HIGHER CYCLOTOMIC UNITS FOR MOTIVIC COHOMOLOGY

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Abstract. In the present article, we describe specific elements in a motivic cohomology group \( H^1_M(\text{Spec}\, \mathbb{Q}(\zeta_l), \mathbb{Z}(2)) \) of cyclotomic fields, which generate a subgroup of finite index for an odd prime \( l \). As \( H^1_M(\text{Spec}\, \mathbb{Q}(\zeta_l), \mathbb{Z}(1)) \) is identified with the group of units in the ring of integers in \( \mathbb{Q}(\zeta_l) \) and cyclotomic units generate a subgroup of finite index, these elements play similar roles in the motivic cohomology group.

1. Introduction

When \( K = \mathbb{Q}(\zeta_m) \) is a cyclotomic field, Dirichlet’s Unit Theorem implies that the group \( \mathcal{O}_K^\times \) of units in the ring of integers in \( K \) is a finitely generated abelian group of rank \( \phi(m)/2 - 1 \). This is proved by using Dirichlet’s regulator map \( \mathcal{O}_K^\times \) onto a full lattice in a hyperplane in the vector space \( \mathbb{R}^{\phi(m)} \).

In [5], a chain complex for motivic cohomology of a regular local ring \( R \), by Goodwillie and Lichtenbaum, is defined to be the chain complex associated to the simplicial abelian group \( d \mapsto K_0(R\Delta^{d}, \mathbb{G}_m^{\wedge t}) \), together with a shift of degree by \(-t\). Here, \( K_0(R\Delta^d, \mathbb{G}_m^{\wedge t}) \) is the Grothendieck group of the exact category of projective \( R \)-modules with \( t \) commuting.
automorphisms factored by the subgroup generated by classes of the objects one of whose \( t \) automorphisms is the identity map. Walker showed, in Theorem 6.5 of [10], that it agrees with motivic cohomology given by Voevodsky and thus various other definitions of motivic cohomology for smooth schemes over an algebraically closed field.

A higher regulator map is originally invented by A. Borel in [3]. Bloch ([1]) introduced a single-valued analogue \( D_2 \) of the dilogarithm function to describe the regulator map on \( K_3(\mathbb{C}) \) explicitly. The author ([9]) introduced another way to formulate a single-valued dilogarithm function and use it to explicitly define a motivic regulator map for \( H^1_M(\text{Spec} \mathbb{C}, \mathbb{Z}(2)) \) defined via Goowillie-Lichtenbaum complex.

The purpose of this paper is to find a set of rational generators of the motivic cohomology \( H^1_M(\text{Spec} \mathbb{Q}(\zeta), \mathbb{Z}(2)) \) for an odd prime \( l \). As cyclotomic units play such roles in \( H^1_M(\text{Spec} \mathbb{Q}(\zeta), \mathbb{Z}(2)) \approx \mathcal{O}_{\mathbb{Q}(\zeta)}^\times \), we term these generators as higher cyclotomic units.

2. The group \( H^1_M(\text{Spec} K, \mathbb{Z}(1)) \)

For a field \( K \), let \( K_0(K \Delta^d, \mathbb{G}_m^A) \) be the abelian group generated by symbols \( (A_1, \ldots, A_t) \) where \( A_1, \ldots, A_t \) are commuting matrices in \( GL_n(K[T_1, \ldots, T_d]) \) for some \( n \geq 1 \) subject to the relations:

\[
(A_1, \ldots, A_t) = (C^{-1}A_1C, \ldots, C^{-1}A_tC) \quad \text{for any } C \in GL_n(K[T_1, \ldots, T_d]),
\]

\[
(A_1, \ldots, A_t) + (B_1, \ldots, B_t) = \left( \begin{array}{cccc}
A_1 & 0 & \cdots & 0 \\
0 & B_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & B_t
\end{array} \right)
\]

and \( (A_1, \ldots, A_t) = 0 \) if some \( A_i \) is the identity matrix. In particular, by the second relation, any element in \( K_0(K \Delta^d, \mathbb{G}_m^A) \) may be represented by a single symbol \( (A_1, \ldots, A_t) \).

Then, \( H^1_M(\text{Spec} K, \mathbb{Z}(1)) \) is the cokernel of the homomorphism

\[
\partial : K_0(K \Delta^1, \mathbb{G}_m^A) \to K_0(K \Delta^0, \mathbb{G}_m^A).
\]

More explicitly the symbol represented by an invertible \( n \times n \) matrix \( A(T) \) is mapped to \( (A(1)) - (A(0)) \). But, the units in the ring \( K[T] \) is the same as the units in the field \( K \). Therefore, \( \det A(0) = \det A(1) \) in \( K^\times \). Hence determinant induces a map \( H^1_M(\text{Spec} K, \mathbb{Z}(1)) \) onto \( K^\times \).

On the other hand, the Whitehead group \( K_1(K) \) is defined as the quotient group \( GL(K)/E(K) \) where \( E(K) \) is a subgroup of \( GL(K) \) generated by elementary matrices \( e_{ij}(r) \) whose diagonal entries are all 1 and whose \((i, j)\) component is \( r \) and 0 everywhere else. Let \( A(T) \) be
the matrix of the same size as $e_{ij}(r)$ and whose diagonal entries are all 1 and whose $(i, j)$ component is $rT$ and 0 everywhere else. Then $A(0)$ is the identity matrix while $A(1)$ is the elementary matrix $e_{ij}(r)$.

So, any symbol represented by an elementary matrix is in the image of $\partial : K_0(K\Delta^1, \mathbb{G}^\times_m) \to K_0(K\Delta^0, \mathbb{G}^\times_m)$. Therefore, we have a map from $K_1(K)$ onto $H^1_M(\text{Spec}K, \mathbb{Z}(1))$ which fits into a commutative diagram

$$K_1(K) \xrightarrow{\sim} H^1_M(\text{Spec}K, \mathbb{Z}(1))$$

Therefore, $H^1_M(\text{Spec}K, \mathbb{Z}(1)) \simeq K_1(K) \simeq K^\times$. Now define a homomorphism $R\text{Log} : H^1_M(\text{Spec}\mathbb{C}, \mathbb{Z}(1)) \to (\mathbb{R}, +)$ by sending the symbol $A$ to $\log|\det A|$.

If $K$ is a number field, any embedding $\sigma$ of $K$ into $\mathbb{C}$ induces a map $H^1_M(\text{Spec}K, \mathbb{Z}(1)) \to H^1_M(\text{Spec}\mathbb{C}, \mathbb{Z}(1))$. Let $\sigma_1, \ldots, \sigma_r$ be real embeddings of $K$ and $\sigma_{r+1}, \ldots, \sigma_{r+s}$ be complex embeddings of $K$ so that $r_1 + 2r_2 = [K : \mathbb{Q}]$. Then

$$R = (R\text{Log} \circ \sigma_1, \ldots, R\text{Log} \circ \sigma_r, 2R\text{Log} \circ \sigma_{r+1}, \ldots, 2R\text{Log} \circ \sigma_{r+s})$$

is the usual Dirichlet regulator map. $R : \mathcal{O}_K^\times \to \mathbb{R}^{r_1+r_2}$ is a map onto a full lattice in a hyperplane in $\mathbb{R}^{r_1+r_2}$ with a finite kernel. In fact, the kernel is the set of roots of unity in $\mathcal{O}_K^\times$.

3. Generators and Relations in $H^1_M(\text{Spec}K, \mathbb{Z}(2))$

$K_0(\mathbb{C}\Delta^1, \mathbb{G}^\times_m^{\wedge 2})$ can be recognized as the abelian group generated by pairs $(A, B) = \langle (A(T), B(T)) \rangle$ and certain explicit relations, where $A, B$ are commuting matrices in $GL_n(\mathbb{C}[T])$ for $n \geq 0$. On the other hand, $K_0(\mathbb{C}\Delta^2, \mathbb{G}^\times_m^{\wedge 2})$ is recognized as the abelian group generated by the symbols $(A(X, Y), B(X, Y))$ with commuting $A(X, Y), B(X, Y) \in GL_n(\mathbb{C}[X, Y])$ and certain relations, and the boundary map $\partial$ on the Goodwillie-Lichtenbaum motivic complex sends the symbol $(A(X, Y), B(X, Y))$ to $(A(1−T, T), B(1−T, T)) = (A(0, T), B(0, T)) + (A(T, 0), B(T, 0))$ in $K_0(\mathbb{C}\Delta^1, \mathbb{G}^\times_m^{\wedge 2})$. The same symbol $(A, B)$ will denote the element in $K_0(\mathbb{C}\Delta^1, \mathbb{G}^\times_m^{\wedge 2})/\partial K_0(\mathbb{C}\Delta^2, \mathbb{G}^\times_m^{\wedge 2})$ represented by $(A, B)$, by abuse of notation. The motivic cohomology group $H^1_M(\text{Spec}\mathbb{C}, \mathbb{Z}(2))$ is a subgroup of this quotient group.
In $H^1_{\text{M}}(\text{Spec}\mathbb{C}, \mathbb{Z}(2))$, note that we have the following two simple relations for any two commuting matrices $A, B$ in $GL_n(\mathbb{C}[T])$:

\[
-(A(T), B(T)) = (A(1-T), B(1-T))
\]

\[
(A_1(T), B_1(T)) + (A_2(T), B_2(T)) = (A_1(T) \oplus A_2(T), B_1(T) \oplus B_2(T)).
\]

The first relation can be shown by applying the boundary map $\partial$ to the symbol $(A(X), B(X))$ in $K_0(\mathbb{C}\Delta^2, \mathbb{G}^\wedge_m)$ and by noting that $(A, B) = 0$ in $H^1_{\text{M}}(\text{Spec}\mathbb{C}, \mathbb{Z}(2))$ when $A$ and $B$ are constant matrices. The fact that $(A, B) = 0$ for constant matrices $A$ and $B$ is obtained simply by applying the boundary map $\partial$ to the symbol $(A, B)$ in $K_0(\mathbb{C}\Delta^2, \mathbb{G}^\wedge_m)$. Hence, an element of $H^1_{\text{M}}(\text{Spec}\mathbb{C}, \mathbb{Z}(2))$ can be represented by a single expression $(A, B)$, where $A, B$ are commuting matrices in $GL_n(\mathbb{C}[T])$ for some positive integer $n$.

4. Motivic Regulator Map for $H^1_{\text{M}}(\text{Spec}\mathbb{C}, \mathbb{Z}(2))$

For $A \in GL_n(\mathbb{C}[T])$, let $P_A(\lambda)$ be the characteristic polynomial associated with $A$. It is a polynomial in $\lambda$ of degree $n$ with coefficients in $\mathbb{C}[T]$. Let $x$ be a point in $\mathbb{C}$ and $O_x$ be the local ring of germs of analytic functions at $x$. Identifying $T$ with the identity function $\mathbb{C} \to \mathbb{C}$ embeds $\mathbb{C}[T]$ into $O_x$. Then for commuting matrices $A, B \in GL_n(\mathbb{C}[T])$, let $x \in \mathbb{C}$ be such that $P_A(\lambda) = (\lambda - a_1(T))(\lambda - a_2(T)) \cdots (\lambda - a_n(T))$ and $P_B(\lambda) = (\lambda - b_1(T))(\lambda - b_2(T)) \cdots (\lambda - b_n(T))$ for some $a_1(T), \ldots, a_n(T)$ and $b_1(T), \ldots, b_n(T) \in O_x$. Then there exists $S \in GL_n(O_x)$ such that $S^{-1}AS$ and $S^{-1}BS$ are upper triangular matrices in $GL_n(O_x)$, i.e., $A, B$ are simultaneously triangularizable in $GL_n(O_x)$ (\cite{8} or \cite{9}).

Let $(\lambda_1(T), \lambda_2(T), \ldots, \lambda_n(T))$ and $(\mu_1(T), \mu_2(T), \ldots, \mu_n(T))$ be the ordered $n$-tuples of diagonal entries of $S^{-1}AS$ and $S^{-1}BS$. Then, the set of pairs $\{(\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_n, \mu_n)\}$ of elements of $O_x$ is determined only by $A, B$ and $x \in \mathbb{C}$ and is independent of the choice of $S$.

For $A \in GL_n(\mathbb{C}[T])$, let $P_A = P_{A,1}P_{A,2} \cdots P_{A,s}$ be the factorization of the characteristic polynomial $P_A$ of $A$ into irreducible polynomials in $\mathbb{C}[\lambda, T]$. The discriminant $\text{disc}_{A,i}$ of each irreducible polynomial $P_{A,i}$ is a nonzero polynomial in $\mathbb{C}[T]$. Let $S_A = \{z \in \mathbb{C} | \text{disc}_{A,i} = 0 \text{ for some } i\}$. Then $S_A$ is a finite set.

Now divide the unit interval $[0, 1]$ into subintervals $[t_0, t_1], [t_0, t_1], \ldots, [t_{r-1}, t_r]$ such that each open interval $(t_{i-1}, t_i)$ is contained in a simply
connected open subset $U$ of $\mathbb{C} - (S_A \cup S_B)$. Using the analytic continuation, we have the set $\{(\lambda_{i,1}, \mu_{i,1}), \ldots, (\lambda_{i,n}, \mu_{i,n})\}$ of pairs of analytic functions on $U$ which are locally pairs. At each $x \in U$, there is $S \in \text{GL}_n(\mathcal{O}(V))$ for some open neighborhood $V \subseteq U$ of $x$ such that $S^{-1}AS$ and $S^{-1}BS$ are both upper triangular matrices in $\text{GL}_n(\mathcal{O}(V))$. Here, $\mathcal{O}(V)$ denotes the ring of analytic functions on $V$. For each subinterval $(t_{i-1}, t_i)$, let $\{(\lambda_{i,1}, \mu_{i,1}), (\lambda_{i,2}, \mu_{i,2}), \ldots, (\lambda_{i,n}, \mu_{i,n})\}$ be the set of pairs of elements in $\mathcal{O}(U)$ which are locally ordered $n$-tuples of diagonal entries of $S^{-1}AS$ and $S^{-1}BS$. Then $\lambda_{i,l}$ and $\mu_{i,l}$ are smooth maps from $(t_{i-1}, t_i)$ into $\mathbb{C} - \{0\}$ and may be thought of as paths into $\mathbb{C} - \{0\}$.

For paths $\gamma$ and $\sigma$ in $\mathbb{C} - \{0\}$. Let $D(\gamma_1, \gamma_2)$ be the real number defined by

$$D(\gamma, \sigma) = \text{Im} \left( \int_0^1 \log |\gamma(t)| \frac{\sigma'(t)}{\sigma(t)} dt - \int_0^1 \log |\sigma(t)| \frac{\gamma'(t)}{\gamma(t)} dt \right)$$

For two commuting matrices $A, B \in \text{GL}_n(\mathbb{C}[T])$, we define

$$D(A, B) = \sum_{i=1}^n \sum_{l=1}^n D(\lambda_{i,l}, \mu_{i,l})$$

Then the integral which defines each term $D(\lambda_{i,l}, \mu_{i,l})$ is convergent and thus $D$ gives a map from the set of pairs of commuting matrices in $\text{GL}_n(\mathbb{C}[T])$ into $\mathbb{R}$.

For notational convenience, we write

$$D(A, B) = \sum_{i=1}^n D(\lambda_i, \mu_i)$$

where, for each $t$,

$$\{(\lambda_1(t), \mu_1(t)), (\lambda_2(t), \mu_2(t)), \ldots, (\lambda_n(t), \mu_n(t))\}$$

are pairs of eigenvalues of $A(t)$ and $B(t)$, which are piecewise smooth paths.

For any continuous piecewise smooth path $\sigma$ from $[0, 1]$ into $\mathbb{C}$, we may divide the interval $[0, 1]$ into subintervals $[t_0, t_1], [t_0, t_1], \ldots, [t_{r-1}, t_r]$ such that, for each $i = 1, \ldots, r$, $\sigma((t_{i-1}, t_i))$ is contained in an open subset $U$ of $\mathbb{C}$ such that there is $S \in \text{GL}_n(\mathcal{O}(U))$ such that $S^{-1}AS$ and $S^{-1}BS$ are upper triangular matrices in $\text{GL}_n(\mathcal{O}(U))$. Then we may define $D(A(\sigma), B(\sigma))$ as the sum.
\[ D(A(\sigma), B(\sigma)) = \sum_{i=1}^{r} \sum_{l=1}^{n} D(\lambda_{i,l} \circ \sigma, \mu_{i,l} \circ \sigma). \]

A proof of the following theorem was given in [8].

**Theorem 4.1.** With the same notation as above, for two commuting matrices \( A, B \in GL_n(\mathbb{C}[T]) \), we define \( D(A, B) = \sum_{i=1}^{n} D(\lambda_i, \mu_i) \). Then \( D \) gives a homomorphism from \( H^1_M(\text{Spec} \mathbb{C}, \mathbb{Z}(2)) \) into \( \mathbb{R} \). In fact, it is a homomorphism on \( K_0(\mathbb{C} \Delta^1, \mathbb{G}^\Delta_m) / \partial K_0(\mathbb{C} \Delta^2, \mathbb{G}^\Delta_m) \).

We also have the following fundamental properties of our \( D \)-map ([8] or [9]):

(i) (Skew-Symmetry) \( D(A, B) = -D(B, A) \) for commuting matrices \( A, B \in GL_n(\mathbb{C}[T]) \).

(ii) (Vanishing of Constant Matrix) \( D(A, B) = 0 \) if \( A, B \in GL_n(\mathbb{C}[T]) \) are commuting and either \( A \) or \( B \) is in \( GL_n(\mathbb{C}) \).

(iii) (Bilinearity) \( D(A_1A_2, B) = D(A_1, B) + D(A_2, B) \) whenever \( A_1, A_2, B \in GL_n(\mathbb{C}[T]) \) are commute with each other.

(iv) (Vanishing of Matrices with Real Coefficients) \( D(A, B) = 0 \) if \( A, B \in GL_n(\mathbb{R}[T]) \).

**5. Technique of constructing elements in** \( H^1_M(\text{Spec} K, \mathbb{Z}(2)) \)

In [9], the following technical lemma was introduced to construct explicit elements in the motivic cohomology group \( H^1_M(\text{Spec} \mathbb{C}, \mathbb{Z}(2)) \). Let \( K \) be a subfield of \( \mathbb{C} \).

**Lemma 5.1.** Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be elements in \( \mathbb{C} \) not equal to either 0 or 1. Suppose also that \( a_1a_2 \cdots a_n = b_1b_2 \cdots b_n \) and \( (1 - a_1)(1 - a_2) \cdots (1 - a_n) = (1 - b_1)(1 - b_2) \cdots (1 - b_n) \). If all the elementary symmetric functions evaluated at \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) are in \( K \), then there is a matrix \( A(T) \) in \( GL_n(\mathbb{K}[T]) \) such that \( I - A(T) \) is also invertible and the eigenvalues of \( A(0) \) and \( A(1) \) are \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \), respectively.

We use this construction to define a map \( \theta : \mathcal{B}(K) \to H^1_M(\text{Spec} K, \mathbb{Z}(2)) \), which will be used to compare the Bloch’s dilogarithmic map to our motivic regulator map.
The group $\mathcal{B}(K)$ of a field $K$ is defined as the kernel of the homomorphism
\[ \mathcal{A}(K) \xrightarrow{\lambda} K^\times \otimes_{\mathbb{Z}} K^\times \]
where $\mathcal{A}(K)$ is a free abelian group generated by the symbols $[a]$ with $a \in K - \{0, 1\}$, $K^\times \otimes_{\mathbb{Z}} K^\times$ is $K^\times \otimes_{\mathbb{Z}} K^\times$ divided by the subgroup generated by $a \otimes (-a)$ with $a \in K^\times$ and where $\lambda([a]) = a \wedge (1 - a)$ (\cite{H} or [Z]).

Define $\theta_1 : \mathcal{A}(K) \to K_0(K\Delta^1, \mathbb{G}^\wedge_2)$ by $\theta_1([a]) = 2(A(a, T), I - A(a, T))$ for every $a \in K - \{0, 1\}$, where
\[ A(a, T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4a & (4 - a)T + a & (a - 4)T + 4 \end{pmatrix}. \]

Then $\theta_1$ induces a map $\mathcal{A}(K) \to K_0(K\Delta^1, \mathbb{G}^\wedge_2)/\partial K_0(K\Delta^2, \mathbb{G}^\wedge_2)$, which we denote again by $\theta_1$ by abuse of notation.

In [9], it was shown that there exists a map $\theta : \mathcal{B}(K) \to H^1_{\mathcal{M}}(\text{Spec}K, \mathbb{Z}(2))$ as a lifting of $\theta_1$ and we have the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{B}(K) & \xrightarrow{\theta} & \mathcal{A}(K) \\
\downarrow \theta & & \downarrow \theta_1 \\
H^1_{\mathcal{M}}(\text{Spec}K, \mathbb{Z}(2)) & \xhookrightarrow{} & K_0(K\Delta^1, \mathbb{G}^\wedge_2)/\partial K_0(K\Delta^2, \mathbb{G}^\wedge_2)
\end{array}
\]

6. Compatibility With Bloch-Wigner Function

Bloch-Wigner function $D_2 : \mathbb{C} \to \mathbb{R}$ may be defined as below (\cite{B} or [6]). When $|z - \frac{1}{2}| < \frac{1}{2}$, it is given by
\[ D_2(z) = -\text{Im} \int_0^z \log(1 - t) \frac{dt}{t} + \arg(1 - z) \log |z| \]
where the principal branches of log and arg are used. Then it can be shown that $D_2$ as a real analytic function is invariant under the continuation along small loops around 0 and 1. Thus $D_2$ is extended to a single-valued, real analytic function on $\mathbb{C} - \{0, 1\}$. The function $D_2$ extends to a continuous function on all of $\mathbb{C}$ by setting $D_2(0) = D_2(1) = 0$.

Then we have the following basic properties of the Bloch-Wigner function:

(i) $D_2$ vanishes on the real line.
(ii) For any $z \in \mathbb{C}$, we have

$$D_2(z) + D_2(1 - z) = D_2(z) + D_2(1/z) = D_2(z) + D_2(z) = 0.$$ 

(iii) (Duplication Formula (c.f. [4])) For any $z \in \mathbb{C}$, we have

$$D_2(z) + D_2(-z) = \frac{1}{2} D_2(z^2).$$

Then the most important lemma which shows the connection between our $D$-map and the Bloch-Wigner function is as follows ([9])

**Lemma 6.1.** Let $\gamma_1$ be a path from $[0, 1]$ into $\mathbb{C} - \{0, 1\}$ and $\gamma_2(t) = 1 - \gamma_1(t)$ for every $t \in [0, 1]$. Then

$$D(\gamma_1, \gamma_2) = D_2(\gamma_1(1)) - D_2(\gamma_1(0)),$$

where $D$ is as in Section [4]

**Corollary 6.2.** Let $A(T)$ be an invertible matrix in $GL_n(K[T])$ such that $I - A(T)$ is also invertible. Let $A(1)$ and $A(0)$ have eigenvalues $b_1, b_2, \ldots, b_n$ and $a_1, a_2, \ldots, a_n$ in $\mathbb{C}$, respectively. Then

$$D(A(T), I - A(T)) = \sum_{i=1}^{n} D_2(b_i) - \sum_{i=1}^{n} D_2(a_i).$$

**Proposition 6.3.** The Bloch-Wigner function $D_2 : B(K) \to \mathbb{R}$ is the composite $D \circ \theta$ where The map $\theta : B(K) \to H^1_{\text{adm}}(\text{Spec}K, \mathbb{Z}(2))$ is given in Section [8]

**Proof.** In the construction of $\theta$, the matrix $A(a, T)$ was such that

$$D(A(T), I - A(T)) = D_2(-2) + D_2(2) + D_2(a)$$

$$- D_2(4) - D_2(\sqrt{a}) - D_2(-\sqrt{a})$$

$$= D_2(a) - D_2(\sqrt{a}) - D_2(-\sqrt{a}) = \frac{1}{2} D_2(a).$$

by the Duplication Formula of $D_2$. Hence, $\theta_1([a]) = 2(A, I - A)$ will yield $D_2(a)$ under $D$. \qed
7. Higher Cyclotomic Units

Let \( \zeta_m \) be a primitive \( m \)-th root of unity where \( m \) is an odd positive integer, and let \( K = \mathbb{Q}(\zeta_m) \) be a cyclotomic field.

Let \( \mathbb{Z}_D = \text{Ker} \left( D : K_0(K \Delta^1, \mathbb{G}^\wedge_{m^2}) \to \mathbb{R} \right) \). The the image \( \partial \mathbb{Z}_D \) of \( \mathbb{Z}_D \) under the boundary homomorphism \( \partial : K_0(K \Delta^1, \mathbb{G}^\wedge_{m^2}) \to K_0(K \Delta^0, \mathbb{G}^\wedge_{m^2}) \).

Then we have the following lemma ([9]).

**Lemma 7.1.** \( \partial \mathbb{Z}_D \) contains elements of the following forms and for any element of these forms, we may find an explicit \( z \in \mathbb{Z}_D \) whose image under \( \partial \) is equal to the element.

(i) \( (AB, C) - (A, C) - (B, C) \) and \( (C, AB) - (C, A) - (C, B) \), for commuting matrices \( A, B, C \in \text{GL}_n(K) \);

(ii) \( (x,1-x) - (y,1-y) \), for \( x, y \in K \cap \mathbb{R}^+ - \{1\} \).

**Proof.** (i) Let \( A(T) \) be the \( 2n \times 2n \) matrix

\[
\begin{pmatrix}
0 & I \\
-AB & T(I + AB) + (1 - T)(A + B)
\end{pmatrix}
\]

Then, \( A(T) \) is in \( GL_{2n}(K[T]) \), \( (A(T), C \oplus C) \) is in \( \mathbb{Z}_D \) since \( C \) is a constant matrix. But, the boundary of \( (A(T), C \oplus C) \) is \( (I \oplus AB, C \oplus C) - (A \oplus B, C \oplus C) = (AB, C) - (A, C) - (B, C) \). The proof for \( (C, AB) - (C, A) - (C, B) \) is similar.

For (ii), note that Bloch-Wigner function vanishes on the real line and that a square root of a positive real number is a real number. Apply Lemma 5.1 to \( a_1 = x, a_2 = \sqrt{y}, a_3 = -\sqrt{y}, b_1 = -\sqrt{x}, b_2 = \sqrt{x}, b_3 = y \) to get \( A(T) \in GL_3((K \cap \mathbb{R})[T]) \). Then \( z = 2(A(T), I - A(T)) \) is in
But, by the theory of rational canonical form, $\partial z$ is equal to

\[
2 \left( (y, 1 - y) + \left( \begin{array}{cc} 0 & 1 \\ x & 0 \end{array} \right), \left( \begin{array}{cc} 1 & -1 \\ -x & 1 \end{array} \right) \right) \\
- \left( (x, 1 - x) + \left( \begin{array}{cc} 0 & 1 \\ y & 0 \end{array} \right), \left( \begin{array}{cc} 1 & -1 \\ -y & 1 \end{array} \right) \right) \\
= \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right), \left( \begin{array}{cc} 1 - y & 0 \\ 0 & 1 - y \end{array} \right) \right) - \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right), \left( \begin{array}{cc} 1 & -1 \\ -y & 1 \end{array} \right) \\
- \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right), \left( \begin{array}{cc} 1 - x & 0 \\ 0 & 1 - x \end{array} \right) + \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right), \left( \begin{array}{cc} 1 & -1 \\ -x & 1 \end{array} \right) \\
= \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ y & 1 \end{array} \right) - \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ x & 1 \end{array} \right) \\
= \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right), \left( \frac{1 - y}{1 - y} \right) \left( \begin{array}{cc} 1 & 1 \\ y & 1 \end{array} \right) \left( \frac{1 - y}{1 - y} \right)^{-1} \\
- \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right), \left( \frac{-x}{1 - x} \right) \left( \begin{array}{cc} 1 & 1 \\ x & 1 \end{array} \right) \left( \frac{-x}{1 - x} \right)^{-1} \\
= \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ y - 1 & 2 \end{array} \right) - \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ x - 1 & 2 \end{array} \right) \right).
\]

By taking the boundary of the element

\[
\left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ y - 1 & (2 - y)T + 2(1 - T) \end{array} \right) \\
- \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ x - 1 & (2 - x)T + 2(1 - T) \end{array} \right) \right),
\]
which is in \( \mathbf{Z}_D \) by the fundamental property (iv) of the \( D \)-map in Section 4. We see that
\[
\partial z = \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right) \cdot \left( \begin{array}{ccc} 0 & 1 & 1 \\ y-1 & 2-y \end{array} \right) - \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right) \cdot \left( \begin{array}{ccc} 0 & 1 & 1 \\ x-1 & 2-x \end{array} \right)
\]
\[
= \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right) \cdot \left( \begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 0 & 1 & 1 \\ y-1 & 2-y \end{array} \right) - \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right) \cdot \left( \begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 0 & 1 & 1 \\ x-1 & 2-x \end{array} \right)
\]
\[
= \left( \begin{array}{cc} y & 0 \\ 0 & y \end{array} \right) \cdot \left( \begin{array}{cc} 1-y & 0 \\ 0 & 1 \end{array} \right) - \left( \begin{array}{cc} x & 0 \\ 0 & x \end{array} \right) \cdot \left( \begin{array}{cc} 1-x & 0 \\ 0 & 1 \end{array} \right)
\]
\[
= (y, 1-y) - (x, 1-x)
\]
in modulo \( \partial \mathbf{Z}_D \). So, (ii) is the boundary of \( 2(A(T), I - A(T)) \). \( \square \)

PROPOSITION 7.2. \((m\text{-th Roots of Unity})\) If \( \zeta_m \) is a primitive \( m \)-the root of unity for an odd integer \( m > 0 \), there exists an explicit element \( h(\zeta_m) \) in \( H^1_M(\text{Spec} \mathbf{Q}(\zeta_m), \mathbf{Z}(2)) \) whose value under the dilogarithm \( D \) is equal to \( mD^2(\zeta_m) \).

\begin{proof}
Let \( \zeta \) be a primitive \( 2m \)-th root of unity such that \( \zeta^2 = \zeta_m \). Then
\[
a_1 = 4, \ a_2 = \zeta, \ a_3 = -\zeta, \ b_1 = -2, \ b_2 = 2, \ b_3 = \zeta^2
\]
satisfy the conditions of Lemma 5.1 with \( K = \mathbf{Q}(\zeta_m) \). Actually,
\[
a_1 = x^2, \ a_2 = y, \ a_3 = -y, \ b_1 = -x, \ b_2 = x, \ b_3 = y^2
\]
for any \( x, y \in K \) would do. Let \( A(T) = A(\zeta^2, T) \) where \( A(a, T) \) is the matrix used to define \( \theta_T \) in Section 5. Then by the calculation in the proof of Proposition 6.3, we have \( 2D(A(T), I - A(T)) = D_2(\zeta_m) \) and thus \( 2mD(A(T), I - A(T)) = mD_2(\zeta_m) \).

Now the only possible problem is that its image \( 2m(A(1), I - A(1)) - 2(A(0), I - A(0)) \) under \( \partial \) might not be 0 in \( K_0(\mathbf{Q}(\zeta_m) \Delta^0, \mathbf{G}_m^\wedge) \), so \( 2m(A(T), I - A(T)) \) might not represent an element in \( H^1_M(\text{Spec} \mathbf{Q}(\zeta_m), \mathbf{Z}(2)) \). So we need to find an element \( z \) in \( K_0(\mathbf{Q}(\zeta_m) \Delta^1, \mathbf{G}_m^\wedge) \) whose image under the boundary map \( \partial \) is equal to \( 2m(A(0), I - A(0)) - 2m(A(1), I - A(1)) \) and \( D(z) = 0 \). Then, \( 2m(A(T), I - A(T)) - z \) would
represent an element of \( H^1_M(\text{Spec}\mathbb{Q}(\zeta_m), \mathbb{Z}(2)) \) and its value under \( D \) would be \( mD_2(\zeta_m) \). But,

\[
2(A(1), I - A(1)) - 2(A(0), I - A(0)) \\
= 2(-2, 3) + 2(2, -1) + 2(\zeta^2, 1 - \zeta^2) - 2(4, -3) \\
- 2\left(\begin{pmatrix} 0 & 1 \\ \zeta^2 & 0 \end{pmatrix}^m, \begin{pmatrix} 1 & -1 \\ -\zeta^2 & 1 \end{pmatrix}\right).
\]

Therefore, it is enough to prove that \( 2mw \) is in \( \partial\mathbb{Z}_D \), where

\[
w = (-2, 3) + (2, -1) + (\zeta^2, 1 - \zeta^2) - (4, -3) - \left(\begin{pmatrix} 0 & 1 \\ \zeta^2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -\zeta^2 & 1 \end{pmatrix}\right).
\]

But,

\[
2mw = m((4, 3) + (2, 1) - (4, 9)) + (1, 1 - \zeta^2) \\
- \left(\begin{pmatrix} 0 & 1 \\ \zeta^2 & 0 \end{pmatrix}^m, \begin{pmatrix} 1 & -1 \\ -\zeta^2 & 1 \end{pmatrix}\right)
\]

modulo \( \partial\mathbb{Z}_D \) by Lemma 7.1 (i). Here

\[
\begin{pmatrix} 0 & 1 \\ \zeta^2 & 0 \end{pmatrix}^m = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^2 \end{pmatrix}^m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

So,

\[
2mw = m((4, 3) - (4, 9)) = -m(4, 3)
\]

modulo \( \partial\mathbb{Z}_D \), again by Lemma 7.1 (i). But if we apply Lemma 7.1 (ii) with \( x = 2 \) and \( y = 3 \) and multiply by 2, we get \( (4, 3) = 0 \) modulo \( \partial\mathbb{Z}_D \). Therefore, \( 2mw = 0 \) modulo \( \partial\mathbb{Z}_D \). Hence, by the proof of Lemma 7.1 there exists an explicit \( z\mathbb{Z}_D \) such that \( h(\zeta_m) = 2m(A(T), I - A(T)) - z \) has the required property.

Note that we were able to construct an element \( h(\zeta_m) \) in \( H^1_M(\text{Spec}\mathbb{Q}(\zeta_m), \mathbb{Z}(2)) \) whose image under \( D \) is \( mD_2(\zeta_m) \), where \( \zeta_m \) is a primitive \( m \)-th root of unity.

Now, let \( m = l \) be an odd prime and let \( \{\sigma_1, \bar{\sigma}_1, \ldots, \sigma_{r_2}, \bar{\sigma}_{r_2}\} \), where \( r_2 = \phi(l)/2 \), be the set of the complex embeddings of \( \mathbb{Q}(\zeta_l) \). Then, we have a homomorphism \( \overline{D} \) from \( H^1_M(\text{Spec}\mathbb{Q}(\zeta_l), \mathbb{Z}(2)) \) into \( \mathbb{R}^{r_2} \) which is defined by

\[
\overline{D}(a) = (D\sigma_1(a), \ldots, D\sigma_{r_2}(a)).
\]

If \( \zeta_l \) is an \( l \)-th primitive root of unity, then the element \( l[\zeta_l] \in \mathcal{A}(K) \) is mapped to \( l(\zeta_l \wedge (1 - \zeta_l)) = \zeta_l^l \wedge (1 - \zeta_l) = 0 \) under the homomorphism
\[ \lambda : \mathcal{A}(K)K^\times \wedge_\mathbb{Z} K^\times \text{ as in Section } 5. \]

Therefore, \( l[\zeta_l] \) is an element of the Bloch’s group \( \mathcal{B}(K) \).

Theorem 7.2.4 in [2] states that the images of \( l[\sigma_1(\zeta)], l[\sigma_2(\zeta)], \ldots, l[\sigma_r(\zeta)] \) in \( \mathcal{B}(K) \) under the given map \( \mathcal{B}(K) \to K_3(\mathbb{Q}(\zeta))_{\mathbb{Q}} \) form a basis of the target group and after the Borel’s regulator map, their images generate a lattice of maximal rank in \( \mathbb{R}_+^r \). Therefore, we obtain the following theorem.

**Theorem 7.3.** (Rational Generators of \( H^1_{\mathcal{M}}(\text{Spec} \mathbb{Q}(\zeta_m), \mathbb{Z}(2)) \)) For an odd prime \( l \), \( h(\sigma_1 \zeta_m), \ldots, h(\sigma_r \zeta_l) \) rationally generates \( H^1_{\mathcal{M}}(\text{Spec} \mathbb{Q}(\zeta_m), \mathbb{Z}(2)) \), i.e., they generate a subgroup of finite index in \( H^1_{\mathcal{M}}(\text{Spec} \mathbb{Q}(\zeta_m), \mathbb{Z}(2)) \).

Note that by the construction of our map \( \theta : \mathcal{B}(K) \to H^1_{\mathcal{M}}(\text{Spec} \mathbb{Q}(\zeta_l), \mathbb{Z}(2)) \) in Section 5 \( \theta(l[\zeta_l]) \) is equal to \( h(\zeta_l) \) modulo an element \( z \) whose value under \( D \) is 0, i.e., a torsion element.

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