Abstract. Let $D$ be an integral domain, $S$ be a saturated multiplicative subset of $D$ such that $D_S$ is a factorial domain, $\{X_\alpha\}$ be a nonempty set of indeterminates, and $D[[X_\alpha]]$ be the polynomial ring over $D$. We show that $S$ is a splitting (resp., almost splitting, $t$-splitting) set in $D$ if and only if every nonzero prime $t$-ideal of $D$ disjoint from $S$ is principal (resp., contains a primary element, is $t$-invertible). We use this result to show that $D \setminus \{0\}$ is a splitting (resp., almost splitting, $t$-splitting) set in $D[[X_\alpha]]$ if and only if $D$ is a GCD-domain (resp., UMT-domain with $Cl(D[[X_\alpha]])$ torsion, UMT-domain).

1. Introduction

Let $D$ be an integral domain with quotient field $K$, and let $F(D)$ be the set of nonzero fractional ideals of $D$. For each $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup\{I_v \mid J \in F(D), J \subseteq I, and J$ is finitely generated}. An ideal $I \in F(D)$ is called a $t$-ideal if $I_t = I$, and a $t$-ideal is a maximal $t$-ideal if it is maximal among proper integral $t$-ideals. It is well known that each nonzero principal ideal is a $t$-ideal; each proper integral $t$-ideal is contained in a maximal $t$-ideal; a prime ideal minimal over a $t$-ideal is a $t$-ideal; and each maximal...
456 G.W. Chang

t-ideal is a prime ideal. We say that an \( I \in \mathbf{F}(D) \) is t-invertible if 
\((II^{-1})_t = D\); equivalently, if \( II^{-1} \not\subseteq P \) for every maximal t-ideal \( P \) of 
\( D \). Let \( T(D) \) be the group of t-invertible fractional t-ideals of \( D \) under 
the t-multiplication \( A \ast B = (AB)_t \), and let \( \text{Prin}(D) \) be its subgroup of 
principal fractional ideals. The \((t-)\text{class group} \) of \( D \) is an abelian group 
\( \text{Cl}(D) = T(D)/\text{Prin}(D) \). The readers can refer to [12] for any undefined 
notation or terminology.

Let \( S \) be a saturated multiplicative subset of an integral domain \( D \). As in [3], we say that \( S \) is a t-splitting set if for each \( 0 \neq d \in D \), we have 
\( dD = (AB)_t \) for some integral ideals \( A, B \) of \( D \), where \( A_t \cap sD = sA_t \) for 
all \( s \in S \) and \( B_t \cap S \neq \emptyset \). We say that \( S \) is an almost splitting set of \( D \) if 
for each \( 0 \neq d \in D \), there is an integer \( n = n(d) \geq 1 \) such that \( d^n = s^a \) for 
some \( s \in S \) and \( a \in N(S) \), where \( N(S) = \{ 0 \neq x \in D \mid (x, s')_t = D \) 
for all \( s' \in S \} \). A splitting set is an almost splitting set in which \( n = n(d) = 1 \) for every \( 0 \neq d \in D \). Let \( \overline{S} \) be the saturation of a multiplicative 
set \( S \) of \( D \). Note that a splitting set is saturated, while both t-splitting 
sets and almost splitting sets need not be saturated. Also, note that \( S \) is 
t-splitting (resp., almost splitting) if and only if \( \overline{S} \) is; so we always 
assume that \( S \) is saturated. It is known that an almost splitting set is 
t-splitting [7, Proposition 2.3]; hence

\[
\text{splitting set} \Rightarrow \text{almost splitting} \Rightarrow \text{t-splitting set}.
\]

Moreover, if \( Cl(D) \) is torsion, then a t-splitting set is almost splitting [7, 
Corollary 2.4] and if \( Cl(D) = 0 \), then splitting set \( \iff \) almost splitting \( \iff \) 
t-splitting set.

Let \( X \) be an indeterminate over \( D \) and \( D[X] \) be the polynomial ring 
over \( D \). An upper to zero over \( D \) and \( D[X] \) be the polynomial ring 
over \( D \). An upper to zero in \( D[X] \) is a nonzero prime ideal \( Q \) of \( D[X] \) 
with \( Q \cap D = (0) \), and \( D \) is called a \( \text{UMT-domain} \) if each upper to 
zero in \( D[X] \) is a maximal t-ideal of \( D[X] \). We say that \( D \) is a \( \text{Pr"{u}fer} \) 
v-multiplication domain (\( \text{Pr}v\text{MD} \)) if each nonzero finitely generated ideal 
of \( D \) is t-invertible. As in [15], we say that \( D \) is an \( \text{almost GCD-domain} \) 
(\( \text{AGCD-domain} \)) if for each nonzero finitely generated ideal of \( D \) is t-invertible. As in [15], we say that \( D \) is an \( \text{almost GCD-domain} \) 
(\( \text{AGCD-domain} \)) if for each \( 0 \neq a, b \in D \), there is an integer 
\( n = n(a,b) \geq 1 \) such that \( a^nD \cap b^nD \) is principal. Clearly, GCD-domains 
are AGCD-domains. It is known that AGCD-domains are UMT-domains 
with torsion class group [5, Lemma 3.1]; \( D \) is a \( \text{PuMD} \) if and only if \( D \) 
is an integrally closed UMT-domain [13, Proposition 3.2]; and \( D \) is a 
GCD-domain if and only if \( D \) is a \( \text{PuMD} \) and \( Cl(D) = 0 \) [6, Corollary 
1.5].
In [9, Theorem 2.8], the authors proved that if $D_S$ is a principal ideal domain (PID), then $S$ is a $t$-splitting set of $D$ if and only if every nonzero prime ideal of $D$ disjoint from $S$ is $t$-invertible. They used this result to show that $D \setminus \{0\}$ is a $t$-splitting set of $D[X]$ if and only if $D$ is a UMT-domain [9, Corollary 2.9]. Also, in [8, Theorem 2], the author showed that if $D_S$ is a PID, then $S$ is an almost splitting set of $D$ if and only if every nonzero prime ideal of $D$ disjoint from $S$ contains a primary element. (A nonzero element $a \in D$ is said to be primary if $aD$ is a primary ideal.) The purpose of this paper is to show that the results of [9, Theorem 2.8] and [8, Theorem 2] are also true when $D_S$ is a factorial domain (note that a PID is a factorial domain). Precisely, we show that if $D_S$ is a factorial domain, then $S$ is a splitting (resp., almost splitting, $t$-splitting) set in $D$ if and only if every nonzero prime $t$-ideal of $D$ disjoint from $S$ is principal (resp., contains a primary element, is $t$-invertible). Let $\{X_\alpha\}$ be a nonempty set of indeterminates over $D$, and note that $D[\{X_\alpha\}]_{D \setminus \{0\}}$ is a factorial domain. Hence, we then use the results we obtained in this paper to show that $D \setminus \{0\}$ is a splitting (resp., almost splitting, $t$-splitting) set in $D[\{X_\alpha\}]$ if and only if $D$ is a GCD-domain (resp., UMT-domain and $Cl(D[\{X_\alpha\}])$ is torsion, UMT-domain).

2. Main Results

Let $D$ be an integral domain, $D^* = D \setminus \{0\}$, $\{X_\alpha\}$ be a nonempty set of indeterminates over $D$, and $D[\{X_\alpha\}]$ be the polynomial ring over $D$.

We begin this section with nice characterizations of splitting sets, almost splitting sets, and $t$-splitting sets which appear in [2, Theorem 2.2], [4, Proposition 2.7], and [3, Corollary 2.3], respectively.

**Lemma 1.** Let $S$ be a saturated multiplicative subset of $D$.

1. $S$ is a splitting (resp., $t$-splitting) set of $D$ if and only if $dD_S \cap D$ is principal (resp., $t$-invertible) for every $0 \neq d \in D$.

2. $S$ is an almost splitting set of $D$ if and only if for every $0 \neq d \in D$, there is a positive integer $n = n(d)$ such that $d^nD_S \cap D$ is principal.

Note that if $D_S$ is a PID, then every nonzero prime ideal $P$ of $D$ disjoint from $S$ has height-one, and thus $P$ is a $t$-ideal. Hence, our first result is a generalization of [9, Theorem 2.8] that if $D_S$ is a PID, then $S$ is a $t$-splitting set in $D$ if and only if every nonzero prime ideal of $D$ disjoint
from $S$ is $t$-invertible. The proof is similar to those of [9, Theorem 2.8] and [8, Theorem 2].

**Theorem 2.** Let $D$ be an integral domain and $S$ be a saturated multiplicative subset of $D$ such that $D_S$ is a factorial domain. Then $S$ is a $t$-splitting set in $D$ if and only if every prime $t$-ideal of $D$ disjoint from $S$ is $t$-invertible.

**Proof.** ($\Rightarrow$) Assume that $S$ is a $t$-splitting set of $D$, and let $P$ be a prime $t$-ideal of $D$ with $P \cap S = \emptyset$. Then $(PD_S)_t = PD_S$ [3, Theorem 4.9], and hence $PD_S = pD_S$ for some $p \in P$ since $D_S$ is a factorial domain. Thus, by Lemma 1, $P = PD_S \cap D = pD_S \cap D$ is $t$-invertible.

($\Leftarrow$) Let $0 \neq d \in D$. Then $dD_S = p_1^{e_1} \cdots p_k^{e_k}D_S$ for some $p_i \in D$ and positive integers $e_i$ such that every $p_i$ is a prime element in $D_S$ and $p_iD_S \neq p_jD_S$ if $i \neq j$. Let $P_i$ be the prime ideal of $D$ such that $P_iD_S = p_iD_S$. Clearly, $P_i$ is minimal over $dD$, and hence $P_i$ is a $t$-ideal. Moreover, $P_i \cap S = \emptyset$; so $P_i$ is $t$-invertible by assumption (and hence a maximal $t$-ideal [13, Proposition 1.3]). Note that $(P_i^{e_i})_t$ is $P_i$-primary [1, Lemma 1] because $P_i$ is a maximal $t$-ideal. Also, $(P_i^{e_i})_tD_S = p_i^{e_i}D_S$, and thus $P_i^{e_i}D_S \cap D = (P_i^{e_i})_t$ and $(P_i^{e_i})_t$ is $t$-invertible. Hence

$$dD_S \cap D = p_1^{e_1} \cdots p_k^{e_k}D_S \cap D$$

$$= (p_1^{e_1}D_S \cap \cdots \cap p_k^{e_k}D_S) \cap D$$

$$= (P_1^{e_1}D_S \cap \cdots \cap P_k^{e_k}D_S) \cap D$$

$$= (P_1^{e_1}D_S \cap D) \cap \cdots \cap (P_k^{e_k}D_S \cap D)$$

$$= (P_1^{e_1})_t \cap \cdots \cap (P_k^{e_k})_t$$

$$= ((P_1^{e_1})_t \cdots (P_k^{e_k})_t)_t.$$ 

Thus, $S$ is a $t$-splitting set by Lemma 1. $\square$

The next result is a generalization of [9, Corollary 2.9] that $D^*$ is a $t$-splitting set in $D[X]$, where $X$ is an indeterminate over $D$, if and only if $D$ is a UMT-domain.

**Corollary 3.** $D^*$ is a $t$-splitting set in $D[\{X_\alpha\}]$ if and only if $D$ is a UMT-domain.

**Proof.** ($\Rightarrow$) Let $X \in \{X_\alpha\}$, and let $P$ be a nonzero prime ideal of $D[X]$ with $P \cap D = (0)$. Then $P$ is a prime $t$-ideal of $D[X]$, and hence $Q := P[Y]$, where $Y = \{X_\alpha\} \setminus \{X\}$, is a prime $t$-ideal of $D[\{X_\alpha\}]$ [11, Lemma 2.1(1)] such that $Q \cap D^* = \emptyset$. Hence, $Q$ is $t$-invertible by Theorem 2 because $D[\{X_\alpha\}]_{D^*}$ is a factorial domain. Note that $D[\{X_\alpha\}] =
Let $Q$ be a prime $t$-ideal of $D[[X_n]]$ such that $Q \cap D^* = \emptyset$. Since $Q \neq (0)$, there are $X_1, \ldots, X_n \in \{X_n\}$ such that $Q \cap D[X_1, \ldots, X_{n-1}] = (0)$, but $Q \cap D[X_1, \ldots, X_n] \neq (0)$. Let $R = D[X_1, \ldots, X_{n-1}]$ and $P = Q \cap R[X_n]$. Then $R$ is a UMT-domain [11, Theorem 2.4] and $P$ is an upper to zero in $R[X_n]$. Hence, $P$ is a $t$-invertible prime $t$-ideal. Let $Z = \{X_n\} \setminus \{X_1, \ldots, X_n\}$, and note that $P[Z] \subseteq Q$ and $P[Z]$ is a $t$-invertible prime $t$-ideal of $D[\{X_n\}]$ (see the proof of $(\Rightarrow)$ above). Hence, $P[Z]$ is a maximal $t$-ideal of $D[\{X_n\}]$, and thus $Q = P[Z]$ and $Q$ is $t$-invertible. Thus, by Theorem 2, $D^*$ is a $t$-splitting set.

We next give an almost splitting set analog of Theorem 2. Even though the proof is a word for word translation of the proof of [8, Theorem 2], we give it for the completeness of this paper.

**Theorem 4.** Let $D$ be an integral domain and $S$ be a saturated multiplicative subset of $D$ such that $D_S$ is a factorial domain. Then $S$ is an almost splitting set in $D$ if and only if every prime $t$-ideal of $D$ disjoint from $S$ contains a primary element.

**Proof.** $(\Rightarrow)$ Assume that $S$ is an almost splitting set of $D$, and let $P$ be a prime $t$-ideal of $D$ disjoint from $S$. Then $PD_S = pD_S$ for some $p \in P$ (see the proof of Theorem 2), and since $S$ is almost splitting, by Lemma 1, there is a positive integer $n$ such that $P \supseteq p^nD_S \cap D = qD$ for some $q \in D$. Clearly, $q$ is a primary element. Thus, $P$ contains a primary element $q$.

$(\Leftarrow)$ Let $0 \neq d \in D$. Then $dD_S = p_1^{e_1} \cdots p_k^{e_k} D_S$, where every $e_i$ is a positive integer and the $p_i$'s are non-associate prime elements in $D_S$ (see the proof of Theorem 2). Let $P_i$ be the prime ideal of $D$ such that $P_iD_S = p_iD_S$. Then $P_i$ is a $t$-ideal of $D$ and $P_i \cap S = \emptyset$; so $P_i$ contains a primary element $q_i$. Clearly, $q_iD_S = p_i^{n_i} D_S$ for some positive integer $n_i$. Let $n = n_1 \cdots n_k$ and $m_i = \frac{n}{n_i}e_i$. Then $p_i^{m_i}D_S = q_i^{m_i} D_S$, and
hence
\[
d^n D_S \cap D = ((p_1^{m_1}) D_S \cap \cdots \cap (p_k^{m_k}) D_S) \cap D
\]
\[
= ((q_1^{m_1} D_S) \cap \cdots \cap (q_k^{m_k} D_S)) \cap D
\]
\[
= (q_1^{m_1} D_S \cap \cdots \cap (q_k^{m_k} D_S) \cap D)
\]
\[
= (q_1^{m_1} D \cap \cdots \cap (q_k^{m_k} D)
\]
\[
= (q_1^{m_1} \cdots q_k^{m_k}) D,
\]
where the fourth and last equalities follow from the fact that each \(q_i^{m_i}\) is a primary element with \(\sqrt{q_i^{m_i} D} \neq \sqrt{q_j^{m_j} D}\) for \(i \neq j\). Therefore, \(S\) is an almost splitting set by Lemma 1.

Let \(N(D^*) = \{ f \in D[X_\alpha] \mid (f, d)_v = D[X_\alpha] \text{ for all } d \in D^* \}\). It is clear that \((f, d)_v = D[X_\alpha] \text{ for all } d \in D^* \text{ if and only if } c(f)_v = D\),

where \(c(f)\) is the ideal of \(D\) generated by the coefficients of \(f\). Hence,
\[
Cl(D[X_\alpha]_{N(D^*)}) = 0 \quad [14, \text{Theorem 2.14}].
\]
The next result is a generalization of [5, Theorem 2.4].

**Corollary 5.** \(D^*\) is an almost splitting set in \(D[X_\alpha]\) if and only if \(D\) is a UMT-domain and \(Cl(D[X_\alpha])\) is torsion.

**Proof.** \((\Rightarrow)\) If \(D^*\) is an almost splitting set in \(D[X_\alpha]\), then \(Cl(D[X_\alpha]_{D^*}) = Cl((D[X_\alpha])_{N(D^*)}) = 0\). Thus, \(Cl(D[X_\alpha])\) is torsion \([7, \text{Theorem 2.10(2)}]\). Also, since almost splitting sets are \(t\)-splitting sets, \(D\) is a UMT-domain by Corollary 3.

\((\Leftarrow)\) Assume that \(D\) is a UMT-domain and \(Cl(D[X_\alpha])\) is torsion. Then \(D^*\) is a \(t\)-splitting set by Corollary 3, and since \(Cl(D[X_\alpha])\) is torsion, \(D^*\) is an almost splitting set. \(\square\)

**Corollary 6.** If \(D\) is integrally closed, then \(D^*\) is an almost splitting (resp., a \(t\)-splitting) set in \(D[X_\alpha]\) if and only if \(D\) is an AGCD-domain (resp., a \(PvMD\)).

**Proof.** Note that \(Cl(D[X_\alpha]) = Cl(D)\) \([10, \text{Corollary 2.13}]\); an integrally closed UMT-domain is a \(PvMD\); and an integrally closed AGCD-domain is a \(PvMD\) with torsion class group. Hence, the result follows directly from Corollaries 3 and 5.

**Theorem 7.** Let \(D\) be an integral domain and \(S\) be a saturated multiplicative subset of \(D\) such that \(D_S\) is a factorial domain. Then \(S\) is a splitting set in \(D\) if and only if every prime \(t\)-ideal of \(D\) disjoint from \(S\) is principal.
Proof. (⇒) Let $P$ be a prime $t$-ideal of $D$ with $P \cap S = \emptyset$. Then $PD_S = pD_S$ for some prime element $p$ of $D_S$ (see the proof of Theorem 2), and thus $PD_S \cap D = pD_S \cap D$ is principal by Lemma 1.

(⇐) An argument similar to the proof (⇐) of Theorem 4 shows that $dD_S \cap D$ is principal for every $0 \neq d \in D$. Thus, by Lemma 1, $S$ is a splitting set.

Let $X$ be an indeterminate over $D$. In [9, p. 77] (cf. [2, Example 4.7]), it was noted that $D^*$ is a splitting set in $D[X]$ if and only if $D$ is a GCD-domain.

**Corollary 8.** $D^*$ is a splitting set in $D[\{X_\alpha\}]$ if and only if $D$ is a GCD-domain.

**Proof.** If $D^*$ is a splitting set in $D[\{X_\alpha\}]$, then $Cl(D) = Cl(D[\{X_\alpha\}]) = 0$ [2, Corollary 3.8] because $Cl(D[\{X_\alpha\}]_{D^*}) = Cl((D[\{X_\alpha\}])_{N(D^*)}) = 0$. Hence, $D$ is integrally closed [10, Corollary 2.13] and $D$ is a UMT-domain by Corollary 3. Thus, $D$ is a GCD domain because $D$ is an integrally closed UMT-domain with $Cl(D) = 0$. Conversely, assume that $D$ is a GCD-domain. Then $D^*$ is a $t$-splitting set in $D[\{X_\alpha\}]$ by Corollary 3 and $Cl(D[\{X_\alpha\}]) = Cl(D) = 0$. Thus, $D^*$ is a splitting set.

Let $S$ be a saturated multiplicative subset of an integral domain $D$ such that $D_S$ is a factorial domain. The proofs of Theorems 2, 4, and 7 show that $S$ is splitting (resp., almost splitting, $t$-splitting) if and only if for every nonzero prime element $p$ of $D_S$, the ideal $pD_S \cap D$ is principal (resp., contains a primary element, $t$-invertible).

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**References**


Department of Mathematics
Incheon National University
Incheon 406-772, Korea

E-mail: whan@incheon.ac.kr