ON \( k \)-QUASI-CLASS \( A \) CONTRACTIONS

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Abstract. A bounded linear Hilbert space operator \( T \) is said to be \( k \)-quasi-class \( A \) operator if it satisfy the operator inequality
\[
T^*k|T|^2T^k \geq T^k|T|^2T^k
\]
for a non-negative integer \( k \). It is proved that if \( T \) is a \( k \)-quasi-class \( A \) contraction, then either \( T \) has a nontrivial invariant subspace or \( T \) is a proper contraction and the nonnegative operator
\[
D = T^k(|T|^2 - |T|^2)T^k
\]
is strongly stable.

1. Introduction

Let \( B(\mathcal{H}) \) denote the algebra of bounded linear operators on an infinite dimensional complex Hilbert space \( \mathcal{H} \). For any operator \( T \) in \( B(\mathcal{H}) \) set, as usual, \( |T| = (T^*T)^{1/2} \) and \( [T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2 \) (the self-commutator of \( T \)), and consider the following standard definitions: \( T \) is hypernormal if \( |T^*|^2 \geq |T|^2 \) (i.e., if self-commutator \( [T^*, T] \) is non-negative or, equivalently, if \( \|T^*x\| \leq \|Tx\| \) for every \( x \) in \( \mathcal{H} \)), \( p \)-hyponormal if \( (T^*T)^p \geq (TT^*)^p \) for some \( p \in (0, 1] \), and \( T \) is called paranormal if \( \|T^2x\| \geq \|Tx\|^2 \) for all unit vector \( x \in \mathcal{H} \). Following [13] and [4] we say that \( T \in B(\mathcal{H}) \) belongs to class \( A \) if \( |T|^2 \geq |T^2| \). We shall denote classes of hyponormal operators, \( p \)-hyponormal operators, paranormal operators, and class \( A \) operators by \( \mathcal{H} \), \( \mathcal{H}(p) \), \( \mathcal{P}\mathcal{N} \), and \( \mathcal{A} \).
respectively. It is well known that

\begin{equation}
\mathcal{H} \subset \mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{P}\mathcal{N}.
\end{equation}

In [8] authors considered an extension of the notion of class \( \mathcal{A} \) operators; we say that \( T \in B(\mathcal{H}) \) is \( k \)-quasi-class \( \mathcal{A} \) operator if

\[ T^{*k}|T|^2T^k \geq T^{*k}|T|^2T^k \]

for non-negative integer \( k \); when \( k = 1 \), it is called the quasi-class \( \mathcal{A} \) operator. We shall denote the set of \( k \)-quasi-class \( \mathcal{A} \) operators by \( \mathcal{QA}(k) \). Class \( \mathcal{QA}(k) \) properly contains class \( \mathcal{A} \) and quasi-class \( \mathcal{A} \).

It is well known that

\begin{equation}
\mathcal{H} \subset \mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{QA} \subset \mathcal{QA}(k).
\end{equation}

In view of inclusions (1) and (2), it seems reasonable to expect that the operators in class \( \mathcal{QA} \) are paranormal. But there exists an example of a class \( \mathcal{QA} \) operator which is not paranormal ([8]).

Recall, [10], that a contraction \( A \) (i.e., if \( \|A\| \leq 1 \), which means that \( \|Ax\| \leq \|x\| \) for every \( x \in \mathcal{H} \)) is said to be a proper contraction if \( \|Ax\| < \|x\| \) for every nonzero \( x \in \mathcal{H} \). A strict contraction (i.e., a contraction \( A \) such that \( \|A\| < 1 \)) is a proper contraction, but a proper contraction is not necessarily a strict contraction. C. S. Kubrusly and N. Levan [10] have proved that if a hyponormal contraction \( A \) has no nontrivial invariant subspace, then

(a) \( A \) is a proper contraction and
(b) its self-commutator \([A^*, A]\) is a strict contraction.

Recently B. p. Duggal, I. H. Jeon and C. S. Kubrusly [2] showed that if \( A \) is a class \( \mathcal{A} \) contraction, then either \( A \) has a nontrivial invariant subspace or \( A \) is a proper contraction and the non-negative operator \( D = |A^2| - |A|^2 \) is strongly stable (i.e., the power sequence \( \{D^n\} \) converges strongly to 0). Very recently B. P. Duggal and authors [3] extend these results to contractions in \( \mathcal{QA} \). In this paper, we extend these results to contractions in \( \mathcal{QA}(k) \), which generalizes results proved for contractions in \( \mathcal{QA} \) [2].

2. Results

We begin with well known following lemma;
**Lemma 2.1.** (see, [13]) An operator $T \in QA(k)$ has a following matrix representation if $\text{ran}(T^k)$ is not dense

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$ on $\overline{\text{ran}(T^k)} \oplus \ker(T^{\ast k})$, where $A \in \mathcal{A}$, $C$ is a nilpotent with order $k$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

**Lemma 2.2.** If $T \in QA(k)$ is a contraction, then the non–negative operator $D = T^{\ast k}(|T^2| - |T|^2)T^k$ is a contraction such that the power sequence $\{D^n\}$ converges strongly to a projection $P$ satisfying $T^{k+1}P = 0$.

**Proof.** Set $R = D^{\frac{1}{2}}$. Then, for every $x \in \mathcal{H}$, we have

$$\langle D^{n+1}x, x \rangle = \langle R^{n+1}x, R^{n+1} \rangle = \langle DR^n x, R^n x \rangle = \langle T^{\ast k}(|T^2| - |T|^2)T^k R^n x, R^n x \rangle = \langle |T|^2 T^k R^n x, T^k R^n x \rangle - \langle |T|^2 T^k R^n x, T^k R^n x \rangle \leq \| |T^2|^{\frac{1}{2}} T^k R^n x \|^2 \leq \|R^n x\|^2 \quad (T \text{ is contraction})$$

$$= \langle D^n x, x \rangle,$$

which implies that $D$ is a contraction. Evidently, the sequence $\{D^n\}$ being a monotonic decreasing sequence of non–negative contractions.

Therefore $\{D^n\}$ converges strongly to a projection $P$. Now we should be show that $T^{k+1}P = 0$. Since

$$\|R^n x\|^2 - \|R^{n+1} x\|^2 = \|R^n x\|^2 - \langle |T^2| T^k R^n x, T^k R^n x \rangle + \|T^{k+1} R^n x\|^2 \leq \langle 1 - T^{\ast k} |T^2| T^k R^n x, R^n x \rangle + \|T^{k+1} R^n x\|^2 \geq \|T^{k+1} R^n x\|^2 \quad (T \text{ is contraction}),$$

we have that

$$\sum_{n=0}^{m} \|T^{k+1} R^n x\|^2 \leq \sum_{n=0}^{m} \|R^n x\|^2 - \sum_{n=0}^{m} \|R^{n+1} x\|^2 = \|x\|^2 - \|R^{m+1} x\|^2 \leq \|x\|^2$$

for every $x \in \mathcal{H}$ and non–negative integer $m$. Hence $\|T^{k+1} R^n x\| \rightarrow 0$ as $n \rightarrow \infty$. Consequently, we have

$$T^{k+1}Px = T^{k+1} \lim_{n \rightarrow \infty} D^n x = \lim_{n \rightarrow \infty} T^{k+1} R^{2n} x = 0,$$

for every $x \in \mathcal{H}$. Hence $T^2 P = 0$. \qed
Recall that $T \in B(\mathcal{H})$ is a $C_0$-contraction (resp., $C_1$-contraction) if $||T^n x||$ converges to 0 for all $x \in \mathcal{H}$ (resp., does not converge to 0 for all non-trivial $x \in \mathcal{H}$); $T$ is of class $C_0$, or $C_1$, if $T^*$ is of class $C_0$, respectively $C_1$. All combinations are allowed, leading to the classes $C_{00}$, $C_{01}$, $C_{10}$ and $C_{11}$ of contractions [11, Page 72]. Duggal, Jeon and Kubrusly [2] showed that the following lemma;

**Lemma 2.3.** If a class $A$ contraction $T$ has no nontrivial invariant subspace, then (a) $T$ is a proper contraction and (b) the non-negative operator $D = |T^2| - |T|^2$ is a strongly stable contraction (so that $D \in C_{00}$).

Using the above lemmas we can show that the following theorem;

**Theorem 2.4.** If $T \in QA(k)$ is a contraction with no non–trivial invariant subspace for non-negative integer $k$, then: (a) $T$ is a proper contraction; (b) the non–negative operator $D = T^*k(|T^2| - |T|^2)T^k$ is a strongly stable contraction (and hence of class $C_{00}$).

*Proof. We may assume that $T$ is non-zero.

(a) If either of $T^{-1}(0)$ or $\text{ran}(T^k)$ is non–trivial (i.e., $T^{-1}(0) \neq \{0\}$ or $\text{ran}(T^k) \neq \mathcal{H}$), then $T$ has a non–trivial invariant subspace. Hence, if $T \in QA(k)$ has no non–trivial invariant subspace, then $T$ is injective and $\text{ran}(T^k) = \mathcal{H}$) Consequently, $T$ must be class $A$ operator. The proof now follows from Lemma 2.3.

(b) If $T \in QA(k)$ is a contraction, then by Lemma 2.2 $D$ is a contraction, $\{D^n\}$ converges strongly to a projection $P$ and $T^{k+1}P = 0$. Therefore we have $PT^{*k+1} = 0$. Suppose $T$ has no non–trivial invariant subspace. Since $P^{-1}(0)$ is a non-zero invariant subspace for $T$ whenever $PT^{*k+1} = 0$, we must have $P^{-1}(0) = \mathcal{H}$, hence $P$ must be zero and so $\{D^n\}$ converges strongly to 0, that is, $D$ is a strongly stable contraction. Since $D$ is a self-adjoint, $D \in C_{00}$. 

It is well known that a self-adjoint operator is a proper contraction if and only if it is a $C_{00}$-contraction. Hence, we have the following from Theorem 2.4.

**Corollary 2.5.** If $T \in QA(k)$ is a contraction with no non–trivial invariant subspace for non-negative integer $k$, then both $T$ and $T^*$ are proper contractions.
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