A REFINED ENUMERATION OF $p$-ARY LABELED TREES

SEUNGHYUN SEO† AND HEESSUNG SHIN‡

Abstract. Let $T_n^{(p)}$ be the set of $p$-ary labeled trees on \{1, 2, \ldots, n\}. A maximal decreasing subtree of an $p$-ary labeled tree is defined by the maximal $p$-ary subtree from the root with all edges being decreasing. In this paper, we study a new refinement $T_n^{(p)}_{n,k}$ of $T_n^{(p)}$, which is the set of $p$-ary labeled trees whose maximal decreasing subtree has $k$ vertices.

1. Introduction

Let $p$ be a fixed integer greater than 1. A $p$-ary tree $T$ is a tree such that:

(i) Either $T$ is empty or has a distinguished vertex $r$ which is called the root of $T$, and

(ii) $T - r$ consists of a weak ordered partition $(T_1, \ldots, T_p)$ of $p$-ary trees.

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A 2-ary (resp. 3-ary) tree is called binary (resp. ternary) tree. Figure 1 exhibits all the ternary tree with 3 vertices. A full $p$-ary tree is a $p$-ary tree, where each vertex has either 0 or $p$ children. It is well known (see [6, 6.2.2 Proposition]) that the number of full $p$-ary trees with $n$ internal vertices is given by the $n$th order-$p$ Fuss-Catalan number $C_n^{(p)} = \frac{1}{pn+1} \binom{pn+1}{n}$. Clearly a full $p$-ary tree $T$ with $m$ internal vertices corresponds to a $p$-ary tree with $m$ vertices by deleting all the leaves in $T$, so the number of $p$-ary trees with $n$ vertices is also $C_n^{(p)}$.

![Figure 1. All 12 ternary trees with 3 vertices](image)

An $p$-ary labeled tree is a $p$-ary tree whose vertices are labeled by distinct positive integers. In most cases, a $p$-ary labeled tree with $n$ vertices is identified with an $p$-ary tree on the vertex set $[n] := \{1, 2, \ldots, n\}$. Let $\mathcal{T}_n^{(p)}$ be the set of $p$-ary labeled trees on $[n]$. Clearly the cardinality of $\mathcal{T}_n^{(p)}$ is given by

$$|\mathcal{T}_n^{(p)}| = n! C_n^{(p)} = (pn)_{n-1},$$

where $m_{(k)} := m(m - 1) \cdots (m - k + 1)$ is a falling factorial.

For a given $p$-ary labeled tree $T$, a maximal decreasing subtree of $T$ is defined by the maximal $p$-ary subtree from the root with all edges being decreasing, denoted by $\text{MD}(T)$. Figure 2 illustrates the maximal decreasing subtree of a given ternary tree $T$. Let $\mathcal{T}_{n,k}^{(p)}$ be the set of $p$-ary labeled trees on $[n]$ with its maximal decreasing subtree having $k$ vertices.

In this paper we present a formula for $|\mathcal{T}_{n,k}^{(p)}|$, which makes a refined enumeration of $\mathcal{T}_n^{(p)}$, or a generalization of equation (1). Note that a similar refinement for rooted labeled trees and ordered labeled trees were done before (see [4,5]), but the $p$-ary case is much more complicated and has quite different features.
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Figure 2. The maximal decreasing subtree of the ternary labeled tree \( T \)

2. Main results

From now on we will consider labeled trees only. So we will omit the word “labeled”. Recall that \( \mathcal{T}_{n,k}^{(p)} \) is the set of \( p \)-ary trees on \([n]\) with its maximal decreasing ordered subtree having \( k \) vertices. Let \( \mathcal{Y}_{n,k}^{(p)} \) be the set of \( p \)-ary trees \( T \) on \([n]\), where \( T \) is given by attaching additional \((n - k)\) increasing leaves to a decreasing tree with \( k \) vertices. Let \( \mathcal{F}_{n,k}^{(p)} \) be the set of (non-ordered) forests on \([n]\) consisting of \( k \) \( p \)-ary trees, where the \( k \) roots are not ordered. In Figure 3, the first two forests are the same, but the third one is a different forest in \( \mathcal{F}_{4,2}^{(2)} \).

Figure 3. Forests in \( \mathcal{F}_{4,2}^{(3)} \)

Define the numbers

\[
\begin{align*}
t(n, k) &= |\mathcal{T}_{n,k}^{(p)}|, \\
y(n, k) &= |\mathcal{Y}_{n,k}^{(p)}|, \\
f(n, k) &= |\mathcal{F}_{n,k}^{(p)}|.
\end{align*}
\]
We will show that a $p$-ary tree can be “decomposed” into a $p$-ary tree in $\bigcup_{n,k} Y^{(p)}_{n,k}$ and a forest in $\bigcup_{n,k} F^{(p)}_{n,k}$. Thus it is important to count the numbers $y(n,k)$ and $f(n,k)$.

**Lemma 2.1.** For $0 \leq k < n$, the number $y(n,k)$ satisfies the recursion:

$$y(n+1,k+1) = \sum_{m=0}^{p} \binom{n}{m} p(m) (kp - n + m + 1) y(n-m,k)$$

with the following boundary conditions:

$$y(n,n) = \prod_{j=0}^{n-1} (1 + (p-1)j) \quad \text{for } n \geq 1$$

$$y(n,k) = 0 \quad \text{for } k < \max\left(\frac{n-1}{p}, 1\right).$$

**Proof.** Consider a tree $Y$ in $\bigcup_{n,k} Y^{(p)}_{n+1,k+1}$. The tree $Y$ with $n+1$ vertices consists of its maximal decreasing tree with $k+1$ vertices and the number of increasing leaves is $n-k$. Note that the vertex 1 is always contained in $\text{MD}(Y)$.

If the vertex 1 is a leaf of $Y$, consider the tree $Y'$ by deleting the leaf 1 from $Y$. The number of vertices in $Y'$ and $\text{MD}(Y')$ are $n$ and $k$, respectively. So the number of possible trees $Y'$ is $y(n,k)$. Since we cannot attach the vertex 1 to $n-k$ increasing leaves of $Y'$, there are $kp - (n-1)$ ways of recovering $Y$. Thus the number of $Y$ with the leaf 1 is

$$y(n+1,k+1) = \sum_{m=0}^{p} \binom{n}{m} p(m) (kp - n + m + 1) y(n-m,k).$$

If the vertex 1 is not a leaf of $Y$, then the vertex 1 has increasing leaves $\ell_1, \ldots, \ell_m$, where $1 \leq m \leq p$. Consider the tree $Y''$ obtained by deleting $\ell_1, \ldots, \ell_m$ from $Y$. Clearly 1 is a leaf of $Y''$ and the number of vertices in $Y''$ and $\text{MD}(Y'')$ are $n-m+1$ and $k+1$, respectively. Thus by (5), the number of possible trees $Y''$ is $(kp - (n-m) + 1) \cdot y(n-m,k)$. To recover $Y$ is to relabel all the vertices except 1 of $Y''$ with the label set $\{2, 3, \ldots, n+1\} \setminus \{\ell_1, \ldots, \ell_m\}$ and to attach the leaves $\ell_1, \ldots, \ell_m$ to the vertex 1 of $Y''$. Clearly $\ell_1, \ldots, \ell_m$ is a subset of $\{2, 3, \ldots, n+1\}$. It is obvious that a way of attaching $\ell_1, \ldots, \ell_m$ to vertex 1 can be regarded as an injection from $\ell_1, \ldots, \ell_m$ to $[p]$. Thus the number of $Y$ without the
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Table 1. $y(n, k)$ with $p = 2$

Leaf 1 is

$$y(n, k) = \binom{n}{k} \binom{p}{m} m! (kp - (n - m) + 1) \cdot y(n - m, k).$$

Since $m$ may be the number from 1 to $p$ and substituting $m = 0$ in (6) yields (5), we have the recursion (2).

Since $\mathcal{Y}^{(p)}_{n,m}$ is the set of decreasing $p$-ary trees on $[n]$, the equation (3) holds (see [1]). If the inequality $pk - (k - 1) < n - k$ holds, $\mathcal{Y}^{(p)}_{n,k}$ should be empty. For $n \geq 1$ and $k = 0$, $\mathcal{Y}^{(p)}_{n,k}$ is also empty. Thus the equation (4) also holds.

The table for $y(n, k)$ with $p = 2$ is shown in Table 1.

Now we calculate $f(n, k)$ which is the number of forests on $[n]$ consisting of $k$ $p$-ary trees, where the $k$ components are not ordered. Here we use the convention that the empty product is 1.

**Lemma 2.2.** For $0 \leq k \leq n$, we have

$$f(n, k) = \binom{n}{k} pk \prod_{i=1}^{n-k-1} (pn - i) \quad \text{if } n > k,$$

else $f(n, n) = 1$.

**Proof.** Consider a forest $F$ in $\mathcal{F}^{(p)}_{n,k}$. The forest $F$ consists of (non-ordered) $p$-ary trees $T_1, \ldots, T_k$ with roots $r_1, r_2, \ldots, r_k$, where $r_1 < r_2 < \cdots < r_k$. The number of ways for choosing roots $r_1, r_2, \cdots, r_k$ from $[n]$ is equal to $\binom{n}{k}$. From the reverse Prüfer algorithm (RP Algorithm) in [3],
the number of ways for adding \( n - k \) vertices successively to \( k \) roots \( r_1, r_2, \ldots, r_k \) is equal to

\[
 pk(pn-1)(pn-2) \cdots (pn-n+k+1)
\]

for \( 0 < k < n \), thus the equation (7) holds. For \( 0 = k < n \), \( \mathcal{F}_{n,0}^{(p)} \) is empty, so \( f(n,0) = 0 \) included in (7). For \( 0 \leq k = n \), \( \mathcal{F}_{n,n}^{(p)} \) is the set of forests with no edges, so \( f(n,n) = 1 \).

Since the number \( y(n,k) \) is determined by the recurrence relation (2) in Lemma 2.1, we can count the number \( t(n,k) \) with the following theorem.

**Theorem 2.3.** For \( n \geq 1 \), we have

\[
 t(n,k) = \sum_{m=k}^{n} \binom{n}{m} \frac{m-k}{n-k} (pn-pk)_{(n-m)} y(m,k) \quad \text{if} \quad 1 \leq k < n,
\]

else \( t(n,n) = \prod_{j=0}^{n-1} (pj-j+1) \), where \( a(\ell) := a(a-1) \cdots (a-\ell+1) \) is a falling factorial.

*Proof.* Given a \( p \)-ary tree \( T \) in \( \mathcal{T}_{n,k}^{(p)} \), let \( Y \) be the subtree of \( T \) consisting of \( \text{MD}(T) \) and its increasing leaves. If \( Y \) has \( m \) vertices, then \( Y \) is a subtree of \( T \) with \( (m-k) \) increasing leaves. Also, the induced subgraph \( Z \) of \( T \) generated by the \( (n-k) \) vertices not belonging to \( \text{MD}(T) \) is a (non-ordered) forest consisting of \( (m-k) \) \( p \)-ary trees whose roots are increasing leaves of \( Y \). Figure 4 illustrates the subgraph \( Y \) and \( Z \) of a given ternary tree \( T \).

Now let us count the number of \( p \)-ary trees \( T \in \mathcal{T}_{n,k}^{(p)} \) with \( |V(Y)| = m \) where \( V(Y) \) is the set of vertices in \( Y \). First of all, the number of ways for selecting a set \( V(Y) \subset [n] \) is equal to \( \binom{n}{m} \). By attaching \( (m-k) \) increasing leaves to a decreasing \( p \)-ary tree with \( k \) vertices, we can make a \( p \)-ary trees on \( V(Y) \). So there are exactly \( y(m,k) \) ways for making such a \( p \)-ary subtree on \( V(Y) \). Since all the roots of \( Z \) are determined by \( Y \), by the definition of \( \mathcal{F}_{n,k}^{(p)} \) and Lemma 2.2, the number of ways for constructing the other parts on \( V(T) \setminus V(\text{MD}(T)) \) is equal to

\[
 f(n-k,m-k) / \binom{n-k}{m-k} = \frac{m-k}{n-k} (pn-pk)_{(n-m)}.
\]

Since the range of \( m \) is \( k \leq m \leq n \), the equation (8) holds.
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Figure 4. Decomposition of $T$ into $Y$ and $Z$

Table 2. $t(n, k)$ with $p = 2$

Finally, $T^{(p)}(n, n)$ is the set of decreasing $p$-ary trees on $[n]$, so

$$t(n, n) = y(n, n) = \prod_{j=0}^{n-1} (pj - j + 1)$$

holds for $n \geq 1$.

The sequence $t(n, k)$ with $p = 2$ is listed in Table 2. Note that each row sum is equal to $n!C_n^{(p)}$ with $p = 2$.

Remark. Due to Lemma 2.1 and Theorem 2.3, we can calculate $t(n, k)$ for all $n, k$. In particular we express $t(n, k)$ as a linear combination of $y(k, k), y(k + 1, k), \ldots, y(n, k)$. However a closed form, a recurrence relation, or a (double) generating function of $t(n, k)$ have not been found yet.
References


Department of Mathematics Education
Kangwon National University
Chuncheon 200-701, Korea
E-mail: shyunseo@kangwon.ac.kr

Department of Mathematics
Inha University
Incheon 402-751, Korea
E-mail: shin@inha.ac.kr