COMBINATORIAL INTERPRETATIONS OF THE ORTHOGONALITY RELATIONS FOR SPIN CHARACTERS OF $\tilde{S}_n$

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1. Introduction

The projective representations of the symmetric groups were originally studied by Schur. In his fundamental paper[6], Schur derived degree and character formulas for projective representations of the symmetric groups remarkably similar in style to the corresponding formulas for ordinary representations due to Frobenius. Morris[3] derived a recurrence for evaluation of spin characters, which is an analogue of the well-known Murnaghan-Nakayama formula for ordinary characters of the symmetric group $S_n$. Stembridge[8] then gave a combinatorial reformulation for Morris’ recurrence using shifted rim hook tableaux,

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In this paper we give combinatorial interpretations for the orthogonality relations of spin characters of $\tilde{S}_n$ based on Stembridge’s combinatorial reformulation for Morris’ rule.

In section 2, we outline the definitions and notation used in this paper. Section 3 reviews the basic properties of a group $\tilde{S}_n$ and draw some relations between the irreducible spin characters of $\tilde{S}_n$ and symmetric functions. In section 4, we give combinatorial interpretations for the orthogonality relations of spin characters of $\tilde{S}_n$.

2. Definitions

We use standard notation $\mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ for the set of all positive integers, the ring of integers, the field of rational numbers and the field of complex numbers, respectively.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of the nonnegative integer $n$, denoted $\lambda \vdash n$ or $|\lambda| = n$, so $\lambda$ is a weakly decreasing sequence of positive integers summing to $n$. We say each term $\lambda_i$ is a part of $\lambda$ and $n$ is the weight of $\lambda$. The number of nonzero parts is called the length of $\lambda$ and is written $\ell = \ell(\lambda)$. Let $\mathcal{P}_n$ be the set of all partitions of $n$ and $\mathcal{P}$ be the
Combinatorial interpretations of the orthogonality for spin characters of \( \tilde{S}_n \) set of all partitions. We also denote

\[
\begin{align*}
OP &= \{ \mu \in \mathcal{P} \mid \text{every part of } \mu \text{ is odd} \}, \\
OP_n &= \{ \mu \in \mathcal{P}_n \mid \text{every part of } \mu \text{ is odd} \}, \\
DP &= \{ \mu \in \mathcal{P} \mid \mu \text{ has all distinct parts} \}, \\
DP_n &= \{ \mu \in \mathcal{P}_n \mid \mu \text{ has all distinct parts} \}, \\
DP^+_n &= \{ \mu \in \mathcal{D}_P \mid n - \ell(\mu) \text{ is even} \}, \\
DP^-_n &= \{ \mu \in \mathcal{D}_P \mid n - \ell(\mu) \text{ is odd} \}.
\end{align*}
\]

We sometimes abbreviate the partition \( \lambda \) with the notation \( 1^{j_1} 2^{j_2} \ldots \), where \( j_i \) is the number of parts of size \( i \). Sizes which do not appear are omitted and if \( j_i = 1 \), then it is not written. Thus, a partition \( (5, 3, 2, 2, 1) \vdash 15 \) can be written \( 1^2 3^3 5 \).

For each \( \lambda \in \mathcal{D}_P \), a \textit{shifted diagram} \( D'_\lambda \) of shape \( \lambda \) is defined by

\[
D'_\lambda = \{(i, j) \in \mathbb{Z}^2 \mid i \leq j \leq \lambda_j + i - 1, 1 \leq i \leq \ell(\lambda) \}.
\]

And for \( \lambda, \mu \in \mathcal{D}_P \) with \( D'_\mu \subseteq D'_\lambda \), a \textit{shifted skew diagram} \( D'_{\lambda/\mu} \) is defined as the set-theoretic difference \( D'_\lambda \setminus D'_\mu \). Figure 2.1 and Figure 2.2 show \( D'_\lambda \) and \( D'_{\lambda/\mu} \) respectively when \( \lambda = (9, 7, 4, 2) \) and \( \mu = (5, 3) \).

![Figure 2.1](image1.png) ![Figure 2.2](image2.png) ![Figure 2.3](image3.png)

A shifted skew diagram \( \theta \) is called a \textit{single rim hook} if \( \theta \) is connected and contains no \( 2 \times 2 \) block of cells. If \( \theta \) is a single rim hook, then its \textit{head} is the upper rightmost cell in \( \theta \) and its \textit{tail} is the lower leftmost cell in \( \theta \). See Figure 2.3.

A \textit{double rim hook} is a shifted skew diagram \( \theta \) formed by the union of two single rim hooks both of whose tails are on the main diagonal. If \( \theta \) is a double rim hook, we denote by \( A[\theta] \) (resp., \( \alpha_1[\theta] \)) the set of diagonals of length two (resp., one). Also let \( \beta_1[\theta] \) (resp., \( \gamma_1[\theta] \)) be a single rim hook in \( \theta \) which starts on the upper (resp., lower) of the two main diagonal cells and ends at the head of \( \alpha_1[\theta] \). The tail of \( \beta_1[\theta] \) (resp., \( \gamma_1[\theta] \)) is called the \textit{first tail} (resp., \textit{second tail}) of \( \theta \) and the head of \( \beta_1[\theta] \) or \( \gamma_1[\theta] \) (resp., \( \gamma_2[\theta], \beta_2[\theta] \), where \( \beta_2[\theta] = \theta \setminus \beta_1[\theta] \) and \( \gamma_2[\theta] = \theta \setminus \gamma_1[\theta] \)) is
called the first head (resp., second head, third head) of $\theta$. Hence we have the following descriptions for a double rim hook $\theta$:

$$
\theta = A[\theta] \cup \alpha_1[\theta] \\
= \beta_1[\theta] \cup \beta_2[\theta] \\
= \gamma_1[\theta] \cup \gamma_2[\theta].
$$

A double rim hook is illustrated in Figure 2.4. We write $A, \alpha_1$, etc. for $A[\theta], \alpha_1[\theta]$, etc. when there is no confusion.

A shifted rim hook tableau of shape $\lambda \in DP$ and content $\rho = (\rho_1, \ldots, \rho_m)$ is defined recursively. If $m = 1$, a rim hook with all 1’s and shape $\lambda$ is a shifted rim hook tableau. Suppose $P$ of shape $\lambda$ has content $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ and the cells containing the $m$’s form a rim hook inside $\lambda$. If the removal of the $m$’s leaves a shifted rim hook tableau, then $P$ is a shifted rim hook tableau. We define a shifted skew rim hook tableau in a similar way. If $P$ is a shifted rim hook tableau, we write $\kappa_P(r)$ (or just $\kappa(r)$) for a rim hook of $P$ containing $r$.

If $\theta$ is a single rim hook then the rank $r(\theta)$ is one less than the number of rows it occupies and the weight $w(\theta) = (-1)^{r(\theta)}$; if $\theta$ is a double rim hook then the rank $r(\theta)$ is $|A[\theta]|/2 + r(\alpha_1[\theta])$ and the weight $w(\theta)$ is $2(-1)^{r(\theta)}$.

The weight of a shifted rim hook tableau $P$, $w(P)$, is the product of the weights of its rim hooks. The weight of a shifted skew rim hook tableau is defined in a similar way.
Figure 2.5 shows an example of a shifted rim hook tableau $P$ of shape $(5, 4, 1)$ and content $(5, 1, 4)$. Here $r(\kappa(1)) = 1$, $r(\kappa(2)) = 0$ and $r(\kappa(3)) = 1$. Also $w(\kappa(1)) = -2$, $w(\kappa(2)) = 1$ and $w(\kappa(3)) = -1$. Hence $w(P) = (-2) \cdot (1) \cdot (-1) = 2$.

Let $P$ be a shifted rim hook tableau. We denote by $P^1$ (resp., $P^2$) one of the tableaux obtained from $P$ by circling or not circling the first tail (resp., second tail) of each double rim hook in $P$. The $P^1$ (resp., $P^2$) is called a first (resp., second) tail circled rim hook tableau. Similarly $P^1_2$ is obtained from $P$ by circling or not circling the first tail and second tail of each double rim hook in $P$ and is called a tail circled rim hook tableau. We use the notation $|\cdot|$ to refer to the uncircled version; e.g., $|P^1| = |P^2| = |P^2_1| = P$. See Figure 2.6 and Figure 2.7 for examples of first and second tail circled rim hook tableaux, respectively. Figure 2.8 shows tail circled rim hook tableaux $P^2_1$ when a shifted rim hook tableau $P$ is given.

$$\begin{array}{c|c|c}
\hline
P & P^1 & P^1_2 \\
\hline
1 & 1 & 1 \\
1 & 1 & 1 \\
3 & 3 & 3 \\
\hline
1 & 1 & 1 \\
1 & 1 & 1 \\
3 & 3 & 3 \\
\hline
1 & 1 & 1 \\
1 & 1 & 1 \\
3 & 3 & 3 \\
\hline
1 & 1 & 1 \\
1 & 1 & 1 \\
3 & 3 & 3 \\
\hline
\end{array}$$

Figure 2.8

We now define a new weight function $w'$ for first or second tail circled rim hook tableaux. If $\tau$ is a rim hook of $P^1$ or $P^2$, we define $w'(\tau) = (-1)^{r(\tau)}$. The weights $w'(P^1)$ and $w'(P^2)$ are the product of the weights of rim hooks in $P^1$ and $P^2$, respectively. For a tail circled rim hook tableau $P^1_2$, we define $w''(P^1_2) = 1$.

For each double rim hook $\tau$ of a rim hook tableau $P$, there are two first circled rim hooks $\tau_1, \tau_2$ such that $w(\tau) = w'(\tau_1) + w'(\tau_2)$. This fact implies the following:

**Proposition 2.1.** Let $\gamma \in OP$. Then we have

$$\sum_P w(P) = \sum_{P^1} w'(P^1),$$

where the left-hand sum is over all shifted rim hook tableaux $P$ of shape $\lambda/\mu$ and content $\gamma$, while the right-hand sum is over all shifted first tail circled rim hook tableaux $P^1$ of shape $\lambda/\mu$ and content $\gamma$. 
We can get the similar identity using shifted second tail circled rim hook tableaux

3. Symmetric functions and irreducible spin characters of $\tilde{S}_n$

We consider the ring $\mathbb{Z}[x_1, x_2, \ldots]$ of formal power series with integer coefficients in the infinite variables $x_1, x_2, \ldots$. Note that the symmetric functions form a subring of $\mathbb{Z}[x_1, x_2, \ldots]$. Let $\Lambda(x)$, or simply $\Lambda$, be the ring of symmetric functions of $x_1, x_2, \ldots$. Define $\mathbb{Z}$-modules $\Lambda^k$ by $\Lambda^k(x) = \{ f \in \Lambda \mid f \text{ is homogeneous of degree } k \}$. Then we have $\Lambda = \prod_{k \geq 0} \Lambda^k$.

Let $r$ be a positive integer. The $r$th power sum $p_r$ is defined by $p_r = \sum_{i \geq 1} x_i^r$.

By convention, we set $p_0 = 1$ and $p_r = 0$ for $r < 0$. Extend the definition of this symmetric function to all partitions by $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$.

We now define a group $\tilde{S}_n$ and draw some connections between the irreducible spin characters of $\tilde{S}_n$ and symmetric functions.

For $n > 1$ let $\tilde{S}_n$ be the group generated by $t_1, t_2, \ldots, t_{n-1}, -1$ subject to relations

\begin{align*}
t_i^2 &= -1 \quad \text{for } i = 1, 2, \ldots, n-1, \\
t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2, \\
t_i t_j &= -t_j t_i \quad \text{for } |i - j| > 1 \ (i, j = 1, 2, \ldots, n-1).
\end{align*}

Note that $|\tilde{S}_n| = 2n!$. Since $-1$ is a central involution, Schur’s lemma implies that an irreducible representation of $\tilde{S}_n$ must represent $-1$ by either the scalar 1 or $-1$. The representation of the former type is an ordinary representation of $S_n$, while one of the latter type will correspond to a projective representation of $S_n$, as we will see later. A representation $T$ of $\tilde{S}_n$ is called a spin representation of $\tilde{S}_n$ if the group element $-1$ is represented by scalar $-1$, i.e., if $T(-1) = -1$.

To describe the characters of spin representations of $\tilde{S}_n$ we consider the structure of the conjugacy classes of $\tilde{S}_n$. Let $\theta_n : \tilde{S}_n \to S_n$ be an epimorphism defined by $t_i \mapsto s_i$, where $s_i$ is an adjacent transposition
Combinatorial interpretations of the orthogonality for spin characters of $\tilde{S}_n$. For each partition $\mu = (\mu_1, \ldots, \mu_\ell)$ of $n$, we choose a specific element $\sigma_\mu$ such that $\theta_n(\sigma_\mu)$ is of cycle-type $\mu$ as follows: Define $\sigma_\mu = \pi_1 \pi_2 \ldots \pi_\ell$, where $\pi_j = t_{a+1} t_{a+2} \ldots t_{a+\mu_j-1}$ for $1 \leq j \leq \ell = \ell(\mu)$. For example, if $\mu = (3, 3, 2) \vdash 8$, then $\sigma_\mu = t_1 t_2 t_4 t_5 t_7 \in \tilde{S}_8$ and $\theta_8(\sigma_\mu) = (123)(456)(78) \in S_8$.

Since $\ker(\theta_n) = \{ \pm 1 \}$, every $\sigma \in \tilde{S}_n$ is conjugate to $\sigma_\mu$ or $-\sigma_\mu$ for some partition $\mu$ of $n$.

**Theorem 3.1. (Schur)** Let $\mu$ be a partition of $n$. Then the elements $\sigma_\mu$ and $-\sigma_\mu$ are not conjugate in $\tilde{S}_n$ iff either $\mu \in \text{OP}_n$ or $\mu \in \text{DP}_n$.

**Proof.** See [1] or [7].

Let $\Omega_Q = \prod_{n \geq 0} \Omega^n_Q$ denote the graded subring of $\Lambda_Q = \Lambda \otimes \mathbb{Q}$ generated by $1, p_1, p_3, \ldots$ and let $\Omega = \Lambda \cap \Omega_Q$ denote the $\mathbb{Z}$-coefficient graded subring of $\Omega_Q$. Clearly $\{ p_\lambda \mid \lambda \in \text{OP}_n \}$ forms a basis of $\Omega^n_Q$ and $\dim_Q \Omega^n_Q = |\text{OP}_n|$.

Define an inner product $[\ , \ ]$ on $\Omega^n_Q$ by setting $[p_\lambda, p_\mu] = z_\lambda 2^{-\ell(\lambda)} \delta_{\lambda\mu}$ for $\lambda, \mu \in \text{OP}_n$, where

$$\delta_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 3.2. (Mac)**

1. $\{ Q_\lambda \mid \lambda \in \text{DP}_n \}$ is a basis of $\Omega^n_Q$.
2. $[P_\lambda, Q_\mu] = \delta_{\lambda\mu}$,

where $P_\lambda$ (resp., $Q_\lambda$) is the Hall-Littlewood symmetric $P$-function (resp., $Q$-function) corresponding to a partition $\lambda \in \text{DP}$.

**Proof.** See [2].

We now describe the irreducible spin characters of $\tilde{S}_n$ using the Hall-Littlewood symmetric functions $P_\lambda$ and $Q_\lambda$.
Theorem 3.3. (Schur) Define a class function $\varphi^\lambda$ for each $\lambda \in DP_n^+$ by

$$
\varphi^\lambda(\sigma_\mu) = \begin{cases} 
[2^{-\ell(\lambda)/2}Q_\lambda, 2^{\ell(\mu)/2}p_\mu] & \text{if } \mu \in OP_n, \\
0 & \text{otherwise}
\end{cases}
$$

and define a pair of class functions $\varphi_\pm^\lambda$ for each $\lambda \in DP_n^-$ via

$$
\varphi_\pm^\lambda(\sigma_\mu) = \begin{cases} 
\frac{1}{\sqrt{2}}[2^{-\ell(\lambda)/2}Q_\lambda, 2^{\ell(\mu)/2}p_\mu] & \text{if } \mu \in OP_n, \\
\pm i^{(n-\ell(\lambda)+1)/2} \sqrt{\frac{1}{2}}z_\lambda & \text{if } \mu = \lambda, \\
0 & \text{otherwise},
\end{cases}
$$

where $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ if $\lambda = 1^{m_1} 2^{m_2} \ldots$.

Then the class functions $\varphi^\lambda(\lambda \in DP_n^+)$ and $\varphi_\pm^\lambda(\lambda \in DP_n^-)$ are the irreducible spin characters of $\tilde{S}_n$.

Proof. See [1] or [7].

Although Theorem 3.3 determines the irreducible spin characters $\varphi^\lambda$, it is difficult to use Theorem 3.3 to evaluate $\varphi^\lambda(\sigma_\mu)$ explicitly for $\mu \in OP$.

But Morris has derived a recurrence for the evaluation of these characters which is similar to the well-known Murnaghan-Nakayama formula for ordinary characters of $S_n$.

In 1990 Stembridge [8] gave a combinatorial reformulation for Morris’ recurrence using shifted tableaux, rather than the machinery of Hall-Littlewood functions used by Morris. We now describe Stembridge’s interpretation for Morris’ rule.

Lemma 3.4. (Stembridge) Let $k$ be an odd number and $|\lambda/\mu| = k$. Then

1. $[Q_{\lambda/\mu}, p_k] = 0$ unless $\lambda/\mu$ is a rim hook.
2. $[Q_{\lambda/\mu}, p_k] = (-1)^r$ if $\lambda/\mu$ is a single rim hook of rank $r$.
3. $[Q_{\lambda/\mu}, p_k] = 2(-1)^r$ if $\lambda/\mu$ is a double rim hook of rank $r$.

Proof. See [8].

Theorem 3.5. (Stembridge) For any $\gamma \in OP$, we have

$$
[Q_{\lambda/\mu}, p_\gamma] = \sum_S w(S),
$$

where the sum is over all shifted rim hook tableaux $S$ of shape $\lambda/\mu$ and content $\gamma$. 
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**Proof.** Since the $P_\lambda$’s and $Q_\lambda$’s are dual bases, we have

$$prP_\mu = \sum_{\lambda \in DP} [p_rP_\mu, Q_\lambda]P_\lambda$$ for any odd integer $r$.

By iterating this expansion successively for $r = \gamma_1, \ldots, \gamma_\ell$, we find

$$[p_{\gamma_1}P_\mu, Q_\lambda] = \sum_{\lambda_1} [p_{\gamma_1}P_{\lambda_1}, Q_{\lambda_1}] \cdots [p_{\gamma_\ell}P_{\lambda_{\ell-1}}, Q_{\lambda_{\ell-1}}]$$

where $\mu = \lambda^0, \lambda = \lambda^\ell$. Since $[Q_{\lambda/\mu}, P_\nu] = [Q_\lambda, P_\mu P_\nu]$ and the $P_\nu$’s span $\Omega_Q$, $[Q_{\lambda/\mu}, f] = [Q_\lambda, fP_\mu]$ for any $f \in \Omega_Q$, and therefore

$$[Q_{\lambda/\mu}, p_{\gamma}] = \sum_{\lambda_1} [Q_{\lambda_1/\lambda^0}, p_{\gamma_1}] \cdots [Q_{\lambda_{\ell-1}/\lambda^{\ell-1}}, p_{\gamma_{\ell-1}}].$$

Note that $Q_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$. Thus the only nonzero contributions to $[Q_{\lambda/\mu}, p_{\gamma}]$ in this expansion occur when $\lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^\ell$ and $|\lambda^i| - |\lambda^{i-1}| = \gamma_i (1 \leq i \leq \ell)$. Hence it suffices to evaluate $[Q_{\lambda/\mu}, p_k]$ for all skew shapes $\lambda/\mu$ of weight $k$ ($k$ odd), and the description of $[Q_{\lambda/\mu}, p_k]$ in Lemma 3.4 gives a complete proof of Theorem 3.5.

**Example 3.6.** Consider $\lambda = (6, 3, 2, 1), \gamma = (5, 3, 3, 1)$. There are four shifted rim hook tableaux of shape $\lambda$ and content $\gamma$. See Figure 3.1. Since $w(T_1) = w(T_2) = w(T_3) = -2$ and $w(T_4) = 4$, $[Q_\lambda, p_\gamma] = -2$. Therefore Theorem 3.5 implies that

$$\varphi^\lambda(\sigma_\gamma) = [Q_\lambda, p_\gamma] = -2.$$

![Figure 3.1](image.png)

4. Combinatorial interpretations of the orthogonality relations for spin characters of $\tilde{S}_n$

Recall that there are two kinds of orthogonality relations for characters of a group $G$. See [5] for detail. First we give combinatorial interpretation for the orthogonality relation of the first kind for spin characters of $\tilde{S}_n$. 
Theorem 4.1. (Orthogonality relation of the first kind) Let $G$ be a group of order $g$. If $\chi$ and $\psi$ are irreducible characters of a group $G$. Then

$$\frac{1}{g} \sum_{x \in G} \chi(x) \psi(x^{-1}) = \delta_{\chi\psi}.$$ 

Using Stembridge’s combinatorial interpretation for Morris’ rule given in Theorem 3.5, orthogonality relation of the first kind for $\tilde{S}_n$ in Theorem 4.1 can be reformulated as follows:

Corollary 4.2. (Orthogonality relation of the first kind for $\tilde{S}_n$) Let $\lambda, \mu \in DP_n$. Then

$$\sum_{\text{triples } (P,Q,\sigma)} 2^{\ell(\text{type}(\sigma))} w(P) w(Q) = \delta_{\lambda\mu} 2^{\ell(\lambda)n!},$$

where the sum is over triples $(P, Q, \sigma)$, with $P$ a shifted rim hook tableau of shape $\lambda$, $Q$ a shifted rim hook tableau of shape $\mu$ and $\sigma \in S_n$, which satisfy $\text{type}(\sigma) \in OP_n$, $\text{content}(P) = \text{content}(Q) = \text{content}(\sigma)$.

Given $\lambda \in DP_n$, a permutation tableau of shape $\lambda$ is a filling of the shifted diagram $D'_{\lambda}$ with positive integers $1, 2, \ldots, n$ and a circled permutation tableau is a permutation tableau with main diagonal entry either circled or uncircled. For example,

$$\begin{array}{ccc}
6 & 7 & 1 & 3 \\
5 & 8 & 4 & 2
\end{array}$$

is a circled permutation tableau of shape $(4, 3, 1)$.

Let $\sigma \in S_n$ and write $\sigma$ in cycle form, $\sigma = \sigma_1 \sigma_2 \ldots \sigma_m$, where the cycles $\sigma_i$ are written in increasing order of the largest in the cycle. Recall that content ($\sigma$) is the sequence $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$, where $\rho_i = |\sigma_i| =$length of the cycle $\sigma_i$. If $\sigma \in S_n$, then let $\overline{\sigma}$ be a permutation obtained from $\sigma$ in which each cycle of $\sigma$ is either barred or unbarred. If $\sigma = (42)(8371)$, $\overline{\sigma}$ is one of $(42)(8371)$, $(42)(8371)$, $(42)(8371)$ and $(42)(8371)$.

Now let $\lambda, \mu \in DP_n$ and let

$$\pi_{\lambda} = \text{the set of all circled permutation tableaux of shape } \lambda,$$

$$\Gamma_{\lambda} = \{ (P^1, \overline{\sigma}) \},$$

$$\Psi_{\lambda\mu}^+ = \{ (P^1, Q_2, \overline{\sigma}) \mid w'(P^1) w'(Q_2) = 1 \},$$

$$\Psi_{\lambda\mu}^- = \{ (P^1, Q_2, \overline{\sigma}) \mid w'(P^1) w'(Q_2) = -1 \},$$
Combinatorial interpretations of the orthogonality for spin characters of $\tilde{S}_n$.

where $P$ is a shifted rim hook tableau of shape $\lambda$, $Q$ is a shifted rim hook tableau of shape $\mu$, $\sigma \in S_n$ with type($\sigma$) $\in OP_n$ and content($P$) $=$ content($Q$) $=$ content($\sigma$).

Note that

$$|\pi_\lambda| = 2^{\ell(\lambda)}n!$$

$$|\Gamma_\lambda| = \sum 2^{\ell(\text{type}(\sigma))} |P^1_2| = \sum 2^{\ell(\text{type}(\sigma))} w''(P^1_2),$$

where $P$ is a shifted rim hook tableau of shape $\lambda$. Since the map given by $P^1_2 \rightarrow (P^1_1, P^1_2)$ is clearly a bijection, we get the following theorem from Corollary 4.2.

**Theorem 4.3.** Let $\lambda, \mu \in DP_n$.

(a) If $\lambda \neq \mu$, then there is a bijection between $\Psi^+_{\lambda \mu}$ and $\Psi^-_{\lambda \mu}$.

(b) If $\lambda = \mu$, then there is a bijection between $\Gamma_\lambda$ and $\pi_\lambda$.

Let’s now give combinatorial interpretation for the orthogonality relation of the second kind for spin characters of $\tilde{S}_n$.

**Theorem 4.4.** (Orthogonality relation of the second kind) Let $G$ be a group of order $g$. Let $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(k)}$ be all irreducible characters of $G$. Then

$$\sum_{i=1}^k \chi^{(i)}_\alpha \chi^{(i)}_\beta = \frac{g}{h_\alpha} \delta_{\alpha \beta},$$

where $h_\alpha$ is the number of elements in the conjugacy class $C_\alpha$ of $\alpha$.

Using Schur’s spin character formulas described in Theorem 3.3 and Stembridge’s combinatorial interpretation for spin characters of $\tilde{S}_n$ respectively, orthogonality relation of the second kind for $\tilde{S}_n$ can be described in the following ways.

**Corollary 4.5.** (Orthogonality relation of the second kind for $\tilde{S}_n$)

Let $\mu, \nu \in P_n$. Then

$$\sum_{\lambda \in DP_p^+} \varphi^\lambda(\sigma_\mu) \varphi^\lambda(\sigma_\nu^{-1}) + \sum_{\lambda \in DP_p^-} \varphi^\lambda(\sigma_\mu) \varphi^\lambda(\sigma_\nu^{-1}) = \delta_{\mu \nu} 1^{j_1} 2^{j_2} j_1! j_2! \cdots,$$

where $\mu = 1^{j_1} 2^{j_2} \cdots$.

**Corollary 4.6.** Let $\mu = 1^{j_1} 2^{j_2} \cdots \in OP_n$ and $\nu \in OP_n$. Then

$$\sum_{(\lambda, P, Q)} 2^{n-\ell(\lambda)} w(P) w(Q) = \delta_{\mu \nu} 2^{n-\ell(\mu)} 1^{j_1} 2^{j_2} j_1! j_2! \cdots,$$
where $\lambda \in DP_n$, $P$ is a shifted rim hook tableau of shape $\lambda$ and content $\mu$ and $Q$ is a shifted rim hook tableau of the same shape $\lambda$ and content $\nu$.

We will describe combinatorial objects whose weights represent the both sides of the identity given in Corollary 4.6.

$H = (H_1, H_2, \ldots, H_m)$ is said to be a circled hook permutation of content $\rho = k^m$, and shape $(\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(m)})$ if the following conditions hold:

1. each $H_i$ is a hook tableau of shape $\tau^{(i)}$,
2. $|\tau^{(i)}| = k$ and
3. for each $i$, all cells of $H_i$ except its tail can be circled or uncircled.

Figure 4.1 gives a circled hook permutation of content $5^5$.

$$
\begin{array}{cccc}
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
1 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 \\
\end{array}
$$

**Figure 4.1**

Using shifted first tail circled rim hook tableaux and circled hook permutations defined in the above, Corollary 4.6 implies the following theorem.

**THEOREM 4.7.** There is a bijection between positive pairs $(P, Q)$, where $P$ is a shifted (first tail circled) rim hook tableau of shape $\lambda$ and content $\mu$ and $Q$ is a circled shifted rim hook tableau of the same shape $\lambda$ and content $\nu$, and,

(a) if $\mu \neq \nu$, negative pairs of $(\hat{P}, \hat{Q})$, where $\hat{P}$ is a shifted (first tail circled) rim hook tableau of shape $\hat{\lambda}$ and content $\mu$ and $\hat{Q}$ is a circled shifted rim hook tableau of the same shape $\hat{\lambda}$ and content $\nu$, or,

(b) if $\mu = \nu$, the union of the set of negative pairs of $(\hat{P}, \hat{Q})$, where $\hat{P}$ is a shifted (first tail circled) rim hook tableau of shape $\hat{\lambda}$ and content $\mu$ and $\hat{Q}$ is a circled shifted rim hook tableau of the same shape $\hat{\lambda}$ and content $\nu$, with the set of circled hook permutations of content $\mu$.

It will be very interesting to construct bijections directly described in Theorem 4.3 and Theorem 4.7.
Combinatorial interpretations of the orthogonality for spin characters of $\tilde{S}_n$

References


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