CONTINUITY OF THE SPECTRUM ON (class\(A\))^*  

JAE WON LEE AND IN HO JEON^*

Abstract. Let (class\(A\))^* denotes the class of operators satisfying \(|T^2| \geq |T^*|^2\). In this paper, we show that the spectrum is continuous on (class\(A\))^*.

1. Introduction

Let \(L(\mathcal{H})\) denotes the algebra of bounded linear operators on a complex infinite dimensional Hilbert space \(\mathcal{H}\). Recall [1] that \(T \in L(\mathcal{H})\) is called hyponormal if \(T^*T \geq TT^*\), and \(T\) is called \(^*-\)paranormal if \n
\[ ||T^2x|| \geq ||T^*x||^2 \]

for all unit vector \(x \in \mathcal{H}\). Recently, B. P. Duggal, I. H. Jeon, and I. H. Kim [7] consider a following class of operators; we say that an operator \(T \in L(\mathcal{H})\) belongs to (class\(A\))^* if \n
\[ |T^2| \geq |T^*|^2. \]

For brevity, we shall denote classes of hyponormal operators, \(^*-\)paranormal operators, and (class\(A\))^* operators by \(\mathcal{H}, \mathcal{PN}^*\), and (class\(A\))^*, respectively. From [7] it is well known that \n
\[ \mathcal{H} \subset (\text{class}\(A\))^* \subset \mathcal{PN}^*. \]

---

^*Corresponding author.


This paper was supported by Research Fund, Kumoh National Institute of Technology.

2010 Mathematics Subject Classification: 47B20.

Key words and phrases: \(^*-\)class \(A\) operator, spectral continuity.

This is an Open Access article distributed under the terms of the Creative commons Attribution Non-Commercial License (http://creativecommons.org/licenses/by-nc/3.0/) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.
Let $\mathcal{K}$ denote the set of all compact subsets of the complex plane $\mathbb{C}$. Equipping $\mathcal{K}$ with the Hausdorff metric, one may consider the spectrum $\sigma$ as a function $\sigma : \mathcal{L}(\mathcal{H}) \to \mathcal{K}$ mapping operators $T \in \mathcal{L}(\mathcal{H})$ into their spectrum $\sigma(T)$. It is known that the function $\sigma$ is upper semicontinuous, but has points of discontinuity [8, p.56]. Studies identifying sets $\mathcal{C}$ of operators for which $\sigma$ becomes continuous when restricted to $\mathcal{C}$ has been carried out by a number authors (see, for example, [3, 4, 5, 6, 9]).

Given an operator $T \in \mathcal{L}(\mathcal{H})$, let $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathcal{H} \setminus T\mathcal{H})$. $T$ is upper semi-Fredholm if $T\mathcal{H}$ is closed and $\alpha(T) < \infty$, and then the index of $T$, $\text{ind}(T)$, is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. $T$ is said to be Fredholm if $T\mathcal{H}$ is closed and the deficiency indices $\alpha(T)$ and $\beta(T)$ are (both) finite.

Let $T^0 \in \mathcal{L}(\mathcal{K})$ denote the Berberian extension of an operator $T \in \mathcal{L}(\mathcal{H})$. Then the Berberian extension theorem [2] says that given an operator $T \in \mathcal{L}(\mathcal{H})$ there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an isometric *-isomorphism $T \to T^0 \in \mathcal{L}(\mathcal{K})$ preserving order such that $\sigma(T) = \sigma(T^0)$ and $\sigma_p(T) = \sigma_a(T) = \sigma(T)$. Here $\sigma_p$ and $\sigma_a$ denote, respectively, the point spectrum and the approximate point spectrum. In the following, we shall denote the set of accumulation points (resp. isolated points) of $\sigma(T)$ by $\text{acc}\sigma(T)$ (resp. $\text{iso}\sigma(T)$).

The aim of this paper is to give a proof of the following theorem.

**Theorem 1.1.** The spectrum $\sigma$ is continuous on $(\text{class}\mathcal{A})^*$.

To prove the theorem we adopt the Berberian technique used in [6] and we, in a sense, try to approach in a little different way.

### 2. Proof of Theorem 1.1

Since the function $\sigma$ is upper semi-continuous [8], if $\{A_n\} \subset \mathcal{L}(\mathcal{H})$ is a sequence which converges in the operator norm topology to $A \in \mathcal{L}(\mathcal{H})$ then

$$\limsup \sigma(A_n) \subseteq \sigma(A). \quad (2.1)$$

Thus to prove the theorem it would suffice to prove that if $\{A_n\} \subset (\text{class}\mathcal{A})^*$ is a sequence of operators such that $\lim_{n \to \infty} ||A_n - A|| = 0$ for
some operator $A \in (\text{class} \mathcal{A})^*$, then
\begin{equation}
\sigma(A) \subseteq \liminf_n \sigma(A_n).
\end{equation}

We first consider the following lemma, which actually is proved in [9, Lemma 2], but for the completeness we give a proof.

**Lemma 2.1.** Let $\{A_n\} \subset \mathcal{L}(\mathcal{H})$ be a sequence which converges in the operator norm topology to $A \in \mathcal{L}(\mathcal{H})$. Then
\begin{equation}
\sigma_a(A) \subseteq \liminf_n \sigma(A_n) \Rightarrow \sigma(A) \subseteq \liminf_n \sigma(A_n).
\end{equation}

**Proof.** Suppose that $\lambda \notin \liminf_n \sigma(A_n)$. Then there exists a $\delta > 0$, a neighbourhood $\mathcal{N}_\delta(\lambda)$ of $\lambda$ and a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that $\sigma(A_{n_k}) \cap \mathcal{N}_\delta(\lambda) = \emptyset$ for every $k \geq 1$. This implies that $A_{n_k} - \mu$ is Fredholm and $\text{ind}(A_{n_k} - \mu) = 0$ for every $\mu \in \mathcal{N}_\delta(\lambda)$. Since $\lambda \notin \sigma_a(A)$ by the assumption, then $A - \lambda$ is left invertible, hence upper semi-Fredholm with $\alpha(A - \lambda) = 0$. Then
\[
||(A_{n_k} - \lambda) - (A - \lambda)|| \to 0 \quad \text{as} \quad n \to 0
\]
and the continuity of the index implies that $\text{ind}(A - \lambda) = 0$, and so $A - \lambda$ is Weyl. Since $\alpha(A - \lambda) = 0$, it follows that $\lambda \notin \sigma(A)$.

It is well known that, from an argument of Newburgh [10, Lemma 3],
\begin{equation}
\lambda \in \text{iso} \sigma(A) \Rightarrow \lambda \in \liminf_n \sigma(A_n).
\end{equation}
Indeed, if $\lambda \in \text{iso} \sigma(A)$, then for every neighbourhood $\mathcal{N}(\lambda)$ of $\lambda$ there exists a positive integer $N$ such that $\sigma(A_n) \cap \mathcal{N}(\lambda) \neq \emptyset$ for all $n > N$.

Now, we consider corresponding the Berberian extensions to $A$ and the sequence $\{A_n\}$ as mentioned above, and then have that
\[
\sigma(A) = \sigma(A^o), \sigma(A_n) = \sigma(A_n^o) \quad \text{and} \quad \sigma_a(A) = \sigma_a(A^o) = \sigma_p(A^o).
\]
Since if $T \in (\text{class} \mathcal{A})^*$ then $T^o \in (\text{class} \mathcal{A})^*$, we have that
\begin{equation}
\sigma(A) \subseteq \liminf_n \sigma(A_n) \iff \sigma(A^o) \subseteq \liminf_n \sigma(A_n^o).
\end{equation}

To complete the proof of the theorem we show the following lemma in the view of Lemma 2.1.

**Lemma 2.2.** Let $\{A_n\} \subset (\text{class} \mathcal{A})^*$ be a sequence which converges in the operator norm topology to $A \in (\text{class} \mathcal{A})^*$. Then
\begin{equation}
\sigma_a(A^o) \subseteq \liminf_n \sigma(A_n^o).
\end{equation}
Proof. If $\lambda \in \sigma_p(A^o) = \sigma_p(A^r)$, then $(A^o - \lambda)^{-1}(0)$ is a reducing subspace of $A^o$ [7, Lemma 2.2], and so we have a representation of $A^o$,

$$A^o = \lambda \oplus B \text{ on } \mathcal{K} = (A^o - \lambda)^{-1}(0) \oplus \{(A^o - \lambda)^{-1}(0)\}^\perp$$

Evidently, $B - \lambda$ is upper semi-Fredholm and $\alpha(B - \lambda) = 0$. There exists an $\epsilon > 0$ such that $B - (\lambda - \mu_o)$ is upper semi-Fredholm with $\text{ind}(B - (\lambda - \mu_o)) = \text{ind}(B - \lambda)$ and $\alpha(B - (\lambda - \mu_o)) = 0$ for every $\mu_o$ satisfying $0 < |\mu_o| < \epsilon$. Choose $0 < \epsilon < \delta$ and set $\mu = \lambda - \mu_o$ ($0 < |\mu_o| < \epsilon$). (Here $\delta > 0$ as in proof of Lemma 2.1) Then $B - \mu$ is upper semi-Fredholm, $\text{ind}(B - \mu) = \text{ind}(B - \lambda)$ and $\alpha(B - \mu) = 0$. This implies that

$$A^o - \mu = \lambda - \mu \oplus B - \mu$$

is upper semi-Fredholm,

$$\text{ind}(A^o - \mu) = \text{ind}(B - \mu) \text{ and } \alpha(A^o - \mu) = 0.$$ 

Assume to the contrary that $\lambda \notin \liminf_n \sigma(A^o_n)$, then evidently, $A^o_n - \mu$ is Fredholm, with $\text{ind}(A^o_n - \mu) = 0$, and

$$\lim_{n \to \infty} \|(A^o_n - \mu) - (A^o - \mu)\| = 0.$$ 

It follows from the continuity of the index that $\text{ind}(A^o - \mu) = 0$ and $A^o - \mu$ is Fredholm. Since $\alpha(A^o - \mu) = 0$, $\mu \notin \sigma(A^o)$ for every $\mu$ in a deleted $\epsilon$-neighbourhood of $\lambda$. This contradicts to (2.4). Hence we must have that $\lambda \in \liminf_n \sigma(A^o_n)$.

\[\square\]

References

Continuity of the spectrum on (class $A$)*


Department of Applied Mathematics
Kumoh National Institute of Technology
Gumi 730-701, Korea
E-mail: ljaewon@mail.kumoh.ac.kr

Department of Mathematics Education
Seoul National University of Education
Seoul 137-742, Korea
E-mail: jihmath@snue.ac.kr