $0 < \theta < \pi$. From $0 \neq f(re^{i\theta}) - f(re^{-i\theta}) = 2i\text{Im}\{f(re^{i\theta})\}$, we have $\text{Im}\{f(re^{i\theta})\} \neq 0$. Since $\text{Im}\{f(re^{i\theta})\}$ is a continuous function of $\theta$, it must be of same sign in the interval $0 < \theta < \pi$. Thus

$$
\frac{2}{\pi} \int_0^\pi \left| \text{Im}\{f(re^{i\theta})\}\right| \sin \theta \ d\theta
= \left| \frac{2}{\pi} \int_0^\pi \text{Im}\{f(re^{i\theta})\} \sin \theta \ d\theta \right|
= |c_1|
= \left| r + \frac{b_1 - a_1}{r} \right|.
$$

Substituting this into (2.2), we have

$$
|c_n| \leq n \left| r + \frac{b_1 - a_1}{r} \right|
$$

where $c_1 = r + \frac{b_1 - a_1}{r}$ and $c_n = \frac{b_n - a_n}{r^n}$ for $n \geq 2$. Letting $r \rightarrow 1$, we obtain $|b_n - a_n| \leq n|1 + b_1 - a_1|$ for $n \geq 2$. \qed
3. Starlike analytic functions

Definition 3.1. A function $H(z)$ is starlike if each radial line from the origin hits the boundary $\partial H(\Delta)$ in exactly one point of $\mathbb{C}\setminus\{0\}$.

Let $\Sigma^*$ be the set of all harmonic, orientation-preserving, univalent mappings $f \in \Sigma$ which have starlike analytic functions $h + g$.

Theorem 3.2. If $f \in \Sigma^*$, then $\sum_{k=1}^{\infty} k|a_k + b_k|^2 \leq 1$.

Proof. A starlike function $H(z) = h + g = z + \sum_{k=1}^{\infty} (a_k + b_k)z^{-k}$ is characterized by the condition

$$\frac{\partial}{\partial \theta} \{\text{arg} H(re^{i\theta})\} > 0$$

for $r > 1$. But $\text{arg} H(re^{i\theta}) = \text{Im} \{\log H(re^{i\theta})\}$, so that

$$\frac{\partial}{\partial \theta} \{\text{Im} \{\log H(re^{i\theta})\}\} = \text{Im} \left\{ \frac{\partial}{\partial \theta} \log H(re^{i\theta}) \right\} = \text{Re} \left\{ \frac{zH'}{H} \right\} > 0.$$

From this, we have that

$$\left| \frac{1 - \frac{zH'}{H}}{1 + \frac{zH'}{H}} \right| < 1.$$

Thus

(3.1) \hspace{1cm} |H - zH'|^2 < |H + zH'|^2.

An integration of the left side of (3.1) gives

$$\frac{1}{2\pi} \int_0^{2\pi} |H(re^{i\theta}) - re^{i\theta}H'(re^{i\theta})|^2 \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (H(re^{i\theta}) - re^{i\theta}H'(re^{i\theta}))(H(re^{i\theta}) - re^{i\theta}H'(re^{i\theta})) \, d\theta$$

$$= \sum_{k=1}^{\infty} (k + 1)^2|a_k + b_k|^2r^{-2k}.$$
An integration of the right side of (3.1) gives
\[
\frac{1}{2\pi} \int_{0}^{2\pi} |H(re^{i\theta}) + re^{i\theta}H'(re^{i\theta})|^2 \, d\theta
= \frac{1}{2\pi} \int_{0}^{2\pi} (H(re^{i\theta}) + re^{i\theta}H'(re^{i\theta}))(H(re^{i\theta}) + re^{i\theta}H'(re^{i\theta})) \, d\theta
= 4r^2 + \sum_{k=1}^{\infty} (1 - k)^2|a_k + b_k|^2r^{-2k}.
\]
Therefore
\[
\frac{1}{2\pi} \int_{0}^{2\pi} |H(re^{i\theta}) - re^{i\theta}H'(re^{i\theta})|^2 \, d\theta
< \frac{1}{2\pi} \int_{0}^{2\pi} |H(re^{i\theta}) + re^{i\theta}H'(re^{i\theta})|^2 \, d\theta
\]
implies that
\[
\sum_{k=1}^{\infty} (k + 1)^2|a_k + b_k|^2r^{-2k} < 4r^2 + \sum_{k=1}^{\infty} (1 - k)^2|a_k + b_k|^2r^{-2k}.
\]
Simplify this, then we obtain
\[
\sum_{k=1}^{\infty} 4k|a_k + b_k|^2r^{-2k} < 4r^2
\]
for \(r > 1\). Letting \(r \to 1\), we have that
\[
\sum_{k=1}^{\infty} k|a_k + b_k|^2 \leq 1.
\]

**Theorem 3.3.** If \(f \in \Sigma^*\), then analytic function \(H(z) = h(z) + g(z)\) is univalent.

*Proof.* Let \(G(\zeta) = \{H(1/\zeta)\}^{-1}\) for \(|\zeta| < 1\). Then
\[
G(\zeta) = \zeta - (a_1 + b_1)\zeta^3 - (a_2 + b_2)\zeta^4 + \cdots
\]
is analytic in \(|\zeta| < 1\) and satisfies that
\[
(3.2) \quad \text{Re} \left\{ \frac{\zeta G'(\zeta)}{G(\zeta)} \right\} = \text{Re} \left\{ \frac{zH'(z)}{H(z)} \right\} > 0.
\]
If $G(\zeta_0) = 0$ at some point $0 < |\zeta_0| < 1$, then $\zeta[G'(\zeta)/G(\zeta)]$ has a simple pole at $\zeta_0$. This means that $\text{Re} \{\zeta[G'(\zeta)/G(\zeta)]\}$ takes on arbitrarily large negative values, contradicting to (3.2). Thus $G(\zeta)$ has no zeros in $|\zeta| < 1$ other than a simple zero at the origin. Let $0 < r < 1$. Since $G(\zeta)$ has one zero and no poles in $|\zeta| \leq r$, the argument principle tells us that $\Delta_{|\zeta|=r} \arg G(\zeta) = 2\pi$. That is, the circle $|\zeta| = r$ is mapped by $G(\zeta)$ onto a closed contour $C_r$ that winds around the origin once. Since $\arg G(\zeta)$ increases with $\arg \zeta$, the curve cannot intersect itself. Hence $C_r$ is a simple closed contour. That is, $G(\zeta)$ is univalent on the circle $|\zeta| = r$ and therefore $G(\zeta)$ is univalent in $|\zeta| \leq r$. Since $r$ is arbitrary, the function $G(\zeta)$ is univalent in the unit disk $\mathbb{D} = \{\zeta : |\zeta| < 1\}$. This implies that $H(z)$ is univalent.

**Theorem 3.4.** If $f \in \Sigma^*$, then $|a_n + b_n| \leq \frac{1}{\sqrt{n}}$.

**Proof.** $f \in \Sigma^*$ implies that $H(z) = h + g = z + \sum_{k=1}^{\infty} a_k + b_k)z^{-k}$ is univalent analytic function in $\Delta$ by Theorem 3.3. Thus we get $|a_1 + b_1| \leq 1$ from [1] and $\sum_{k=1}^{\infty} k|a_k + b_k|^2 \leq 1$ from Theorem 3.2.

$$n|a_n + b_n|^2 \leq 1 - |a_1 + b_1|^2 \leq 1$$

for $n \geq 2$ and so $|a_n + b_n| \leq \frac{1}{\sqrt{n}}$. □

**Corollary 3.5.** If $f \in \Sigma^*$ and $\text{Re}\{a_1 + b_1\} \leq \frac{n^2 - 1}{n^2 + 1}$ for $t > 0$, then

$$\text{Re}\{t(a_1 + b_1) - (a_n + b_n)\} \leq t \quad \text{for} \quad n \geq 2.$$ 

**Proof.** In the proof of Theorem 3.4, we know that $n|a_n + b_n|^2 \leq 1 - |a_1 + b_1|^2 \leq 1$ for $n \geq 2$ and so $|a_n + b_n| \leq \frac{\sqrt{1 - |a_1 + b_1|^2}}{\sqrt{n}}$. Hence

$$\text{Re}\{t(a_1 + b_1) - (a_n + b_n)\} \leq t \text{Re}\{a_1 + b_1\} + \frac{1}{\sqrt{n}} \sqrt{1 - |a_1 + b_1|^2}$$

$$\leq t \text{Re}\{a_1 + b_1\} + \frac{1}{\sqrt{n}} \sqrt{1 - (\text{Re}\{a_1 + b_1\})^2}.$$ 

Let $x = \text{Re}\{a_1 + b_1\}$, then $\text{Re}\{t(a_1 + b_1) - (a_n + b_n)\} \leq tx + \frac{1}{\sqrt{n}} \sqrt{1 - x^2}$. The function $F(x) = tx + \frac{1}{\sqrt{n}} \sqrt{1 - x^2}$ is increasing for $-1 \leq x \leq \frac{n^2 - 1}{n^2 + 1}$ and therefore $\text{Re}\{t(a_1 + b_1) - (a_n + b_n)\} \leq t$ for $n \geq 2$. □

**Corollary 3.6.** If $f \in \Sigma^*$ and $\text{Re}\{a_1 + b_1\} \leq \frac{n^2 - 1}{n^2 + 1}$, then

$$\text{Re}\{n(a_1 + b_1) - (a_n + b_n)\} \leq n.$$
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Proof. Set \( t = n \) in Corollary 3.5.

COROLLARY 3.7. If \( f \in \Sigma^* \), then \( Re\{n(a_1 + b_1) - (a_n + b_n)\} \leq n \) for all \( n \) sufficiently large depending on \( f \).

Proof. Fix \( f \). If \( Re\{a_1 + b_1\} = 1 \), then \( a_n + b_n = 0 \) for all \( n \geq 2 \) by Theorem 3.2 and the result holds for all \( n \geq 2 \). If \( Re\{a_1 + b_1\} < 1 \), then \( Re\{a_1 + b_1\} \leq \frac{nn^2-1}{nn^2+1} \) for all \( n \) sufficiently large since \( (n^3-1)/(n^3+1) \to 1 \) as \( n \to \infty \). In this case the result follows from Corollary 3.6.

References


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