NOTE ON AVERAGE OF CLASS NUMBERS OF CUBIC FUNCTION FIELDS

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Abstract. Let $k = \mathbb{F}_q(T)$ be the rational function field over a finite field $\mathbb{F}_q$, where $q \equiv 1 \mod 3$. In this paper, we determine asymptotic values of average of ideal class numbers of some family of cubic Kummer extensions of $k$.

1. Introduction and statement of result

Let $k = \mathbb{F}_q(T)$ be the rational function field over a finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[T]$. Average values of $L$-functions associated to orders in quadratic extensions of $k$ are obtained by Hoffstein and Rosen [3] when $q$ is odd and by Chen [2] when $q$ is even. Rosen [6] generalized some results of [3] to general Kummer extensions of $k$ of degree $\ell$, where $\ell$ is a prime divisor of $q - 1$, and determined average values of ideal class numbers of Kummer extensions of $k$ of degree 3. Prime [5] obtained an $L$-function average over imaginary quadratic extensions of $k$ with prime discriminants. Recently, Bae, Jung and Kang [1] extended the result of Chen to general Artin-Schreier extensions of $k$ and determined average values of ideal class numbers of Artin-Schreier extensions of $k$ of degree 3. They also extended the result of Prime to general Kummer extensions of $k$ and obtained a similar formulas of average values of prime $L$-functions.
of Artin-Schreier extensions of \( k \) of degree 2. The aim of this paper is to determine asymptotic values of average of ideal class numbers of some family of cubic Kummer extensions of \( k \).

To state our main result, we introduce some notations. Assume that \( q \equiv 1 \) mod 3. A monic irreducible polynomial in \( \mathbb{A} \) will be called a prime polynomial. Any (geometric) cubic Kummer extension \( K \) of \( k \) can be written as \( K = k(\sqrt[3]{D}) \) for some cubic power free polynomial \( D \in \mathbb{A} \). We say that \( K/k \) is ramified imaginary, real or inert imaginary according as the infinite prime \( \infty \) of \( k \) ramifies, splits completely or is inert in \( K \). Let \( \mathcal{O}_K \) be the integral closure of \( \mathbb{A} \) in \( K \). Write \( h(\mathcal{O}_K) \), \( d(\mathcal{O}_K) \) and \( R(\mathcal{O}_K) \) for the ideal class number, the discriminant and the regulator of \( \mathcal{O}_K \), respectively. For \( K = k(\sqrt[3]{D}) \), we have \( d(\mathcal{O}_K) = \text{rad}(D)^2 \) (see (2.1)), where \( \text{rad}(D) \) is the product of distinct prime divisors of \( D \), and \( R(\mathcal{O}_K) = 1 \) if \( K/k \) is imaginary. Let \( \mathcal{A} \) be the family of all ramified imaginary cubic Kummer extensions \( K \) of \( k \) such that \( d(\mathcal{O}_K) \) is a square of a prime polynomial and \( \mathcal{A}_n = \{ K \in \mathcal{A} : \deg d(\mathcal{O}_K) = 2n \} \) for each positive integer \( n \) with \( 3 \nmid n \). Similarly, replacing “ramified imaginary” by “real” or “inert imaginary” and “\( 3 \nmid n \)” by “\( 3 | n \)” , we define \( \mathcal{B}, \mathcal{B}_n \) or \( \mathcal{C}, \mathcal{C}_n \), respectively. We determine asymptotic values of average of ideal class numbers \( h(\mathcal{O}_K) \) (or times regulator \( R(\mathcal{O}_K) \)) as \( K \) varies over \( \mathcal{A}, \mathcal{B}, \) or \( \mathcal{C} \). Our main result is the following theorem.

**Theorem 1.1.** Assume that \( q \equiv 1 \) mod 3. Then we have

1. as \( n \to \infty \) with \( 3 \nmid n \),

\[
\frac{1}{\#\mathcal{A}_n} \sum_{K \in \mathcal{A}_n} h(\mathcal{O}_K) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} q^{n-1} + O \left( 4^n q^{2} \right),
\]

2. as \( n \to \infty \) with \( 3 | n \),

\[
\frac{1}{\#\mathcal{B}_n} \sum_{K \in \mathcal{B}_n} h(\mathcal{O}_K)R(\mathcal{O}_K) = \frac{\zeta_{\mathbb{A}}(2)^3\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)^2} q^{n-2} + O \left( 4^n q^{2} \right),
\]

3. as \( n \to \infty \) with \( 3 | n \),

\[
\frac{1}{\#\mathcal{C}_n} \sum_{K \in \mathcal{C}_n} h(\mathcal{O}_K) = \frac{3\zeta_{\mathbb{A}}(3)^2\zeta_{\mathbb{A}}(4)}{\zeta_{\mathbb{A}}(6)^2} q^{n-2} + O \left( 4^n q^{2} \right),
\]

where \( \zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{-s}} \) is the zeta function of \( \mathbb{A} \).
2. L-functions of Kummer extensions and Class number formula

In this section we recall some basic results on Kummer extensions of \( k \). For more details, we refer to [6, §1]. Let \( \mathbb{A}^+ \) be the subset of \( \mathbb{A} \) consisting of all monic polynomials and \( \mathcal{P}(\mathbb{A}) \) be the set of prime polynomials in \( \mathbb{A} \). Let \( \mathbb{A}_n^+ = \{ N \in \mathbb{A}^+ : \deg N = n \} \) and \( \mathcal{P}_n(\mathbb{A}) = \mathcal{P}(\mathbb{A}) \cap \mathbb{A}_n^+ \) for \( n \geq 1 \). Assume that \( q \) is a power of an odd prime. Let \( \ell \) be a prime divisor of \( q - 1 \). For an \( \ell \)-th power free polynomial \( D \in \mathbb{A} \), let \( \mathcal{O}_k \) be the integral closure of \( \mathbb{A} \) in \( K = k(\sqrt[\ell]{D}) \). Let \( h(\mathcal{O}_k), d(\mathcal{O}_k) \) and \( R(\mathcal{O}_k) \) be the ideal class number, the discriminant and the regulator of \( \mathcal{O}_D \), respectively.

By [6, Theorem 1.2], we have

\[
(2.1) \quad d(\mathcal{O}_k) = \text{rad}(D)^{\ell - 1},
\]

where \( \text{rad}(D) \) is the product of distinct prime divisors of \( D \). The decomposition of the infinite prime \( \infty_k \) of \( k \) in \( K = k(\sqrt[\ell]{D}) \) is determined as follows:

- \( \infty_k \) ramifies in \( K \) if and only if \( \ell \) does not divide \( \deg D \). In this case \( K/k \) is called a ramified imaginary extension.
- \( \infty_k \) splits completely in \( K \) if and only if \( \ell \) divides \( \deg D \) with \( \text{sgn}(D) \in \mathbb{F}_q^{*\ell} \). In this case \( K/k \) is called a real extension.
- \( \infty_k \) is inert in \( K \) if and only if \( \ell \) divides \( \deg P \) with \( \text{sgn}(D) \notin \mathbb{F}_q^{*\ell} \). In this case \( K/k \) is called an inert imaginary extension.

We note that \( R_{\mathcal{O}_k} = 1 \) if \( K/k \) is imaginary. To defined the character \( \chi_D \), we first need to fix an isomorphism \( \omega \) between the group of \( \ell \)-th roots of unity in \( \mathbb{C} \) and the group of \( \ell \)-th roots of unity in \( \mathbb{F}_q \). For a prime polynomial \( P \in \mathcal{P}(\mathbb{A}) \), define \( \chi_D(P) = 0 \) if \( P \mid D \), and if \( P \nmid D \), \( \chi_D(P) \in \mathbb{C}^* \) is defined by

\[
D^{[\frac{P}{P}] - 1} \equiv \omega(\chi_D(P)) \mod P.
\]

Now we extend the definition to all of \( \mathbb{A}^+ \) by multiplicativity. Then the \( L \)-function \( L(s, \chi_D^i) \) associated to \( \chi_D^i \) \( (0 \leq i \leq \ell - 1) \) is defined by

\[
L(s, \chi_D^i) = \sum_{N \in \mathbb{A}^+} \frac{\chi_D^i(N)}{|N|^s}.
\]
It is well known ([6, Lemma 2.1]) that $L(s, \chi_D^i)$ is a polynomial in $q^{-s}$ of degree at most $\deg D - 1$ for $1 \leq i \leq \ell - 1$. Thus, we can write

$$L(s, \chi_D^i) = \sum_{n=0}^{\deg D - 1} \sum_{N \in \mathcal{A}_+^*} \chi_D^i(N)q^{-ns}.$$ 

For $K = k(\sqrt[3]{\gamma})$, we have the following class number formula (see [6, Theorem 1.3]):

$$\prod_{i=1}^{\ell-1} L(1, \chi_D^i) = \begin{cases} 
q^{\ell-1} \frac{h(O_K)}{\sqrt{|d(O_K)|}} & \text{if } K/k \text{ is ramified imaginary,} \\
(q-1)^{\ell-1} \frac{h(O_K)R(O_K)}{\sqrt{|d(O_K)|}} & \text{if } K/k \text{ is real,} \\
q^{\ell-1} \frac{h(O_K)}{\sqrt{|d(O_K)|}^{(q-1)/2}} & \text{if } K/k \text{ is inert imaginary.}
\end{cases}$$

(2.2)

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Assume that $q \equiv 1 \mod 3$. Let $\gamma$ be a generator of $\mathbb{F}_q^*$ and $\gamma_j = \gamma^j$ for $j \geq 0$. Then $\{\gamma_0 = 1, \gamma_1, \gamma_2\}$ forms a complete set of representatives of $\mathbb{F}_q^*/\mathbb{F}_q^{*3}$. By Kummer theory, it is easy to see that any cubic Kummer extension $K$ of $k$ such that the discriminant $d(O_K)$ is a square of a prime polynomial can be written uniquely as $K = k(\sqrt[3]{\gamma_jP})$ with $P \in \mathcal{P}(\mathcal{A})$ and $0 \leq j \leq 2$. For $K = k(\sqrt[3]{\gamma_jP})$, by (2.1), we have $d(O_K) = P^2$ and $K/k$ is ramified imaginary, splits completely or inert imaginary according as $3 \nmid \deg P$, $3 \mid \deg P$ with $j = 0$ or $3 \mid \deg P$ with $j = 1, 2$, respectively. Moreover, by (2.2), we have

$$\prod_{i=1}^{2} L(1, \chi_{\gamma_jP}^i) = \begin{cases} 
q^{1-\deg P} h(O_K) & \text{if } 3 \nmid \deg P, \\
(q-1)^2 q^{-\deg P} h(O_K) R(O_K) & \text{if } 3 \mid \deg P \text{ and } j = 0, \\
q^{-\deg P} (q^{3-1}) h(O_K) & \text{if } 3 \mid \deg P \text{ and } j = 1, 2.
\end{cases}$$

(3.1)

Hence, for a positive integer $n$ with $3 \nmid n$, we have

$$\mathcal{A}_n = \{k(\sqrt[3]{\gamma_jP}) : P \in \mathcal{P}_n(\mathcal{A}), 0 \leq j \leq 2\}$$
and
\begin{equation}
\frac{1}{\#A_n} \sum_{K \in A_n} h(O_K) \sum_{P \in \mathcal{P}_n(\mathbb{A})} \sum_{j=0}^{2} h(O_{k(\sqrt[3]{\gamma_j} P)}).
\end{equation}

For a positive integer $n$ with $3 \mid n$, we have
\begin{equation}
\mathcal{B}_n = \{k(\sqrt[3]{P}) : P \in \mathcal{P}_n(\mathbb{A})\}, \quad \mathcal{C}_n = \{k(\sqrt[3]{\gamma_j P}) : P \in \mathcal{P}_n(\mathbb{A}), j = 1, 2\},
\end{equation}

\begin{equation}
\frac{1}{\#B_n} \sum_{K \in B_n} h\langle O_K \rangle R\langle O_K \rangle = \frac{1}{\#P_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} h\langle O_{k(\sqrt[3]{P})} \rangle R\langle O_{k(\sqrt[3]{P})} \rangle,
\end{equation}

and
\begin{equation}
\frac{1}{\#C_n} \sum_{K \in C_n} h\langle O_K \rangle = \frac{1}{2\#P_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \sum_{j=1}^{2} h\langle O_{k(\gamma_j P)} \rangle.
\end{equation}

We need a lemma which will be used in the proof of Proposition 3.1.

**LEMMA 3.1.** For any positive integer $n$, $D \in \mathbb{A}^+$ not cubic power and $i = 1, 2$, we have
\begin{equation}
\left| \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{i, \gamma_j P}(D) \right| \leq \frac{(\deg D + 1)}{n} q^n.
\end{equation}

In particular, if $\deg D < n$, then
\begin{equation}
\left| \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{i, \gamma_j P}(D) \right| \leq 2q^n.
\end{equation}

**Proof.** (3.5) follows from Theorem 2.1 in [4] and cubic power reciprocity law. For (3.6), by (3.5), we have
\begin{equation}
\left| \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{i, \gamma_j P}(D) \right| \leq \left(1 + \frac{1}{n}\right) q^n \leq 2q^n.
\end{equation}

\[ \square \]

For a positive integer $n$, let
\begin{equation}
\mathcal{Z}_{n, j}(s) = \frac{1}{\#P_n(\mathbb{A})} \sum_{P \in \mathcal{P}_n(\mathbb{A})} L(s, \chi_{\gamma_j P}) L(s, \chi_{\gamma_j^2 P}) \quad (0 \leq j \leq 2).
\end{equation}
Proposition 3.2. We have

\[ Z_{n,j}(1) = \frac{1}{\# P_n(\mathbb{A})} \sum_{P \in P_n(\mathbb{A})} \prod_{i=1}^{2} L(1, \chi_{\gamma_i}^j) = \frac{\zeta_{\mathbb{A}}(2)\zeta_{\mathbb{A}}(3)^2}{\zeta_{\mathbb{A}}(6)} + O\left(4^nq^{-\frac{n}{2}}\right). \]

Proof. Since \( L(s, \chi_{\gamma_i}^j) \) and \( L(s, \chi_{\gamma_i}^j) \) are polynomials in \( q^{-s} \) of degree \( \leq n-1 \), we can write

\[ \prod_{i=1}^{2} L(s, \chi_{\gamma_i}^j) = \sum_{m=0}^{2n-2} \sum_{m_1+m_2=m}^{(M_1,M_2) \in \mathcal{A}_{m_1} \times \mathcal{A}_{m_2}} \chi_{\gamma_j}^j(M_1M_2^2)q^{-ms}. \]

Thus,

\[ Z_{n,j}(s) = \frac{1}{\# P_n(\mathbb{A})} \sum_{m=0}^{2n-2} \sum_{P \in P_n(\mathbb{A})} a_m(\gamma_j^j)q^{-ms} \]

with

\[ a_m(\gamma_j^j) = \sum_{m_1+m_2=m}^{(M_1,M_2) \in \mathcal{A}_{m_1} \times \mathcal{A}_{m_2}} \chi_{\gamma_j}^j(M_1M_2^2). \]

By [6, Lemma 2.3], we have

\[ |a_m(\gamma_j^j)| \leq \binom{2n-2}{m} q^{\frac{m}{2}}, \]

so

\[ \left| \sum_{m=n}^{2n-2} a_m(\gamma_j^j)q^{-ms} \right| \leq 2^{2n-2}(q^{\frac{1}{2}-\sigma})^n(1-q^{\frac{1}{2}-\sigma})^{-1} \]

for \( s \in \mathbb{C} \) with \( \sigma = \text{Re}(s) > 1/2 \). Hence, we have

\[ \left| \frac{1}{\# P_n(\mathbb{A})} \sum_{P \in P_n(\mathbb{A})} \sum_{m=n}^{2n-2} a_m(\gamma_j^j)q^{-ms} \right| \leq 2^{2n-2}(q^{\frac{1}{2}-\sigma})^n(1-q^{\frac{1}{2}-\sigma})^{-1}. \]
Now, we consider

\[ Z'_{\gamma,j}(s) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{P \in \mathcal{P}_n(\mathbb{A})} a_m(\gamma_j P) q^{-ms} \]

\[ = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{m_1 + m_2 = m \ (M_1, M_2) \in \mathbb{A}_{m_1}^* \times \mathbb{A}_{m_2}^*} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) q^{-ms}. \]

Write \( Z'_{\gamma,j}(s) = \alpha_{\gamma,j}(s) + \beta_{\gamma,j}(s) \) with

\[ \alpha_{\gamma,j}(s) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{m_1 + m_2 = m \ (M_1, M_2) \in \mathbb{A}_{m_1}^* \times \mathbb{A}_{m_2}^*} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) q^{-ms} \]

and

\[ \beta_{\gamma,j}(s) = \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{m_1 + m_2 = m \ (M_1, M_2) \in \mathbb{A}_{m_1}^* \times \mathbb{A}_{m_2}^*} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) q^{-ms}. \]

If \( M_1 M_2^2 \) is not cube, by (3.6), we have

\[ |\sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2)| \leq 2q^{\frac{1}{2}}. \] (3.9)

For \( s \in \mathbb{C} \) with \( \sigma = \text{Re}(s) > \frac{1}{2} \), using (3.9) and the fact that \( \#\mathcal{P}_n(\mathbb{A}) > \frac{q^n}{2^{n+1}} \), we have

\[ |\alpha_{\gamma,j}(s)| \leq \frac{1}{\#\mathcal{P}_n(\mathbb{A})} \sum_{m=0}^{n-1} \sum_{m_1 + m_2 = m \ (M_1, M_2) \in \mathbb{A}_{m_1}^* \times \mathbb{A}_{m_2}^*} \sum_{P \in \mathcal{P}_n(\mathbb{A})} \chi_{\gamma_j P}(M_1 M_2^2) q^{-ms} \]

\[ < 4nq^{\frac{1}{2}} \sum_{m=0}^{n-1} \sum_{m_1 + m_2 = m \ (M_1, M_2) \in \mathbb{A}_{m_1}^* \times \mathbb{A}_{m_2}^*} q^{-m\sigma} \]

\[ < 4nq^{\frac{1}{2}} \left( \frac{1 - q^{n(1-\sigma)}}{(1 - q^{1-\sigma})^2} \right) \to 0 \] (3.10)
as \( n \rightarrow \infty \). Now, we consider \( \beta_{n,j}(s) \). Since \( P \nmid M_1M_2 \) and \( M_1M_2^2 \) is cube, we have \( \chi_{\gamma_jP}(M_1M_2^2) = 1 \). Hence, we have

\[
\beta_{n,j}(s) = \sum_{\substack{m_1+m_2=n \\
(M_1,M_2) \in \mathbb{A}_1 \times \mathbb{A}_2 \\
M_1M_2^2 \text{ : cube}}} |M_1|^{-s} |M_2|^{-s}.
\]

Put

\[
L(s) = \sum_{\substack{(M_1,M_2) \in \mathbb{A}_1 \times \mathbb{A}_2 \\
P|M_1M_2M_1M_2^2 \text{ : cube}}} |M_1|^{-s} |M_2|^{-s}.
\]

Then, as in [6, §2], we have

\[
L(s) = \frac{\zeta_A(3s)^2\zeta_A(2s)}{\zeta_A(6s)}
\]

and

\[
|\beta_{n,j}(s) - L(s)| \leq Cn^2q^{-s(1-3\sigma)}
\]

for \( \sigma = \Re(s) > \frac{1}{3} \) and some constant \( C \) which depends on \( s \) but is independent of \( n \). By (3.10) and (3.11), we have that for \( \sigma = \Re(s) > \frac{1}{3} \),

\[
Z_{n,j}(s) = \frac{\zeta_A(3s)^2\zeta_A(2s)}{\zeta_A(6s)} + O\left(Cn^2q^{-s(1-3\sigma)}\right).
\]

Since \( Cn^2q^{-2s} = o(4^nq^{-s}) \), by (3.8) and (3.12), we have

\[
Z_{n,j}(1) = \frac{1}{\#_P(\mathbb{A})} \sum_{P \in \mathbb{P}_n(\mathbb{A})} \prod_{i=1}^2 L(1, \chi_{\gamma_jP}^i) = \frac{\zeta_A(2)\zeta_A(3)^2}{\zeta_A(6)} + O(4^nq^{-s}).
\]

Let \( P \in \mathbb{P}_n(\mathbb{A}) \). By (3.1), we have that if \( 3 \nmid n \),

\[
\prod_{i=1}^2 L(1, \chi_{\gamma_jP}^i) = q^{-n}h(\mathcal{O}_{k(\sqrt{\gamma_jP})}) \quad (0 \leq j \leq 2)
\]

and, if \( 3 \mid n \),

\[
\prod_{i=1}^2 L(1, \chi_{\gamma_jP}^i) = q^{-n}(q-1)^2h(\mathcal{O}_{k(\sqrt{\gamma_jP})})R(\mathcal{O}_{k(\sqrt{\gamma_jP})})
\]
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\[(3.15) \prod_{i=1}^{2} L(1, \chi_{\gamma_j} P) = \frac{q^n(q^3 - 1)}{3(q - 1)} h(\mathcal{O}_{k(\sqrt[3]{\gamma_j} P)}) = \frac{\zeta_A(2)}{3\zeta_A(4)} q^{-n+2} h(\mathcal{O}_{k(\sqrt[3]{\gamma_j} P)}) \quad (j = 1, 2).\]

As \(n \to \infty\) with \(3 \nmid n\), by (3.2), (3.7) and (3.13), we have

\[\frac{1}{\#A_n} \sum_{K \in A_n} h(\mathcal{O}_K) = \frac{\zeta_A(2)\zeta_A(3)^2}{\zeta_A(6)} q^{n-1} + O \left(4^n q^{\frac{n}{2}}\right),\]

and \(n \to \infty\) with \(3 \mid n\), by (3.3), (3.4), (3.7), (3.14) and (3.15), we have

\[\frac{1}{\#B_n} \sum_{K \in B_n} h(\mathcal{O}_K) R(\mathcal{O}_K) = \frac{\zeta_A(2)\zeta_A(3)^2}{\zeta_A(6)} q^{n-2} + O \left(4^n q^{\frac{n}{2}}\right),\]

\[\frac{1}{\#C_n} \sum_{K \in C_n} h(\mathcal{O}_K) = \frac{3\zeta_A(3)^2\zeta_A(4)}{\zeta_A(6)} q^{n-2} + O \left(4^n q^{\frac{n}{2}}\right).\]

References


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