STATISTICAL CONVERGENCE FOR GENERAL BETA OPERATORS

NAOKANT DEO, MEHMET ALI ÖZARSLAN, AND NEHA BHARDWAJ

Abstract. In this paper, we consider general Beta operators, which is a general sequence of integral type operators including Beta function. We study the King type Beta operators which preserves the third test function \( x^2 \). We obtain some approximation properties, which include rate of convergence and statistical convergence. Finally, we show how to reach best estimation by these operators.

1. Introduction

Three classical operators \( L_n \) (Bernstein operators, Szász-Mirakjan operators and Baskakov operators) preserve \( e_i(x) = x^i (i = 0, 1) \), i.e., \( L_n(e_0; x) = e_0(x) \) and \( L_n(e_1; x) = e_1(x) \), \( n \in \mathbb{N} \). For each of these operators, \( L_n(e_2; x) \neq e_2(x) = x^2 \). In the year 2003, J. P. King [10] presented a non-trivial sequence of positive linear operators \( V_n : C[0, 1] \rightarrow C[0, 1] \), given as follows:

\[
V_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} f \left( \frac{k}{n} \right), 0 \leq x \leq 1,
\]
where $r^*_n(x) : [0, 1] \to [0, 1]$, are defined by

$$r^*_n(x) = \begin{cases} 
  x^2, & n=1, \\
  -\frac{1}{2(n-1)} + \sqrt{\frac{n-1}{n-1}} + \frac{1}{4(n-1)^2}, & n=2,3,.. .
\end{cases}$$

This sequence preserves the test functions $e_0, e_2$ and $V_n(f, x) = r^*_n(x)$ holds. Replacing $r^*_n(x)$ by $e_1$, then we obtain classical Bernstein operators.

Beta operators were introduced by Lupaș [11] and further modified and studied by Khan [9], Upreti [15], Divis [5] and others.

The Beta approximation $\beta_n(f)$ to a function $f : [0, 1] \to \mathbb{R}$ is the operator:

$$(1.1) \quad \beta_n(f; x) = \frac{1}{B(nx, n(1-x))} \int_0^1 t^{nx-1} (1-t)^{n(1-x)-1} f(t) dt$$

where $B(u, v)$ is the well-known beta probability density function

$$B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt; \quad u, v > 0,$$

with the support $(0, 1)$ such that $t$ denotes a value of the random variable $T$, where $n \in \mathbb{N}$, $x \in (0, 1)$ and $f$ is any real measurable, Lebesgue integrable function defined on $[0, 1]$. When $x = 0$ or $x = 1$, then $\beta_n(f, x) = f(x)$ for all $n$.

Now the following Lemmas follow from [16], for the operators $\beta_n$ mentioned by (1.1).

**Lemma 1.1** ([16]). Let $e_i(x) = x^i$, $i = 0, 1, 2$. Then, for each $0 < x < 1$ and $n \in \mathbb{N}$, we have

(i) $\beta_n(e_0; x) = 1$,

(ii) $\beta_n(e_1; x) = x$,

(iii) $\beta_n(e_2; x) = \frac{x(1+nx)}{n+1}$.

**Lemma 1.2** ([5]). For each $0 < x < 1$ and $n \in \mathbb{N}$ and $\varphi_x(t) = t - x$, we have $\beta_n(\varphi_x^2; x) = \frac{x(1-x^2)}{n+1}$.

The aim of this article is to construct a general Beta type operators including the King type Beta operators which preserves the third test function $x^2$. We study some approximation properties, which include rate of convergence and statistical convergence. Finally, we show how to reach best estimation by these operators than the original Beta
operators \( \beta_n(f,x) \). Note that rate of convergence and statistical convergence of many other approximation operators are available in literatures (See [1], [2], [4], [6], [7], [8], [12], [13], [14]).

2. King Type Beta operators

Let \( \{\alpha_n(x)\} \) be a sequence of real-valued continuous functions defined on \([0,1]\) with \(0 < \alpha_n(x) < 1\). Now consider a sequence of positive linear operators:

\[
\hat{\beta}_n(f,x) = \frac{1}{B(n\alpha_n(x), n(1-\alpha_n(x)))} \int_0^1 t^{n\alpha_n(x)-1} (1-t)^{n(1-\alpha_n(x))-1} f(t) dt,
\]

where \(x \in [0,1]\), \(f \in [0,1]\) and \(n \in \mathbb{N}\) (set of natural numbers). If \(\alpha_n(x)\) is replaced by \(e_1\), then we obtain original beta operators (1.1). Note that

**Lemma 2.1.** For each \(0 \leq x \leq 1\) and \(n \in \mathbb{N}\) and \(\phi_x(t) := t - x\), we have

\[
\begin{align*}
(i) & \quad \hat{\beta}_n(e_0; x) = 1, \\
(ii) & \quad \hat{\beta}_n(e_1; x) = \alpha_n(x), \\
(iii) & \quad \hat{\beta}_n(e_2; x) = \frac{\alpha_n(x)(1+n\alpha_n(x))}{n+1}, \\
(iv) & \quad \hat{\beta}_n(\phi_x^2; x) = (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1-\alpha_n(x))}{n+1}.
\end{align*}
\]

Now, if we replace \(\alpha_n(x)\) by

\[
\alpha_n^*(x) = \frac{-1 + \sqrt{1 + 4n(n+1)x^2}}{2n}, \quad x \in [0,1] \text{ and } n \in \mathbb{N},
\]

then the operators \(\hat{\beta}_n\) defined in (2.1) reduce to the operators

\[
\beta_n^*(f;x) = \frac{1}{B(n\alpha_n^*(x), n(1-\alpha_n^*(x)))} \int_0^1 t^{n\alpha_n^*(x)-1} (1-t)^{n(1-\alpha_n^*(x))-1} f(t) dt.
\]

These operators are the King type Beta operators. Furthermore, the following Lemma hold:

**Lemma 2.2.** The operators defined by (2.2) verify the following identities

\[
\begin{align*}
(i) & \quad \beta_n^*(e_0; x) = 1,
\end{align*}
\]
\( \beta^*_n(e_1; x) = \frac{-1 + \sqrt{1 + 4n(n+1)x^2}}{2n}, \)

\( \beta^*_n(e_2; x) = x^2. \)

**Lemma 2.3.** For each \( 0 \leq x \leq 1 \) and \( n \in \mathbb{N} \) and \( \varphi_x(t) = t - x \), we have

\( \beta^*_n(\varphi_x; x) = \frac{\sqrt{1 + 4n(n+1)x^2} - (1 + 2nx)}{2n}, \)

\( \beta^*_n(\varphi^2_x; x) = \frac{(1 + 2nx)x - x\sqrt{1 + 4n(n+1)x^2}}{n}. \)

**3. Rate of Convergence**

In this section we study the rate of convergence of the operators \( \hat{\beta}_n(f; x) \) to \( f(x) \) by means of the modulus of continuity and Peetre’s \( K \)-functional. For \( f \in C[a, b] \), the modulus of continuity of \( f \), denoted by \( \omega(f; \delta) \), is defined to be

\[
\omega(f; \delta) = \sup_{|y-x|<\delta, x,y \in [a,b]} |f(y) - f(x)|.
\]

It is known that for any \( \delta > 0 \) and \( x, y \in [a, b] \), we have

\[
|f(y) - f(x)| \leq \omega(f; \delta) \left( \frac{|y - x|}{\delta} + 1 \right).
\]

**Theorem 3.1.** For every \( f \in C[0, 1] \) and \( 0 \leq x \leq 1 \), we have

\[
\left| \hat{\beta}_n(f; x) - f(x) \right| \leq 2 \omega(f; \delta_{n,x})
\]

where \( \delta_{n,x} = \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1}} \) and \( \omega(f; \delta_{n,x}) \) is the modulus of continuity of \( f \).

**Proof.** Let \( f \in C[0, 1] \) and \( x \in [0, 1] \). Since \( \hat{\beta}_n(e_0, x) = e_0(x) \), from Cauchy-Schwarz inequality for linear positive operators, we obtain for every \( \delta > 0 \) and \( n \in \mathbb{N} \), that

\[
\left| \hat{\beta}_n(f; x) - f(x) \right| \leq \left[ \hat{\beta}_n(e_0; x) + \frac{1}{\delta_{n,x}} \left( \hat{\beta}_n(((e_1 - x)^2); x) \right)^{1/2} \right] \omega(f; \delta_{n,x}).
\]

Choosing \( \delta_{n,x} = \sqrt{\hat{\beta}_n(((e_1 - x)^2); x)} = \sqrt{(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1}}, \) we obtain
\[ |\hat{\beta}_n(f; x) - f(x)| \leq 2\omega(f, \delta_{n,x}). \]

For the King type Beta operators we have the following Corollary at once:

**Corollary 3.2.** For every \( f \in C[0, 1] \) and \( 0 \leq x \leq 1 \), we have
\[ |\hat{\beta}_n(f; x) - f(x)| \leq 2\omega(f, \delta_{n,x}) \]
where \( \delta_{n,x} = \sqrt{\frac{1+2nx-x}{n}}\sqrt{\frac{1+4n(n+1)x^2}{n^2}} \).

Now we give the rate of convergence for the operators \( \hat{\beta}_n(f; x) \) by using the Peetre’s \( K \)-functional in the space \( C^2[0, 1] \). We recall some definitions and notations. The classical Peetre’s \( K \)-functional of a function \( f \in C[0, 1] \) is defined by
\[ K(f, \delta) = \inf \left\{ \| f - g \|_{C[0,1]} + \delta \| g'' \|_{C[0,1]} : g \in C^2[0,1] \right\}, \quad \delta > 0 \]
where \( C^2[0,1] = \{ g \in C[0,1] : g', g'' \in C^2[0,1] \} \).

and the norm
\[ \| f \|_{C^2[0,1]} = \| f \|_{C[0,1]} + \| f' \|_{C[0,1]} + \| f'' \|_{C[0,1]} \]

**Theorem 3.3.** For each \( f \in C[0, 1] \)
\[ |\hat{\beta}_n(f; x) - f(x)| \leq \frac{\alpha_n(x)(1 - \alpha_n(x))}{n+1} \]

**Proof.** Applying Taylor expansion to the function \( g \in C^2[0,1] \), we get
\[ \hat{\beta}_n(g, x) - g(x) = g'(x)\hat{\beta}_n((e_1 - x), x) + \frac{1}{2} \hat{\beta}_n(g''(\xi)(e_1 - x)^2, x); \xi \in (t, x). \]

Hence
\[ |\hat{\beta}_n(g; x) - g(x)| \]
\[ \leq \| g' \|_{C[0,1]} |\hat{\beta}_n((e_1 - x), x)| + \| g'' \|_{C[0,1]} |\hat{\beta}_n((e_1 - x)^2, x)| \]
\[ = \| g' \|_{C[0,1]} |\alpha_n(x) - x| + \| g'' \|_{C[0,1]} |(\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n+1}|. \]
For each $f \in C[0,1]$, we can write
\[
\left| \hat{\beta}_n(f, x) - f(x) \right| \\
\leq \left| \hat{\beta}_n(f, x) - \hat{\beta}_n(g, x) \right| + \left| \hat{\beta}_n(g, x) - g(x) \right| + |g - f| \\
\leq 2 \|g - f\|_{C[0,1]} + \left| \hat{\beta}_n(g; x) - g(x) \right| \\
\leq 2 \|g - f\|_{C[0,1]} \\
+ \left( |\alpha_n(x) - x| + \left| (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right| \right) \|g''\|_{C[0,1]} \\
\leq 2 \left( \|g - f\|_{C[0,1]} + |\alpha_n(x) - x| \\
+ \left| (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right| \|g''\|_{C[0,1]} \right)
\]
Taking infimum over $g \in C^2[0,1]$, we get
\[
\left| \hat{\beta}_n(f, x) - f(x) \right| \\
\leq K \left( f; \left( |\alpha_n(x) - x| + \left| (\alpha_n(x) - x)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \right| \right) \right).
\]

For the King type Beta operators we immediately have the following Corollary:

**Corollary 3.4.** For each $f \in C[0,1]$ 
\[
\left| \hat{\beta}_n(f; x) - f(x) \right| \leq K \left( f; \gamma_{n,x} \right),
\]
where $\gamma_{n,x} = \frac{1}{2n} (2x - 1) \left( 2nx - \sqrt{4n^2x^2 + 4nx^2 + 1} + 1 \right)$. 

**4. Statistical convergence**

In this part of the paper, we use concept of statistical convergence and study the Korovkin type approximation theorem for the operators $\hat{\beta}_n$. Before we present the main results, we shall recall some notation on the statistical convergence.
Let $M$ be any subset of $\mathbb{N}$. The density of $M$ is defined by
\[
\delta (M) = \lim_{n} \frac{1}{n} \sum_{j=1}^{n} \chi_{M}(j)
\]
provided the limit exists, where $\chi_{M}$ is the characteristic function of $M$. A sequence $x = (x_{k})$ is said to be statistical convergence to the number $l$,
\[
\delta \{k \in \mathbb{N}: |x_{k} - l| \geq \varepsilon \} = 0
\]
for every $\varepsilon > 0$ or equivalently there exists a subset $K \subseteq \mathbb{N}$ with $\delta (K) = 1$ and $n_{0}(\varepsilon)$ such that $k > n_{0}$ and $k \in K$ imply that $|x_{k} - l| < \varepsilon$. We write
\[
\text{st} - \lim_{n} x_{k} = l
\]
Assume that for each $x \in [0, 1], (\alpha_{n}(x))_{n \in \mathbb{N}}$ is a sequence in $(0, 1)$ satisfying
\[
\text{(4.1)} \quad \text{st} - \lim_{n} \alpha_{n}(x) = x.
\]
Then we have
\[
\text{(4.2)} \quad \text{st} - \lim_{n} |x - \alpha_{n}(x)| = 0,
\]
and
\[
\text{(4.3)} \quad \text{st} - \lim_{n} \left| \frac{\alpha_{n}(x)(1 - \alpha_{n}(x))}{n + 1} \right| = 0.
\]
Such a sequence $(\alpha_{n}(x))_{n \in \mathbb{N}}$ can be constructed as follows. Choose
\[
\alpha_{n}(x) = \begin{cases} 
2 & \text{if } n = m^{2} \ (m \in \mathbb{N}) \\
\alpha_{n}^{*}(x) & \text{otherwise}
\end{cases}
\]
where
\[
\alpha_{n}^{*}(x) = \frac{-1 + \sqrt{1 + 4n(n + 1)x^{2}}}{2n}, \quad x \in [0, 1] \text{ and } n \in \mathbb{N}.
\]
It is clear that (4.1) is satisfied.

**Theorem 4.1.** For each $x \in [0, 1]$ and for every $f \in C[0, 1]$, we have
\[
\text{st} - \lim_{n} \left| \hat{\beta}_{n} (f; x) - f(x) \right| = 0.
\]
Proof. For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. Now define the sets:

\[ U := \{ n : \delta_{n,x}^2 \geq r \}, \]
\[ U_1 := \{ n : |x - \alpha_n(x)| \geq \sqrt{\frac{r - \varepsilon}{2}} \}, \]
\[ U_2 := \{ n : \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1} \geq \frac{r - \varepsilon}{2} \}, \]

where $\delta_{n,x} := \sqrt{\left(\alpha_n(x) - x\right)^2 + \frac{\alpha_n(x)(1 - \alpha_n(x))}{n + 1}}$. Then it follows that $U \subseteq U_1 \cup U_2$, which gives

\[ \sum_{j=1}^{n} \chi_{U}(j) \leq \sum_{j=1}^{n} \chi_{U_1}(j) + \sum_{j=1}^{n} \chi_{U_2}(j) \]

Multiplying both sides of (4.4) by $\frac{1}{n}$ and letting $n \to \infty$, we get using (4.2) and (4.3) that

\[ \lim_{n \to \infty} \sum_{j=1}^{n} \chi_U(j) = 0. \]

This guarantees that $st-\lim_n \delta_{n,x}^2 = 0$ which implies $st-\lim_n \omega(f, \delta_{n,x}) = 0$. Using Theorem 3.1 completes the proof.

Remark 4.2. If we choose the sequence $(\alpha_n(x))_{n \in \mathbb{N}}$ as in (4.1), then our statistical approximation result (Theorem 4.1) works; however its classical version does not work since $\alpha_n(x) \not\to x$ in the usual sense.

5. Best Error Estimation

Let $\psi_x$ be the first central moment function defined by $\psi_x(y) = y - x$. In order to get a better error estimation on a subinterval $I$ of $[0, 1]$, in the approximation by means of the operators $\beta_n$, we are aimed to find a functional sequence $(s_n)$, $s_n : I \to A$, satisfying

\[ \delta_{n,x}^* := \sqrt{\beta_n(\psi_x^2; u_n(x))} \leq \sqrt{\beta_n(\psi_x^2; x)} =: \delta_{n,x} \quad \text{for } x \in I. \]
By Lemmas 1.2 and 2.1 (d), (5.1) takes the form
\[
\frac{n}{n+1} s_n^2(x) + \left(\frac{1}{n+1} - 2x\right) s_n(x) - \left(\frac{n}{n+1} - 2\right)x^2 - \frac{1}{n+1}x \leq 0.
\]

Let
\[
\Delta_n(x) := \left(\frac{1}{n+1} - 2x\right)^2 + 4\frac{n}{n+1} \left\{\left(\frac{n}{n+1} - 2\right)x^2 + \frac{1}{n+1}x\right\}.
\]

Then it is clear that
\[
\Delta_n(x) \geq 0
\]
and
\[
x + \frac{x}{n} - \frac{1}{2n} \in [0, 1]
\]
hold for every \(x \in I = \left[\frac{1}{4}, \frac{3}{4}\right]\) and for every \(n \geq 1\). Therefore, from (5.2), (5.3) and (5.4), we get
\[
\frac{2x - \frac{1}{n+1} - \sqrt{\Delta_n(x)}}{2\frac{n}{n+1}} \leq s_n(x) \leq \frac{2x - \frac{1}{n+1} + \sqrt{\Delta_n(x)}}{2\frac{n}{n+1}}.
\]

Then \(s_n(x)\) takes its minimum when
\[
s_n(x) := x + \frac{x}{n} - \frac{1}{2n}.
\]

Therefore, for all \(x \in \left[\frac{1}{4}, \frac{3}{4}\right]\), we define a new Beta type operator by
\[
\beta_n^s(f; x) = \beta_n(f; s_n(x)) = \frac{1}{B(ns_n(x), n(1-s_n(x)))} \int_0^1 t^{ns_n(x)-1}(1-t)^{n(1-s_n(x))-1} f(t)dt.
\]

Then, for all \(x \in \left[\frac{1}{4}, \frac{3}{4}\right]\) and \(n \geq 1\), we have
\[
\beta_n^s(\psi_2^2; x) = \frac{x(1-x)}{n} - \frac{1}{4n(n+1)} \leq \frac{x(1-x)}{n+1} = \beta_n(\psi_2^2; x)
\]
which shows that the operators \(\beta_n^s(f; x)\) provides the better estimation than the operators \(\beta_n(f; x)\).
References

Naokant Deo  
Department of Applied Mathematics  
Delhi Technological University  
(Formerly Delhi College of Engineering)  
Bawana Road, Delhi 110042, India  
E-mail: dr_naokant_deo@yahoo.com

Mehmet Ali Özarslan  
Department of Mathematics  
Eastern Mediterranean University  
Faculty of Arts and Sciences  
Gazimagusa, Mersin 10, Turkey.  
E-mail: mehmetali.ozarslan@emu.edu.tr

Neha Bhardwaj  
Department of Applied Science  
ABESIT, Vijaynagar  
Ghaziabad 201009, India.  
E-mail: neha_bhr@yahoo.co.in