THE PRIMITIVE BASES OF THE SIGNED CYCLIC GRAPHS

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Abstract. The base $l(S)$ of a signed digraph $S$ is the maximum number $k$ such that for any vertices $u$, $v$ of $S$, there is a pair of walks of length $k$ from $u$ to $v$ with different signs. A graph can be regarded as a digraph if we consider its edges as two-sided arcs. A signed cyclic graph $\overrightarrow{C}_n$ is a signed digraph obtained from the cycle $C_n$ by giving signs to all arcs. In this paper, we compute the base of a signed cyclic graph $\overrightarrow{C}_n$ when $\overrightarrow{C}_n$ is neither symmetric nor antisymmetric. Combining with previous results, the base of all signed cyclic graphs are obtained.

1. Introduction

A sign pattern matrix $A$ of order $n$ is the $n \times n$ matrix with entries 1, 0 and $-1$. When we compute the entries of the powers of $A$, we use the operation rule that continues to hold the sign of the usual addition and multiplication, that is for any $a \in \{1, 0, -1\}$

\[
1 + 1 = 1; \quad (-1) + (-1) = -1; \quad 1 + 0 = 0 + 1 = 1; \quad (-1) + 0 = 0 + (-1) = -1
\]

\[
0 \cdot a = a \cdot 0 = 0; \quad 1 \cdot 1 = (-1) \cdot (-1) = 1; \quad 1 \cdot (-1) = (-1) \cdot 1 = -1
\]
In this case we contact the ambiguous situations $1 + (-1)$ and $(-1) + 1$, which we will use the notation $\sharp$ as in [3]. Define the addition and multiplication involving the symbol $\sharp$ as follows:

\[-1 + 1 = 1 + (-1) = \sharp; \quad a + \sharp = \sharp + a = \sharp \text{ for any } a \in \{1, -1, \sharp, 0\}\]

\[0 \cdot \sharp = \sharp \cdot 0 = 0; \quad b \cdot \sharp = \sharp \cdot b = \sharp \text{ for any } b \in \{1, -1, \sharp\}.

A generalized sign pattern matrix $A$ of order $n$ is the $n \times n$ matrix with entries $1$, $0$, $-1$ and the ambiguous sign $\sharp$. A least positive integer $l$ such that there is a positive integer $p$ satisfying $A^l = A^{l+p}$ is called the base of $A$, and denoted by $l(A)$. And the least such positive integer $p$ is called to be the period of $A$, and denoted by $p(A)$. A generalized sign pattern matrix $A$ is called powerful if there appears no $\sharp$ entry in any power of $A$. And $A$ is non-powerful if it is not powerful. If a sign pattern matrix $A$ is non-powerful and there is a number $l$ such that every entry of $A^l$ is $\sharp$, then the least such integer $l$ is the base of $A$.

In [3], Li, Hall and Stuart showed that if the sign pattern matrix $A$ is powerful, then $l(A) = l(|A|)$ where $|A|$ is the matrix by assigning each non-zero entry of $A$ to $1$. If $A$ is non-powerful, then the $\sharp$ entry appears and we have a different situation. We introduce a graph theoretic method to study the powers of a sign pattern matrix.

A signed digraph $S$ is a digraph where each arc of $S$ is assigned a sign $1$ or $-1$. The sign of the walk $W$ in $S$, denoted by $\text{sgn}(W)$, is defined to be the product of signs of all arcs in $W$. If two walks $W_1$ and $W_2$ have the same initial points, the same terminal points, the same lengths and different signs, then we say $W_1$ and $W_2$ a pair of SSSD walks. A signed digraph $S$ is powerful if $S$ contains no pair of SSSD walks. So every non-powerful primitive signed digraph contains a pair of SSSD walks.

Let $A = A(S) = (a_{ij})$ be the adjacency matrix of a signed digraph $S$, that is $\text{sgn}(i, j) = \alpha$ if and only if $a_{ij} = \alpha$ where $\alpha = 1$, or $-1$ for an arc $(i, j)$ of $S$. In this case $A$ is a sign pattern matrix which satisfies that $(i, j)$— entry of $A^k = 0$, if and only if there is no walk of length $k$ from $i$ to $j$. Moreover $(i, j)$— entry of $A^k$ is $1$ (or, $-1$), if and only if all walks of length $k$ from $i$ to $j$ are of sign $1$ (or, $-1$). Also $(i, j)$— entry of $A^k$ is $\sharp$ if and only if there is a pair of SSSD walks of length $k$ from $i$ to $j$. Thus we see from the above relations between matrices and graphs that each power of a signed digraph $S$ contains no pair of SSSD walks if and only if the adjacency sign pattern matrix $A(S)$ is powerful. A signed digraph $S$ is also said to be powerful or non-powerful if its adjacency
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sign pattern matrix is powerful or non-powerful respectively. There is an important characterization for powerful irreducible sign pattern matrices given in [2] which will be the starting point of our study on the bases of non-powerful irreducible sign pattern matrices. Let $S$ be a strongly connected signed digraph and $h$ be the index of imprimitivity of $S$ (i.e., $h$ is the greatest common divisor of the lengths of all the cycles of $S$). Then $S$ is powerful if and only if $S$ satisfies the following two conditions:

(A1) All cycles in $S$ with lengths even multiples of $h$ (if any) are positive.

(A2) All cycles in $S$ with lengths odd multiples of $h$ have the same sign.

From now on we assume that $S$ is a primitive non-powerful signed digraph of order $n$. For each pair of vertices $v_i, v_j$ of $S$, we define the local base $l_S(v_i, v_j)$ from $v_i$ to $v_j$ to be the smallest integer $l$ such that for each $k \geq l$, there is a pair of SSSD walks of length $k$ in $S$ from $v_i$ to $v_j$. The base $l(S)$ of $S$ is defined to be $\max \{l_S(v_i, v_j) \mid v_i, v_j \in V(S)\}$. It follows directly from the definitions that $l(S) = l(A)$ where $A$ is the adjacency matrix of $S$. You et al. [7] found upper bounds for the bases of primitive nonpowerful sign pattern matrices and completely characterized extremal cases. Gao, Huang and Shao [1], Shao and Gao [6] and Li and Liu [4] studied the base and local base of the primitive non-powerful signed symmetric digraphs with loops. Liang, Liu and Lai [5] gave the bounds on the $k$−th multiple generalized base index for a class of non-powerful generalized sign pattern matrices. They also characterized the extremal graphs for the (generalized) base for primitive anti-symmetric sign pattern matrices.

Let us assume that $\overline{C}_n$ is a signed digraph of order $n$ which is the cyclic graph $C_n$ on $n$ vertices by assigning signs to each arc such that it becomes a signed digraph. Liang, Liu and Lai [5] proved that the base of anti-symmetric signed cyclic graph $\overline{C}_n$ on $n$ vertices is $2n - 1$. In this paper we find the base of $\overline{C}_n$ when $\overline{C}_n$ is neither symmetric nor anti-symmetric.

Let $Q$ be the canonical cycle in $C_n$. We then can summarize the main contributions of the present paper as follows:

(C1) If the cycle $Q$ and its inverse cycle $-Q$ have the same sign, then the base of $\overline{C}_n$ is $n + 1$.

(C2) If the cycle $Q$ and its inverse cycle $-Q$ have distinct sign, then the base of $\overline{C}_n$ is $n$. 

Consequently the base of all signed cyclic graphs are obtained.

2. Main theorem

In this section we assume that \( n \) is an odd positive integer, \( C_n = (V, E) \) where \( V = \{v_0, v_1, \ldots, v_{n-1}\} \) and \( E = \{\{v_i, v_j\} | j \equiv i+1 \pmod{n}\} \). Thus \( C_n \) is a cyclic graph of odd order. If \( A = \{(v_i, v_j) | \{v_i, v_j\} \in E\} \) and \( f : A \rightarrow \{\pm 1\} \), then \( \overline{C_n} = (V, A, f) \) is a signed digraph. If \( a = (v, w) \in A \), then \( a^{-1} = (w, v) \) is the inverse of \( a \) and \( e = \{v, w\} \) is the underlying edge of \( a \). If \( W = w_0w_1 \cdots w_k \) where \( w_0, w_1, \ldots, w_k \in V \) is a walk of length \( k \) in \( C_n \), then \( -W = w_kw_{k-1} \cdots w_0 \) is the inverse of \( W \). If \( W_1 = v_0v_1 \cdots v_n \) and \( W_2 = v_nv_{n+1} \cdots v_m \) are two walks in a graph, then we use \( W_1 \cup W_2 \) to be the walk \( v_0v_1 \ldots v_m \). We also use the notation \( kW = W + W + \cdots + W \) (\( k \)-times) for a circuit \( W \).

The sign \( f(W) \) of \( W \) is \( f(w_0w_1)f(w_1w_2) \cdots f(w_{k-1}w_k) \). If \( e = \{v, w\} \), then the sign of \( e \) is \( f(vw)f(wv) \). Note that \( \overline{C_n} \) is symmetric when the sign of every edge is 1, and anti-symmetric when the sign of every edge is \(-1\). If \( W = w_0w_1 \cdots w_k \) is a cycle of length \( k \), then \( W' = w_kw_{k+1} \cdots w_0w_1 \cdots w_i \) is a rotation of \( W \) for \( 0 \leq i \leq k \).

**Lemma 1.** If \( W = w_0w_1 \cdots w_k \) is a walk of length \( k \) in an odd cycle \( C_n \) with \( w_0 = w_k \), then for each \( e \in E \), the number of \( i \) such that \( \{w_i, w_{i+1}\} = e \) is congruent to \( k \) modulo 2.

**Proof.** Since \( C_n - e \) is isomorphic to the path \( P_n \), which is bipartite, there are \( V_0, V_1 \subset V \) such that \( V_0 \cup V_1 = V \) and \( V_0 \cap V_1 = \emptyset \) and every edge except \( e \) joins a vertex of \( V_0 \) and a vertex of \( V_1 \). We may assume that the two vertices incident to \( e \) belong to \( V_0 \). For each \( i = 0, 1, \ldots, k - 1 \), the membership of \( w_i \) and \( w_{i+1} \) among \( V_0 \) and \( V_1 \) is changed if and only if \( \{w_i, w_{i+1}\} \neq e \). Since \( w_0 = w_k \), the number of \( i \) such that \( \{w_i, w_{i+1}\} \neq e \) is even. So the number of \( i \) such that \( \{w_i, w_{i+1}\} = e \) is congruent to \( k \) modulo 2. \( \square \)

**Lemma 2.** Every even cycle in an odd cycle \( C_n \) is a 2-cycle.

**Proof.** Let \( Z = w_0w_1 \cdots w_k \) be an even cycle. If \( e = \{w_0, w_1\} \), then by Lemma 1 the number of \( i \) such that \( \{w_i, w_{i+1}\} = e \) is even. Hence there is a \( t \) such that \( t \geq 1 \) and \( \{w_t, w_{t+1}\} = e \). Since \( Z \) is a cycle, \( w_i \neq w_j \) for \( i \neq j \) except \( i = 0, j = k \) or \( i = k, j = 0 \). Hence \( t = 1 \) and \( w_2 = w_0 \). Hence we have \( k = 2 \) and \( Z = w_0w_1w_0 \). \( \square \)
Let $Q$ be the canonical $n$-cycle $v_0v_1\cdots v_{n-1}v_0$ in $C_n$. Then $-Q = v_0v_{n-1}v_{n-2}\cdots v_0$.

**Lemma 3.** Let $C_n$ be a cyclic graph of odd order. Then there are exactly two odd cycles $Q$ and $-Q$ up to a rotation in $C_n$.

**Proof.** If $Z = w_0w_1\cdots w_k$ is an odd cycle, then by Lemma 1, for each edge $e$, the number of $i$ such that $\{w_i, w_{i+1}\} = e$ is odd. Since $Z$ doesn’t visit the same vertex twice, except $w_0 = w_k$, for all edge $e$ of $C_n$, there is exactly one $i$ such that $\{w_i, w_{i+1}\} = e$. Thus $k = n$ and $\{w_0, w_1, \cdots, w_{n-1}\} = V$. We may assume that $w_0 = v_0$. Since $v_1$ and $v_{n-1}$ are only two vertices adjacent to $v_0$, $w_1$ are $v_1$ or $v_{n-1}$. If $w_1 = v_1$, then $w_2$ is $v_1$ or $v_2$. Since $Z$ is a cycle, $w_2 \neq v_0$. Hence $w_2 = v_2$. Similarly we have $w_i = v_i$ for any $3 \leq i \leq n-1$ and $w_n = v_0$. Therefore $Z = Q$. If $w_1 = v_{n-1}$, then by the same method we have $Z = -Q$. \hfill $\square$

**Proposition 1.** If a signed odd cyclic graph $\tilde{C}_n$ is symmetric, then $\tilde{C}_n$ is powerful.

**Proof.** If $Z$ is an even cycle, then by Lemma 2 $Z$ is a $2$-cycle. Hence $Z = w_0w_1w_0$ for some $w_0, w_1 \in V$. Thus $f(Z) = f(w_0w_1)f(w_1w_0)$ is the same with the sign of edge $\{v_0, v_1\}$. Since $\tilde{C}_n$ is symmetric, $f(Z) = 1$. So there is no even cycle of sign $-1$. By Lemma 3 the odd cycles of $C_n$ are $Q$ and $-Q$ up to translation. Since $\tilde{C}_n$ is symmetric, we have $f(-Q) = f(w_0w_{n-1})f(w_{n-1}w_{n-2})\cdots f(w_1w_0) = f(w_0w_1)f(w_1w_2)\cdots f(w_{n-1}w_0) = f(Q)$. Thus all odd cycles in $\tilde{C}_n$ have the same signs. Hence every even cycle in $\tilde{C}_n$ has sign $1$ and every odd cycles, $Q$ and $-Q$, have the same signs. By the characterization of powerful signed digraph provided in introduction, $\tilde{C}_n$ is powerful. \hfill $\square$

It is known [3] that the base of a primitive powerful signed digraph $S$ is equal to the exponent of $S$. Hence we have the following Corollary.

**Corollary 1.** If a signed odd cyclic graph $\tilde{C}_n$ is symmetric, then the base of $\tilde{C}_n$ is $n - 1$.

The following Proposition is due to Liang, Liu and Lai [5].

**Proposition 2.** If a signed odd cyclic graph $\tilde{C}_n$ is anti-symmetric, then $l(\tilde{C}_n) = 2n - 1$.

**Lemma 4.** There is only one walk of length $n - 1$ from $v_0$ to $v_{n-1}$ in an odd cycle $C_n$. 

Proof. If $W = w_0w_1 \cdots w_k$ is a walk of length $n - 1$ from $v_0$ to $v_{n-1}$ in $C_n$, then since $|E| = n$, there is $e \in E$ such that $\{w_i, w_{i+1}\} \neq e$ for all $i = 0, 1, \cdots, n - 2$. If $e \neq \{w_0, w_{n-1}\}$, then since $C_n - e$ is bipartite, there is no walk of even length from $v_0$ to $v_{n-1}$. This contradicts to the fact that $W$ is a walk of even length $n - 1$ from $v_0$ to $v_{n-1}$. Thus $e = \{v_{n-1}, v_0\}$. Since the distance from $v_0$ to $v_{n-1}$ in $C_n - \{v_0, v_{n-1}\}$ is $n - 1$, we have $W = v_0v_1 \cdots v_{n-1}$.

**Lemma 5.** There are exactly two walks $Q$ and $-Q$ of length $n$ from $v_0$ to $v_0$ in an odd cycle $C_n$.

Proof. If $W = w_0w_1 \cdots w_n$ is a walk of length $n$ from $v_0$ to $v_0$ in $C_n$, then $w_{n-1}$ is $v_{n-1}$ or $v_1$. If $w_{n-1} = v_{n-1}$, then by Lemma 4 $w_0w_1 \cdots w_{n-1} = v_0v_1 \cdots v_{n-1}$. Hence $W = v_0v_1 \cdots v_{n-1} = Q$. By the same method, if $w_{n-1} = v_1$, then we have $W = -Q$.

**Proposition 3.** Assume that an odd cycle $\widetilde{C}_n$ is neither symmetric nor anti-symmetric. Then $l(\widetilde{C}_n) = n+1$ if $f(Q) = f(-Q)$, and $l(\widetilde{C}_n) = n$ if $f(Q) = -f(-Q)$.

Proof. Let $v, w \in V$. We may assume that $v = v_0$ and $w = v_t$ for $0 \leq t \leq n-1$. Let $\alpha = n+1$ if $f(Q) = f(-Q)$, and $\alpha = n$ if $f(Q) = -f(-Q)$. Let $e_i = \{v_i, v_{i+1}\}$ for all $i = 0, 1, \cdots, n - 2$ and $e_{n-1} = \{v_{n-1}, v_0\}$. Since $\widetilde{C}_n$ is neither symmetric nor anti-symmetric, there is $s$ such that $0 \leq s \leq n-2$ and $f(v_0v_1v_2) = f(v_0v_2v_1)$. Let $Z = v_0v_1v_2v_3$, $Z_1 = v_0v_1v_2v_3$, and $Z_2 = v_0v_1v_2v_3$. Therefore $f(Z) = -f(Z_1)$ and $f(Z_2)$. Since $n$ is odd, $\alpha \equiv t(\text{mod}2)$ or $\alpha \equiv n - t(\text{mod}2)$. We may assume that $\alpha \equiv t(\text{mod}2)$.

If $t \geq 1$ and $0 \leq s \leq t$, then since $\alpha - t - 2$ is even and $\alpha - t - 2 \geq n - (n - 1) - 2 = -1$, $\alpha - t - 2 = 2k$ for all $k \geq 0$. Let $W_1 = v_0v_1v_2v_3v_4$, and $W_2 = v_0v_sv_{s+1}v_t$. Then $(k+1)Z + W_1 + W_2$ and $kZ + W_1 + Z_1 = Z_2$ are SSSD walks of length $\alpha$ from $v_0$ to $v_t$.

If $t \geq 1$ and $s \leq n - 2$, then since $n - t - 1 = (n - s - 1) + (s - t)$, $s - t \leq \frac{n-t-1}{2}$ or $n - s - 1 \leq \frac{n-t-1}{2}$. Let $X_1 = v_0v_1v_2v_3v_4v_5$, $X_2 = v_0v_1v_2v_3v_4v_5$, and $X_3 = v_0v_{n-1}v_{n-2}v_{n-3}v_{n-4}$. If $s - t \leq \frac{n-t-1}{2}$, since $\alpha - 2s + t - 2$ is even and $\alpha - 2s + t - 2 \geq n - 2s + (2s + 1 - n) - 2 = -1$, $\alpha - 2s + t - 2 = 2k$ for some $k \geq 0$. Then $(k+1)Z + X_1 + X_2 + X_3 + X_4 + X_5 - X_2$ are SSSD walks of length $\alpha$ from $v_0$ to $v_t$. If $n - s - 1 \leq \frac{n-t-1}{2}$, by the similar method with $\alpha = 2k + 2(n - s) + t$, we can show that $(k+1)Z + X_3 - X_3 + X_1$ and $kZ + X_3 + Z_2 - X_3 + X_1$ are SSSD walks of length $\alpha$ from $v_0$ to $v_t$. 


If $t = 0$ and $f(Q) = -f(-Q)$, then $Q$ and $-Q$ are SSSD walks of length $n$ from $v_0$ to $v_1$. So $l(\tilde{C}_n) \leq n = \alpha$. If $t = 0$ and $f(Q) = f(-Q)$, then $s \leq \frac{n-1}{2}$ or $n - s - 1 \leq \frac{n-1}{2}$. Let $Y_1 = v_0v_1 \cdots v_s$ and $Y_2 = v_0v_{n-1}v_{n-2} \cdots v_{s+1}$. Since $\alpha = n + 1$ is even, $\alpha - 2s - 2$ is even. If $s \leq \frac{n-1}{2}$, then since $\alpha = 2s - 2 \geq n + 1 - (n - 1) - 2 = 0$, we have $\alpha - 2s - 2 = 2k$ for some $k \geq 0$. Hence $(k + 1)Z + Y_1 - Y_1$ and $kZ + Y_1 - Z_1 - Y_1$ are SSSD walks of length $n + 1$ from $v_0$ to $v_0$. Similarly $\alpha - 2n - 2s = 2l$ for some $l \geq 0$. If $n - s - 1 \leq \frac{n-1}{2}$, then $(l+1)Z + Y_2 - Y_2$ and $IZ + Y_2 - Z_2 - Y_2$ are SSSD walks of length $n + 1$ from $v_0$ to $v_0$. So $l(\tilde{C}_n) \leq n + 1 = \alpha$.

If $f(Q) = -f(-Q)$, then by Lemma 4 $l(\tilde{C}_n) \geq n$. So $l(\tilde{C}_n) = n = \alpha$. If $f(Q) = f(-Q)$, then by Lemma 5 $Q$ and $-Q$ are only 2 walks of length $n$ from $v_0$ to $v_0$. Since $f(Q) = f(-Q)$, there is no walk of length $n$ from $v_0$ to $v_0$ with sign $-f(Q)$. Thus $l(\tilde{C}_n) \leq n + 1$. As a consequence we have $l(\tilde{C}_n) = n + 1 = \alpha$. 

From Propositions 1, 2 and 3 we conclude the following.

**Theorem 1.** Let $\tilde{C}_n$ be a signed odd cyclic graph of order $n$. Then

$$l(\tilde{C}_n) = \begin{cases} 
  n - 1, & \text{if } \tilde{C}_n \text{ is symmetric;} \\
  2n - 1, & \text{if } \tilde{C}_n \text{ is anti-symmetric;} \\
  n + 1, & \text{if } \tilde{C}_n \text{ is neither anti-symmetric nor symmetric, and } f(Q) = f(-Q); \\
  n, & \text{if } \tilde{C}_n \text{ is neither anti-symmetric nor symmetric, and } f(Q) \neq f(-Q). 
\end{cases}$$

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