ON THE BIRKHOFF INTEGRAL OF FUZZY
MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce the Birkhoff integral of fuzzy mappings in Banach spaces in terms of the Birkhoff integral of set-valued mappings and investigate some properties of the Birkhoff integrals of set-valued mappings and fuzzy mappings in Banach spaces.

1. Introduction

Birkhoff [2] introduced the Birkhoff integral for Banach space valued functions. Birkhoff integrability lies strictly between Bochner and Pettis integrability when the range space $X$ is nonseparable [2, 8]. Lately, several authors [4, 7, 9] have investigated the Birkhoff integral for Banach space valued functions. Several types of integrals of set-valued mappings were introduced by many authors. In particular, Cascales and Rodriguez [3] introduced the Birkhoff integral of $CWK(X)$-valued mappings by means of a certain embedding of $CWK(X)$ into a Banach space. Several authors introduced the integrals of fuzzy mappings in Banach spaces in terms of the integrals of set-valued mappings. In particular, Xue, Ha and Ma [10] and Xue, Wang and Wu [11] introduced integrals of fuzzy mappings in Banach spaces in terms of Aumann-Pettis and Aumann-Bochner integrals of set-valued mappings.
In this paper, we introduce the Birkhoff integral of fuzzy mappings in Banach spaces in terms of the Birkhoff integral of set-valued mappings and investigate some properties of the Birkhoff integrals of set-valued mappings and fuzzy mappings in Banach spaces and obtain convergence theorems for set-valued mappings and fuzzy mappings in Banach spaces.

2. Preliminaries

Throughout this paper, \((\Omega, \Sigma, \mu)\) denotes a complete finite measure space and \((X, \| \cdot \|)\) a Banach space with dual \(X^*\). The closed unit ball of \(X^*\) is denoted by \(B_{X^*}\). \(CL(X)\) denotes the family of all nonempty closed subsets of \(X\) and \(CWK(X)\) the family of all nonempty convex weakly compact subsets of \(X\). For \(A \subseteq X\) and \(x^* \in X^*\), let \(s(x^*, A) = \sup\{x^*(x) : x \in A\}\), the support function of \(A\). For \(A, B \in CL(X)\), let \(H(A, B)\) denote the Hausdorff metric of \(A\) and \(B\) defined by

\[
H(A, B) = \max\left(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right),
\]

where \(d(a, B) = \inf b \in B \|a - b\|\) and \(d(b, A) = \inf_{a \in A} \|a - b\|\). Especially,

\[
H(A, B) = \sup_{x^* \in B_{X^*}} |s(x^*, A) - s(x^*, B)|
\]

whenever \(A, B\) are convex sets.

Note that \((CWK(X), H)\) is a complete metric space with the following properties:

1. \(H(\lambda A, AB) = |\lambda| H(A, B)\) for all \(A, B \in CWK(X)\) and \(\lambda \in \mathbb{R}\);
2. \(H(A + C, B + C) = H(A, B)\) for all \(A, B, C \in CWK(X)\);
3. \(H(A + C, B + D) \leq H(A, B) + H(C, D)\) for all \(A, B, C, D \in CWK(X)\).

The number \(\|A\|\) is defined by \(\|A\| = H(A, \{0\}) = \sup_{x \in A} \|x\|\).

Let \(u : X \to [0, 1]\). We denote \([u]^r = \{x \in X : u(x) \geq r\}\) for \(r \in (0, 1]\) and \([u]^0 = \text{cl}\{x \in X : u(x) > 0\}\). \(u\) is called a generalized fuzzy number on \(X\) if for each \(r \in (0, 1], [u]^r \in CWK(X)\). Let \(\mathcal{F}(X)\) denote the set of all generalized fuzzy numbers on \(X\). For \(u, v \in \mathcal{F}(X)\) and \(\lambda \in \mathbb{R}\), we define \(u + v\) and \(\lambda u\) as follows:

\[
(u + v)(x) = \sup_{x = y + z} \min(u(y), v(z)),
\]

\[
\lambda u(x) = \lambda u(x^*),
\]

where \(\lambda u(x^*)\) is the support function of \(\lambda u\).


\[(\lambda u)(x) = u\left(\frac{1}{\lambda}x\right), \lambda \neq 0\]

\[\lambda u = \hat{0}, \lambda = 0, \text{ where } \hat{0} = \chi_{\{0\}}.\]

For \(u, v \in \mathcal{F}(X)\) and \(\lambda \in \mathbb{R}\), \([u + v]^r = [u]^r + [v]^r\) and \([\lambda u]^r = \lambda [u]^r\) for each \(r \in (0, 1]\). Hence \(u + v, \lambda u \in \mathcal{F}(X)\). For \(u, v \in \mathcal{F}(X)\), we define \(u \leq v\) as follows:

\[u \leq v \text{ if } u(x) \leq v(x) \text{ for all } x \in X.\]

For \(u, v \in \mathcal{F}(X)\), \(u \leq v\) if and only if \([u]^r \subseteq [v]^r\) for each \(r \in (0, 1]\).

Define \(D: \mathcal{F}(X) \times \mathcal{F}(X) \to [0, +\infty]\) by the equation

\[D(u, v) = \sup_{r \in (0, 1]} H([u]^r, [v]^r).\]

Then \(D\) is a metric on \(\mathcal{F}(X)\). The norm \(\|u\|\) of \(u \in \mathbf{F}(X)\) is defined by

\[\|u\| = D(u, \hat{0}) = \sup_{r \in (0, 1]} H([u]^r, \{0\}) = \sup_{r \in (0, 1]}\|[u]^r\|.\]

The mapping \(F: \Omega \to \text{CL}(X)\) is called a set-valued mapping. \(F\) is said to be scalarly measurable if for every \(x^* \in X^*\), the real-valued function \(s(x^*, F(\cdot))\) is measurable. \(F\) is said to be Effros measurable (or measurable for short) if \(F^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma\) for every open subset \(U\) of \(X\). Note that measurability is stronger than scalar measurability.

Let \(F: \Omega \to \text{CL}(X)\). Then the following statements are equivalent:

1. \(F: \Omega \to \text{CL}(X)\) is measurable;
2. \(F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} \in \Sigma\) for every \(A \in \text{CL}(X)\);
3. (Castaing representation) there exists a sequence \((f_n)\) of measurable functions \(f_n: \Omega \to X\) such that \(F(\omega) = \text{cl}\{f_n(\omega)\}\) for all \(\omega \in \Omega\).

\(F: \Omega \to \text{CL}(X)\) is said to be weakly integrably bounded if the real-valued function \(|x^*F|: \Omega \to \mathbb{R}, |x^*F|(\omega) = \sup\{|x^*(x)| : x \in F(\omega)\}\), is integrable for every \(x^* \in X^*\). \(F: \Omega \to \text{CL}(X)\) is said to be integrably bounded if there exists an integrable real-valued function \(h\) such that for each \(\omega \in \Omega\), \(|x| \leq h(\omega)\) for all \(x \in F(\omega)\). \(F: \Omega \to \text{CL}(X)\) is said to be scalarly integrable on \(\Omega\) if for every \(x^* \in X^*\), \(s(x^*, F(\cdot))\) is integrable on \(\Omega\). \(F: \Omega \to \text{CL}(X)\) is said to be scalarly uniformly integrable if the set \(\{s(x^*, F(\cdot)) : x^* \in B_{X^*}\}\) is uniformly integrable. \(F: \Omega \to X\) is called a measurable selector of \(F: \Omega \to \text{CL}(X)\) if \(F\) is measurable and \(f(\omega) \in F(\omega)\) for all \(\omega \in \Omega\).
A measurable set-valued mapping $F : \Omega \to CWK(X)$ is said to be \textit{Pettis integrable} on $\Omega$ if $F : \Omega \to CWK(X)$ is scalarly integrable on $\Omega$ and for each $A \in \Sigma$ there exists $(P) \int_A F d\mu \in CWK(X)$ such that $s(x^*, (P) \int_A F d\mu) = \int_A s(x^*, F) d\mu$ for all $x^* \in X^*$. In this case, $(P) \int_A F d\mu$ is called the \textit{Pettis integral} of $F$ over $A$ [6].

A function $f : \Omega \to X$ is called \textit{summable} with respect to a given countable partition $\Gamma = (A_n)$ of $\Omega$ in $\Sigma$ if $f|_{A_n}$ is bounded whenever $\mu(A_n) > 0$ and the set $J(f, \Gamma) = \left\{ \sum_n \mu(A_n)f(t_n) : t_n \in A_n \right\}$ is made up of unconditionally convergent series.

\textbf{Definition 2.1.[2].} A function $f : \Omega \to X$ is said to be \textit{Birkhoff integrable} on $\Omega$ if for every $\epsilon > 0$ there exists a countable partition $\Gamma$ of $\Omega$ in $\Sigma$ for which $f$ is summable and $\|\cdot\|\text{-diam} (J(f, \Gamma)) < \epsilon$. In this case, the \textit{Birkhoff integral} $(B) \int_\Omega f d\mu$ of $f$ is the only point in the intersection
\[
\cap \left\{ \text{co}(J(f, \Gamma)) : f \text{ is summable with respect to } \Gamma \right\}.
\]

If $f : \Omega \to X$ is Birkhoff integrable on $\Omega$, then $f : \Omega \to X$ is Birkhoff integrable on every $A \in \Sigma$. Birkhoff integrability lies strictly between Bochner and Pettis integrability. If $f : \Omega \to X$ is Birkhoff integrable, then $(B) \int_\Omega f d\mu = (P) \int_\Omega f d\mu$. When the range space $X$ is separable, Birkhoff and Pettis integrability are the same. In the definition of the Birkhoff integral, if the respective series $J(f, \Gamma) = \left\{ \sum_n \mu(A_n)f(t_n) : t_n \in A_n \right\}$ is made up of absolutely convergent series, then $f : \Omega \to X$ is said to be \textit{absolutely Birkhoff integrable} on $\Omega$ [1].

\textbf{Theorem 2.2.[5].} Let $\ell_\infty(B_{X^*})$ be the Banach space of bounded real-valued functions defined on $B_{X^*}$ endowed with the supremum norm...
\[ \| \cdot \|_\infty. \] Then the map \( j : CWK(X) \rightarrow \ell_\infty(B_{X^*}) \) given by \( j(A) := s(\cdot, A) \) satisfies the following properties:

1. \( j(A + B) = j(A) + j(B) \) for every \( A, B \in CWK(X) \);
2. \( j(\lambda A) = \lambda j(A) \) for every \( \lambda \geq 0 \) and \( A \in CWK(X) \);
3. \( H(A, B) = \| j(A) - j(B) \|_\infty \) for every \( A, B \in CWK(X) \);
4. \( j(CWK(X)) \) is closed in \( \ell_\infty(B_{X^*}) \).

**Definition 2.3**. A set-valued mapping \( F : \Omega \rightarrow CWK(X) \) is said to be **Birkhoff integrable** on \( \Omega \) if the composition \( j \circ F : \Omega \rightarrow \ell_\infty(B_{X^*}) \) is Birkhoff integrable on \( \Omega \). In this case, for each \( A \in \Sigma \) there exists a unique element \( \left( B \right) \int_A Fd\mu \in CWK(X) \), that is called the **Birkhoff integral** of \( F \) on \( A \), such that \( j\left( \left( B \right) \int_A Fd\mu \right) = \left( B \right) \int_A \circ Fd\mu \).

**3. Results**

A mapping \( \tilde{F} : \Omega \rightarrow F(X) \) is called a **fuzzy mapping** in a Banach space \( X \). In this case, \( \tilde{F}^r : \Omega \rightarrow CWK(X) \) defined by \( \tilde{F}^r(\omega) = [\tilde{F}(\omega)]^r \) is a set-valued mapping for each \( r \in (0, 1] \). A fuzzy mapping \( \tilde{F} : \Omega \rightarrow F(X) \) is said to be **measurable** (resp., **scalarly measurable**) if \( \tilde{F}^r : \Omega \rightarrow CWK(X) \) is measurable (resp., scalarly measurable) for each \( r \in (0, 1] \).

**Definition 3.1**. A fuzzy mapping \( \tilde{F} : \Omega \rightarrow F(X) \) is said to be **Birkhoff integrable** on \( \Omega \) if for each \( A \in \Sigma \) there exists \( u_A \in F(X) \) such that \( [u_A]^r = \left( B \right) \int_A \tilde{F}^r d\mu \) for each \( r \in (0, 1] \). In this case, \( u_A = \left( B \right) \int_A \tilde{F}d\mu \) is called the **Birkhoff integral** of \( \tilde{F} \) on \( A \).

**Theorem 3.2**. Let \( \tilde{F} : \Omega \rightarrow F(X) \) and \( \tilde{G} : \Omega \rightarrow F(X) \) be Birkhoff integrable on \( \Omega \) and \( \lambda \geq 0 \). Then

1. \( \tilde{F} + \tilde{G} \) is Birkhoff integrable on \( \Omega \) and for each \( A \in \Sigma \)
   \[ (B) \int_A (\tilde{F} + \tilde{G})d\mu = (B) \int_A \tilde{F}d\mu + (B) \int_A \tilde{G}d\mu, \]
2. \( \lambda \tilde{F} \) is Birkhoff integrable on \( \Omega \) and for each \( A \in \Sigma \)
   \[ (B) \int_A \lambda \tilde{F}d\mu = \lambda (B) \int_A \tilde{F}d\mu. \]
Proof. (1) Let \( \tilde{F} : \Omega \to \mathcal{F}(X) \) and \( \tilde{G} : \Omega \to \mathcal{F}(X) \) be Birkhoff integrable on \( \Omega \). Then for each \( A \in \Sigma \) there exist \( u_A, v_A \in \mathcal{F}(X) \) such that \( [u_A]^{r} = (B) \int_{A} \tilde{F}^{r} d\mu \), \( [v_A]^{r} = (B) \int_{A} \tilde{G}^{r} d\mu \) for each \( r \in (0, 1] \). Thus \( j \circ \tilde{F}^{r} \) and \( j \circ \tilde{G}^{r} \) are Birkhoff integrable on \( \Omega \) and \( j([u_A]^{r}) = j((B) \int_{A} \tilde{F}^{r} d\mu) = \int_{A} j \circ \tilde{F}^{r} d\mu \), \( j([v_A]^{r}) = j((B) \int_{A} \tilde{G}^{r} d\mu) = \int_{A} j \circ \tilde{G}^{r} d\mu \) for each \( r \in (0, 1] \) and \( A \in \Sigma \). Hence \( j \circ (\tilde{F} + \tilde{G})^{r} = j \circ (\tilde{F}^{r} + \tilde{G}^{r}) \) is Birkhoff integrable on \( \Omega \) and

\[
[j([u_A + v_A]^{r})](x^{*}) = [j([u_A]^{r}) + j([v_A]^{r})](x^{*})
\]

\[
= [j([u_A]^{r})](x^{*}) + [j([v_A]^{r})](x^{*})
\]

\[
= [j((B) \int_{A} \tilde{F}^{r} d\mu)](x^{*}) + [j((B) \int_{A} \tilde{G}^{r} d\mu)](x^{*})
\]

\[
= [(B) \int_{A} j \circ \tilde{F}^{r} d\mu](x^{*}) + [(B) \int_{A} j \circ \tilde{G}^{r} d\mu](x^{*})
\]

\[
= [(B) \int_{A} j \circ (\tilde{F} + \tilde{G})^{r} d\mu](x^{*})
\]

for each \( x^{*} \in B_{X^{*}}, r \in (0, 1] \) and \( A \in \Sigma \). Hence \( j([u_A + v_A]^{r}) = \int_{A} j \circ (\tilde{F} + \tilde{G})^{r} d\mu \) for each \( r \in (0, 1] \) and \( A \in \Sigma \). Thus \( [u_A + v_A]^{r} = (B) \int_{A} (\tilde{F} + \tilde{G})^{r} d\mu \) for each \( r \in (0, 1] \) and \( A \in \Sigma \). Hence \( \tilde{F} + \tilde{G} \) is Birkhoff integrable on \( \Omega \) and for each \( A \in \Sigma \)

\[
(B) \int_{A} (\tilde{F} + \tilde{G}) d\mu = u_A + v_A = (B) \int_{A} \tilde{F} d\mu + (B) \int_{A} \tilde{G} d\mu.
\]

(2) Let \( \tilde{F} : \Omega \to \mathcal{F}(X) \) be Birkhoff integrable on \( \Omega \) and \( \lambda \geq 0 \). Then there exists \( u_A \in \mathcal{F}(X) \) such that \( [u_A]^{r} = (B) \int_{A} \tilde{F}^{r} d\mu \) for each \( r \in (0, 1] \). Since \( j([\lambda u_A]^{r}) = \lambda j([u_A]^{r}) \) for each \( r \in (0, 1] \) and \( A \in \Sigma \), using the same method as (1) we obtain that \( \lambda \tilde{F} \) is Birkhoff integrable on \( \Omega \) and for each \( A \in \Sigma \)
(B) \int_A \lambda \tilde{F} d\mu = \lambda(B) \int_A \tilde{F} d\mu.

Lemma 3.3. Let \( F : \Omega \to CWK(X) \) and \( G : \Omega \to CWK(X) \) be Birkhoff integrable set-valued mappings. Then

1. if \( F(\omega) = G(\omega) \) \( \mu \)-a.e., then \((B) \int_A F d\mu = (B) \int_A G d\mu \) for each \( A \in \Sigma \);
2. if \( X \) is separable and \((B) \int_A F d\mu = (B) \int_A G d\mu \) for each \( A \in \Sigma \), then \( F(\omega) = G(\omega) \) \( \mu \)-a.e.

Proof. (1) Since \( F : \Omega \to CWK(X) \) and \( G : \Omega \to CWK(X) \) are Birkhoff integrable on \( \Omega \), \( j \circ F \) and \( j \circ G \) are Birkhoff integrable on \( \Omega \) and there exist \((B) \int_A F d\mu, (B) \int_A G d\mu \in CWK(X) \) such that

\[
 j((B) \int_A F d\mu) = (B) \int_A j \circ F d\mu, \quad j((B) \int_A G d\mu) = (B) \int_A j \circ G d\mu
\]
for each \( A \in \Sigma \).

If \( F(\omega) = G(\omega) \) \( \mu \)-a.e., then \((j \circ F)(\omega) = (j \circ G)(\omega) \) \( \mu \)-a.e. Hence

\[
 j((B) \int_A F d\mu) = (B) \int_A j \circ F d\mu = (B) \int_A j \circ G d\mu = j((B) \int_A G d\mu)
\]
for each \( A \in \Sigma \). Thus

\[
 s(x^*, (B) \int_A F d\mu) = [j((B) \int_A F d\mu)](x^*) = [j((B) \int_A G d\mu)](x^*) = s(x^*, (B) \int_A G d\mu)
\]
for each \( x^* \in B_{X^*} \) and \( A \in \Sigma \). Since \((B) \int_A F d\mu, (B) \int_A G d\mu \in CWK(X) \) for each \( A \in \Sigma \), by the separation theorem \((B) \int_A F d\mu = (B) \int_A G d\mu \) for each \( A \in \Sigma \).
If \((B) \int_A F d\mu = (B) \int_A G d\mu\) for each \(A \in \Sigma\), then
\[
(B) \int_A j \circ F d\mu = j((B) \int_A F d\mu) = (B) \int_A j \circ G d\mu
\]
for each \(A \in \Sigma\). Since \(X\) is a separable Banach space, by [2, Theorem 24] \((j \circ F)(\omega) = (j \circ G)(\omega)\) \(\mu\)-a.e. and so \(H(F(\omega), G(\omega)) = \|((j \circ F)(\omega) - (j \circ G)(\omega))\|_\infty = 0\) \(\mu\)-a.e. Hence \(F(\omega) = G(\omega)\) \(\mu\)-a.e.

**Theorem 3.4.** Let \(\tilde{F} : \Omega \to \mathcal{F}(X)\) and \(\tilde{G} : \Omega \to \mathcal{F}(X)\) be Birkhoff integrable on \(\Omega\). If \(\tilde{F}(\omega) = \tilde{G}(\omega)\) \(\mu\)-a.e., then \((B) \int_A \tilde{F} d\mu = (B) \int_A \tilde{G} d\mu\) for each \(A \in \Sigma\).

**Proof.** Since \(\tilde{F} : \Omega \to \mathcal{F}(X)\) and \(\tilde{G} : \Omega \to \mathcal{F}(X)\) are Birkhoff integrable on \(\Omega\), for each \(A \in \Sigma\) there exist \(u_A, v_A\) \(\in \mathcal{F}(X)\) such that \([u_A]^r = (B) \int_A \tilde{F}^r d\mu, [v_A]^r = (B) \int_A \tilde{G}^r d\mu\) for each \(r \in (0, 1]\). If \(\tilde{F}(\omega) = \tilde{G}(\omega)\) \(\mu\)-a.e., then \(\tilde{F}^r(\omega) = \tilde{G}^r(\omega)\) \(\mu\)-a.e. for each \(r \in (0, 1]\). By Lemma 3.3 \([u_A]^r = (B) \int_A \tilde{F}^r d\mu = (B) \int_A \tilde{G}^r d\mu = [v_A]^r\) for each \(r \in (0, 1]\) and \(A \in \Sigma\) and so \((B) \int_A \tilde{F} d\mu = u_A = v_A = (B) \int_A \tilde{G} d\mu\) for each \(A \in \Sigma\).

If \(X\) is separable and \(F : \Omega \to \text{CWK}(X)\) is Birkhoff integrable on \(\Omega\), then
\[
(B) \int_A F d\mu = \left\{ (B) \int_A f d\mu : f \text{ is a Birkhoff integrable selector of } F \right\}
\]
for each \(A \in \Sigma\) [3].

**Lemma 3.5.** Let \(X\) be separable and let \(F : \Omega \to \text{CWK}(X)\) and \(G : \Omega \to \text{CWK}(X)\) be Birkhoff integrable set-valued mappings. If \(F(\omega) \subseteq G(\omega)\) on \(\Omega\), then \((B) \int_A F d\mu \subseteq (B) \int_A G d\mu\) for each \(A \in \Sigma\).

**Proof.** Since \(F : \Omega \to \text{CWK}(X)\) and \(G : \Omega \to \text{CWK}(X)\) are Birkhoff integrable on \(\Omega\) and \(F(\omega) \subseteq G(\omega)\) on \(\Omega\), for each \(A \in \Sigma\)
\[(B) \int_A Fd\mu = \left\{(B) \int_A fd\mu : f \text{ is a Birkhoff integrable selector of } F \right\} \subseteq \left\{(B) \int_A gd\mu : g \text{ is a Birkhoff integrable selector of } G \right\} = (B) \int_A Gd\mu.\]

\[\text{Theorem 3.6. Let } X \text{ be separable and let } \tilde{F} : \Omega \to \mathcal{F}(X) \text{ and } \tilde{G} : \Omega \to \mathcal{F}(X) \text{ be Birkhoff integrable on } \Omega. \text{ If } \tilde{F}(\omega) \leq \tilde{G}(\omega) \text{ on } \Omega, \text{ then } (B) \int_A \tilde{F}d\mu \leq (B) \int_A \tilde{G}d\mu \text{ for each } A \in \Sigma.\]

\[\text{Proof. (1) Since } \tilde{F} : \Omega \to \mathcal{F}(X) \text{ and } \tilde{G} : \Omega \to \mathcal{F}(X) \text{ are Birkhoff integrable on } \Omega, \text{ for each } A \in \Sigma \text{ there exist } u_A, v_A \in \mathcal{F}(X) \text{ such that } [u_A]^r = (B) \int_A \tilde{F}^r d\mu \text{ and } [v_A]^r = (B) \int_A \tilde{G}^r d\mu \text{ for each } r \in (0,1]. \text{ If } \tilde{F}(\omega) \leq \tilde{G}(\omega) \text{ on } \Omega, \text{ then } \tilde{F}^r(\omega) \subseteq \tilde{G}^r(\omega) \text{ on } \Omega \text{ for each } r \in (0,1]. \text{ By Lemma 3.5 } [u_A]^r = (B) \int_A \tilde{F}^r d\mu \subseteq (B) \int_A \tilde{G}^r d\mu = [v_A]^r \text{ for each } r \in (0,1] \text{ and so } (B) \int_A \tilde{F}d\mu = u_A \leq v_A = (B) \int_A \tilde{G}d\mu \text{ for each } A \in \Sigma.\]

\[\text{Lemma 3.7. Let } X \text{ be separable. If } F : \Omega \to \text{CWK}(X) \text{ and } G : \Omega \to \text{CWK}(X) \text{ are measurable, integrably bounded and Birkhoff integrable set-valued mappings, then } H(F,G) \text{ is integrable on } \Omega \text{ and } H(\int_A Fd\mu, (B) \int_A Gd\mu) \leq \int_A H(F,G)d\mu.\]

\[\text{Proof. Since } F : \Omega \to \text{CWK}(X) \text{ and } G : \Omega \to \text{CWK}(X) \text{ are measurable, there exist Castaing representations } (f_n) \text{ and } (g_n) \text{ for } F \text{ and } G. \text{ Since } f_n \text{ and } g_n \text{ are measurable for all } n \in \mathbb{N},\]

\[H(F(\omega),G(\omega)) = \max\left(\sup_{n \geq 1} \inf_{k \geq 1} \|f_n(\omega) - g_k(\omega)\|, \sup_{n \geq 1} \inf_{k \geq 1} \|g_n(\omega) - f_k(\omega)\|\right)\]
is measurable. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are integrably bounded, there exist integrable real-valued functions $h_1$ and $h_2$ on $\Omega$ such that for each $\omega \in \Omega$, $\|x\| \leq h_1(\omega)$ for all $x \in F(\omega)$ and $\|x\| \leq h_2(\omega)$ for all $x \in G(\omega)$. Hence

$$H(F(\omega), G(\omega)) \leq H(F(\omega), \{0\}) + H(G(\omega), \{0\}) \leq h_1(\omega) + h_2(\omega)$$

for each $\omega \in \Omega$. Therefore $H(F, G)$ is integrable on $\Omega$. Since $F : \Omega \to CWK(X)$ and $G : \Omega \to CWK(X)$ are Birkhoff integrable on $\Omega$, $j \circ F$ and $j \circ G$ are Birkhoff integrable on $\Omega$ and there exist $(B) \int_{\Omega} Fd\mu$, $(B) \int_{\Omega} Gd\mu \in CWK(X)$ such that $j((B) \int_{\Omega} Fd\mu) = (B) \int_{\Omega} j \circ Fd\mu$ and $j((B) \int_{\Omega} Gd\mu) = (B) \int_{\Omega} j \circ Gd\mu$. Since $X$ is separable, by [3, Proposition 3.2] $(B) \int_{\Omega} Fd\mu = (P) \int_{\Omega} Fd\mu$ and $(B) \int_{\Omega} Gd\mu = (P) \int_{\Omega} Gd\mu$. Hence

$$H\left((B) \int_{\Omega} Fd\mu, (B) \int_{\Omega} Gd\mu\right) = \left\|j((B) \int_{\Omega} Fd\mu) - j((B) \int_{\Omega} Gd\mu)\right\|_\infty$$

$$= \sup_{x^* \in B_{X^*}} \left|j((B) \int_{\Omega} Fd\mu)(x^*) - j((B) \int_{\Omega} Gd\mu)(x^*)\right|$$

$$= \sup_{x^* \in B_{X^*}} \left|s(x^*, (B) \int_{\Omega} Fd\mu) - s(x^*, (B) \int_{\Omega} Gd\mu)\right|$$

$$= \sup_{x^* \in B_{X^*}} \left|s(x^*, (P) \int_{\Omega} Fd\mu) - s(x^*, (P) \int_{\Omega} Gd\mu)\right|$$

$$= \sup_{x^* \in B_{X^*}} \left|\int_{\Omega} s(x^*, F)d\mu - \int_{\Omega} s(x^*, G)d\mu\right|$$

$$\leq \sup_{x^* \in B_{X^*}} \int_{\Omega} |s(x^*, F) - s(x^*, G)| d\mu$$

$$\leq \int_{\Omega} \sup_{x^* \in B_{X^*}} |s(x^*, F) - s(x^*, G)| d\mu$$

$$= \int_{\Omega} H(F, G)d\mu$$

$$\square$$
A fuzzy mapping \( \tilde{F} : \Omega \to \mathcal{F}(X) \) is said to be \textit{integrably bounded} if there exists an integrable real-valued function \( h \) on \( \Omega \) such that for each \( \omega \in \Omega \), \( \| x \| \leq h(\omega) \) for all \( x \in \tilde{F}^0(\omega) \), where \( \tilde{F}^0(\omega) = \text{cl} \left( \bigcup_{0 < r \leq 1} \tilde{F}^r(\omega) \right) \).

**Theorem 3.8.** Let \( X \) be separable. If \( \tilde{F} : \Omega \to \mathcal{F}(X) \) and \( \tilde{G} : \Omega \to \mathcal{F}(X) \) are measurable, integrably bounded and Birkhoff integrable fuzzy mappings, then \( D(\tilde{F}, \tilde{G}) \) is integrable on \( \Omega \) and

\[
D \left( (B) \int_{\Omega} \tilde{F} d\mu, (B) \int_{\Omega} \tilde{G} d\mu \right) \leq \int_{\Omega} D(\tilde{F}, \tilde{G}) d\mu.
\]

**Proof.** Since \( \tilde{F} : \Omega \to \mathcal{F}(X) \) and \( \tilde{G} : \Omega \to \mathcal{F}(X) \) are measurable, there exist Castaing representations \( (f^r_n) \) and \( (g^r_n) \) for \( \tilde{F}^r \) and \( \tilde{G}^r \) for each \( r \in (0, 1] \). Since \( f^r_n \) and \( g^r_n \) are measurable for all \( n \in \mathbb{N} \),

\[
H(\tilde{F}^r(\omega), \tilde{G}^r(\omega)) = \max \left( \sup_{n \geq 1} \inf_{k \geq 1} \| f^r_n(\omega) - g^r_k(\omega) \|, \sup_{n \geq 1} \inf_{k \geq 1} \| g^r_n(\omega) - f^r_k(\omega) \| \right)
\]

is measurable for each \( r \in (0, 1] \). Hence \( D(\tilde{F}(\omega), \tilde{G}(\omega)) = \sup_{k \geq 1} H(\tilde{F}^{r_k}(\omega), \tilde{G}^{r_k}(\omega)) \) is measurable, where \( \{r_k : k \in \mathbb{N} \} \) is dense in \( (0, 1] \). Since \( \tilde{F} : \Omega \to \mathcal{F}(X) \) and \( \tilde{G} : \Omega \to \mathcal{F}(X) \) are integrably bounded, there exist integrable real-valued functions \( h_1 \) and \( h_2 \) on \( \Omega \) such that for each \( \omega \in \Omega \), \( \| x \| \leq h_1(\omega) \) for all \( x \in \tilde{F}^0(\omega) \) and \( \| x \| \leq h_2(\omega) \) for all \( x \in \tilde{G}^0(\omega) \). Hence

\[
D(\tilde{F}(\omega), \tilde{G}(\omega)) \leq D(\tilde{F}(\omega), 0) + D(\tilde{G}(\omega), 0) \leq h_1(\omega) + h_2(\omega)
\]

for each \( \omega \in \Omega \). Therefore \( D(\tilde{F}, \tilde{G}) \) is integrable on \( \Omega \). By Lemma 3.7

\[
H \left( (B) \int_{\Omega} \tilde{F}^r d\mu, (B) \int_{\Omega} \tilde{G}^r d\mu \right) \leq \int_{\Omega} H(\tilde{F}^r, \tilde{G}^r) d\mu
\]
for each \( r \in (0, 1] \). Hence

\[
D \left( (B) \int_\Omega \tilde{F} d\mu, (B) \int_\Omega \tilde{G} d\mu \right) \\
= \sup_{r \in (0, 1]} H \left( \left[ (B) \int_\Omega \tilde{F} d\mu \right]^r, \left[ (B) \int_\Omega \tilde{G} d\mu \right]^r \right) \\
= \sup_{r \in (0, 1]} H \left( (B) \int_\Omega \tilde{F}^r d\mu, (B) \int_\Omega \tilde{G}^r d\mu \right) \\
\leq \sup_{r \in (0, 1]} \int_\Omega H(\tilde{F}^r, \tilde{G}^r) d\mu \\
\leq \int_\Omega \sup_{r \in (0, 1]} H(\tilde{F}^r, \tilde{G}^r) d\mu \\
= \int_\Omega D(\tilde{F}, \tilde{G}) d\mu.
\]

\( \square \)

**Theorem 3.9.** Let \( F_n : \Omega \to CWK(X) \) be a Birkhoff integrable set-valued mapping for each \( n \in \mathbb{N} \) and let \( F : \Omega \to CWK(X) \). If \((F_n)\) converges uniformly to \( F \) on \( \Omega \), then \( F : \Omega \to CWK(X) \) is Birkhoff integrable on \( \Omega \) and

\[
\lim_{n \to \infty} (B) \int_\Omega F_n d\mu = (B) \int_\Omega F d\mu.
\]

**Proof.** Since \( F_n : \Omega \to CWK(X) \) is Birkhoff integrable on \( \Omega \) for each \( n \in \mathbb{N} \), \( j \circ F_n \) is Birkhoff integrable on \( \Omega \) and there exists \( (B) \int_\Omega F_n d\mu \in CWK(X) \) such that \( j((B) \int_\Omega F_n d\mu) = (B) \int_\Omega j \circ F_n d\mu \) for each \( n \in \mathbb{N} \). Since \((F_n)\) converges uniformly to \( F \) on \( \Omega \), \((j \circ F_n)\) also converges uniformly to \( j \circ F \) on \( \Omega \). By [1, Theorem 4] \( j \circ F \) is Birkhoff integrable on \( \Omega \) and \( \lim_{n \to \infty} (B) \int_\Omega j \circ F_n d\mu = (B) \int_\Omega j \circ F d\mu \). Hence \( F : \Omega \to CWK(X) \) is Birkhoff integrable on \( \Omega \) and
\[
\lim_{n \to \infty} H \left( (B) \int_\Omega F_n d\mu, (B) \int_\Omega F d\mu \right) \\
= \lim_{n \to \infty} \left\| j \left( (B) \int_\Omega F_n d\mu \right) - j \left( (B) \int_\Omega F d\mu \right) \right\|_\infty \\
= \lim_{n \to \infty} \left\| \int_\Omega j \circ F_n d\mu - \int_\Omega j \circ F d\mu \right\|_\infty = 0.
\]

Thus \( \lim_{n \to \infty} (B) \int_\Omega F_n d\mu = (B) \int_\Omega F d\mu \).

A set-valued mapping \( F : \Omega \to CWK(X) \) is said to be absolutely Birkhoff integrable on \( \Omega \) if the composition \( j \circ F : \Omega \to \ell_\infty(B_X^*) \) is absolutely Birkhoff integrable on \( \Omega \).

From [1, Theorem 7] and [1, Corollary 8], we can obtain the following two theorems using the same method in the Theorem 3.9.

**Theorem 3.10.** Let \( F_n : \Omega \to CWK(X) \) be a Birkhoff integrable set-valued mapping for each \( n \in \mathbb{N} \) and let \( F : \Omega \to CWK(X) \) be a set-valued mapping such that \( (F_n) \) converges to \( F \) almost uniformly on \( \Omega \). If there exists an integrable real-valued function \( h \) on \( \Omega \) such that \( \|F_n(\omega)\| \leq h(\omega) \) for all \( n \in \mathbb{N} \) and almost all \( \omega \in \Omega \), then \( F : \Omega \to CWK(X) \) is absolutely Birkhoff integrable on \( \Omega \) and

\[
\lim_{n \to \infty} (B) \int_\Omega F_n d\mu = (B) \int_\Omega F d\mu.
\]

**Theorem 3.11.** Let \( F_n : \Omega \to CWK(X) \) be a Birkhoff integrable set-valued mapping such that \( j \circ F_n \) is measurable for each \( n \in \mathbb{N} \) and let \( F : \Omega \to CWK(X) \) be a set-valued mapping such that \( (F_n) \) converges to \( F \) almost everywhere on \( \Omega \). If there exists an integrable real-valued function \( h \) on \( \Omega \) such that \( \|F_n(\omega)\| \leq h(\omega) \) for all \( n \in \mathbb{N} \) and almost all \( \omega \in \Omega \), then \( F : \Omega \to CWK(X) \) is absolutely Birkhoff integrable on \( \Omega \) and

\[
\lim_{n \to \infty} (B) \int_\Omega F_n d\mu = (B) \int_\Omega F d\mu.
\]

\( \tilde{F} : \Omega \to \mathcal{F}(X) \) is said to be \( j \)-measurable if \( j \circ \tilde{F}^r : \Omega \to \ell_\infty(B_{X^*}) \) is measurable for each \( r \in (0, 1) \).
THEOREM 3.12. Let $X$ be separable and let $\tilde{F}_n : \Omega \rightarrow \mathcal{F}(X)$ be a j-measurable and Birkhoff integrable fuzzy mapping for each $n \in \mathbb{N}$. If $(\tilde{F}_n)$ converges to $\tilde{F} : \Omega \rightarrow \mathcal{F}(X)$ on $\Omega$ and there exists an integrable real-valued function $h$ on $\Omega$ such that $\|\tilde{F}_n^0(\omega)\| \leq h(\omega)$ on $\Omega$ for all $n \in \mathbb{N}$, then $\tilde{F} : \Omega \rightarrow \mathcal{F}(X)$ is Birkhoff integrable on $\Omega$ and

$$\lim_{n \rightarrow \infty} (B) \int_{\Omega} \tilde{F}_n d\mu = (B) \int_{\Omega} \tilde{F} d\mu.$$ 

Proof. Since $(\tilde{F}_n)$ converges to $\tilde{F}$ on $\Omega$, for each $\epsilon > 0$ and $\omega \in \Omega$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow D(\tilde{F}_n(\omega), \tilde{F}(\omega)) < \epsilon$. Hence

$$\|\tilde{F}_n^0(\omega)\| = D(\tilde{F}(\omega), \tilde{0}) \leq D(\tilde{F}(\omega), \tilde{F}_N(\omega)) + D(\tilde{F}_N(\omega), \tilde{0})$$

$$< \|\tilde{F}_N^0(\omega)\| + \epsilon \leq h(\omega) + \epsilon$$

for each $\omega \in \Omega$. Since $\epsilon > 0$ is arbitrary, $\|\tilde{F}_n^0(\omega)\| \leq h(\omega)$ on $\Omega$. Thus $\tilde{F} : \Omega \rightarrow \mathcal{F}(X)$ is integrably bounded. Since $\tilde{F}_n : \Omega \rightarrow \mathcal{F}(X)$ is Birkhoff integrable on $\Omega$ for each $n \in \mathbb{N}$, $\tilde{F}_n^\circ : \Omega \rightarrow \text{CWK}(X)$ is Birkhoff integrable on $\Omega$ for each $n \in \mathbb{N}$ and $r \in (0, 1]$. Since $\tilde{F}_n : \Omega \rightarrow \mathcal{F}(X)$ is j-measurable for each $n \in \mathbb{N}$, $j \circ \tilde{F}_n^\circ : \Omega \rightarrow \ell_\infty(B_{X^*})$ is measurable for each $n \in \mathbb{N}$ and $r \in (0, 1]$. Since $(\tilde{F}_n)$ converges to $\tilde{F}$ on $\Omega$, $(\tilde{F}_n^\circ)$ converges to $\tilde{F}^\circ$ on $\Omega$ for each $r \in (0, 1]$. Since $\|\tilde{F}_n^0(\omega)\| \leq h(\omega)$ on $\Omega$ for each $n \in \mathbb{N}$, $\|\tilde{F}_n^r(\omega)\| \leq h(\omega)$ on $\Omega$ for each $r \in (0, 1]$ and $n \in \mathbb{N}$. By Theorem 3.11, $\tilde{F}^\circ : \Omega \rightarrow \text{CWK}(X)$ is Birkhoff integrable on $\Omega$ for each $r \in (0, 1]$. Let $A \in \Sigma$. Then there exists $M_r \in \text{CWK}(X)$ such that

$$M_r = (B) \int_{A} \tilde{F}_n^\circ d\mu$$

for each $r \in (0, 1]$. For $r_1, r_2 \in (0, 1]$ with $r_1 < r_2$,

$$\tilde{F}_n^{r_1}(\omega) \supseteq \tilde{F}_n^{r_2}(\omega)$$

for each $\omega \in \Omega$. By Lemma 3.5 $M_{r_1} = (B) \int_{A} \tilde{F}_n^{r_1} d\mu \supseteq (B) \int_{A} \tilde{F}_n^{r_2} d\mu = M_{r_2}$. Let $r \in (0, 1]$ and $(r_n)$ be a sequence in $(0, 1]$ such that $r_1 \leq r_2 \leq r_3 \leq \cdots$ and $\lim_{n \rightarrow \infty} r_n = r$. Then $\tilde{F}^r(\omega) = \cap_{n=1}^\infty \tilde{F}^{r_n}(\omega)$ on $\Omega$. By [10, Lemma 4.2], $\lim_{n \rightarrow \infty} s(x^*, \tilde{F}^{r_n}(\omega)) = s(x^*, \tilde{F}^r(\omega))$ on $\Omega$ for each $x^* \in X^*$. Hence $\lim_{n \rightarrow \infty} (j \circ \tilde{F}^{r_n})(\omega) = (j \circ \tilde{F}^r)(\omega)$ on $\Omega$. Since $\|j \circ \tilde{F}^{r_n}(\omega)\|_\infty = \|\tilde{F}^{r_n}(\omega)\| \leq \|\tilde{F}_n^0(\omega)\| \leq h(\omega)$ on $\Omega$ for each $n \in \mathbb{N}$, by [1, Corollary 8] $j \circ \tilde{F}^r : \Omega \rightarrow \ell_\infty(B_{X^*})$ is Birkhoff integrable on $\Omega$ and
\[ \lim_{n \to \infty} \int_A j \circ \tilde{F}^r_n d\mu = (B) \int_A j \circ \tilde{F}^r d\mu. \]

For each \( x^* \in B_{X^*}, \)
\[
|s(x^*, M_{r_n}) - s(x^*, M_r)| = \left| s(x^*, (B) \int_A \tilde{F}^r_n d\mu) - s(x^*, (B) \int_A \tilde{F}^r d\mu) \right|
\]
\[
= \left| j(\int_A \tilde{F}^r_n d\mu)(x^*) - j(\int_A \tilde{F}^r d\mu)(x^*) \right|
\]
\[
= \left| \int_A j \circ \tilde{F}^r_n d\mu(x^*) - \int_A j \circ \tilde{F}^r d\mu(x^*) \right|
\]
\[
\leq \left\| (B) \int_A j \circ \tilde{F}^r_n d\mu - (B) \int_A j \circ \tilde{F}^r d\mu \right\|_{\infty} \to 0 \text{ as } n \to \infty.
\]

Thus \( \lim_{n \to \infty} s(x^*, M_{r_n}) = s(x^*, M_r) \) for each \( x^* \in B_{X^*} \) and so
\[
\lim_{n \to \infty} s(x^*, M_{r_n}) = s(x^*, M_r) \text{ for each } x^* \in X^*. \]
By [10, Lemma 4.2],
\[
M_r = \bigcap_{n=1}^{\infty} M_{r_n}. \]
Let \( M_0 = X. \)
By [10, Lemma 4.1], there exists \( u_A \in \mathcal{F}(X) \) such that \( [u_A]^r = M_r = (B) \int \tilde{F}^r d\mu \) for each \( r \in (0, 1]. \)
Hence
\[ \tilde{F} : \Omega \to \mathcal{F}(X) \]
is Birkhoff integrable on \( \Omega. \)
By Theorem 3.8 and the Lebesgue Convergence Theorem,
\[
D \left( (B) \int \tilde{F}_n d\mu, (B) \int \tilde{F} d\mu \right) \leq \int \left| D(\tilde{F}_n, \tilde{F}) d\mu \right| \to 0 \text{ as } n \to \infty.
\]
Thus \( \lim_{n \to \infty} (B) \int \tilde{F}_n d\mu = (B) \int \tilde{F} d\mu. \)

\[ \square \]

References


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