REDEFINED FUZZY CONGRUENCES ON SEMIGROUPS

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ABSTRACT. We redefine a fuzzy congruence, discuss some properties of the fuzzy congruences, find the fuzzy congruence generated by a fuzzy relation on a semigroup, and give some lattice theoretic properties of the fuzzy congruences on semigroups.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([8]). Subsequently, many researchers ([2], [7], [5], [4]) studied fuzzy relations in various contexts. The original definition of a reflexive fuzzy relation $\mu$ on a set $X$ was $\mu(x,x) = 1$ for all $x \in X$, which seemed to be too strong. Gupta et al. ([3]) suggested a G-reflexive fuzzy relation by generalizing the definition, defined a fuzzy G-equivalence relation, and developed some properties of that relation. Chon ([1]) defined a generalized fuzzy congruence using the G-reflexive fuzzy relation and characterized that congruence. However the generalized fuzzy congruence turned out not to have some crucial properties (see [1]) such that the congruence on a semigroup is not always generated by a fuzzy relation and the collection of all those congruences is not a complete lattice. In this note, we suggest a new reflexive fuzzy relation as $\mu(x,x) \geq \epsilon > 0$ for all $x \in X$ and

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inf_{t \in X} \mu(t, t) \geq \mu(y, z) \text{ for all } y \neq z \in X, \text{ define a fuzzy congruence, and show that the redefined fuzzy congruence has those crucial properties which the generalized fuzzy congruence does not have. Also our work may be considered as a generalization of the studies which Samhan \([6]\) performed based on the original reflexive fuzzy relation.}

In section 2 we redefine a fuzzy congruence and review some basic definitions and properties of fuzzy relations which will be used in the next section. In section 3 we discuss some basic properties of the fuzzy congruences, find the fuzzy congruence generated by a fuzzy relation on a semigroup, show that the collection \(C(S)\) of all fuzzy congruences on a semigroup \(S\) is a complete lattice, and show that if \(S\) is a group, then \(C_k(S) = \{\mu \in C(S) : \mu(c, c) = k \text{ for all } c \in S\}\) is a modular lattice for \(0 < \epsilon \leq k \leq 1\).

\section{Preliminaries}

We redefine a fuzzy congruence and recall some properties of fuzzy relations which will be used in the next section.

\textbf{Definition 2.1.} A function \(B\) from a set \(X\) to the closed unit interval \([0, 1]\) in \(\mathbb{R}\) is called a \textit{fuzzy subset} of \(X\). For every \(x \in X\), \(B(x)\) is called a \textit{membership grade} of \(x\) in \(B\). A fuzzy relation \(\mu\) in a set \(Z\) is a fuzzy subset of \(Z \times Z\).

The original definition of a fuzzy reflexive relation \(\mu\) in a set \(X\) was \(\mu(x,x) = 1\) for all \(x \in X\). Gupta et al. \([3]\) defined a G-reflexive fuzzy relation \(\mu\) in a set \(X\) by \(\mu(x,x) > 0\) for all \(x \in X\) and \(\inf_{t \in X} \mu(t, t) \geq \mu(x, y)\) for all \(x, y \in X\) such that \(x \neq y\). But the fuzzy congruence defined from the G-fuzzy reflexive relation does not have some crucial properties (see \([1]\)). We redefine the fuzzy congruence for a settlement of these problems.

\textbf{Definition 2.2.} Let \(\mu\) be a fuzzy relation in a set \(X\). \(\mu\) is \textit{reflexive} in \(X\) if \(\mu(x, x) \geq \epsilon > 0\) and \(\inf_{t \in X} \mu(t, t) \geq \mu(x, y)\) for all \(x, y \in X\) such that \(x \neq y\). \(\mu\) is \textit{symmetric} in \(X\) if \(\mu(x, y) = \mu(y, x)\) for all \(x, y\) in \(X\). The composition \(\lambda \circ \mu\) of two fuzzy relations \(\lambda, \mu\) in \(X\) is the fuzzy subset of \(X \times X\) defined by
\[
(\lambda \circ \mu)(x, y) = \sup_{z \in X} \min(\lambda(x, z), \mu(z, y)).
\]
A fuzzy relation $\mu$ in $X$ is \textit{transitive} in $X$ if $\mu \circ \mu \subseteq \mu$. A fuzzy relation $\mu$ in $X$ is called a \textit{fuzzy equivalence relation} if $\mu$ is reflexive, symmetric, and transitive.

Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set $X$. Then it is easy to see that the composition $\circ$ is associative, $\mathcal{F}_X$ is a monoid under the operation of composition $\circ$, and a fuzzy equivalence relation is an idempotent element of $\mathcal{F}_X$.

\textbf{Definition 2.3.} Let $\mu$ be a fuzzy relation in a set $X$. $\mu$ is called \textit{fuzzy left (right) compatible} if $\mu(x,y) \leq \mu(zx,zy)$ ($\mu(x,y) \leq \mu(xz,yz)$) for all $x, y, z \in X$. A fuzzy equivalence relation on $X$ is called a \textit{fuzzy left congruence (right congruence)} if it is fuzzy left compatible (right compatible). A fuzzy equivalence relation on $X$ is called a \textit{fuzzy congruence} if it is a fuzzy left and right congruence.

\textbf{Definition 2.4.} Let $\mu$ be a fuzzy relation in a set $X$. $\mu^{-1}$ is defined as a fuzzy relation in $X$ by $\mu^{-1}(x,y) = \mu(y,x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations $\mu$ and $\nu$. The following Proposition 2.5, Proposition 2.6, and Proposition 2.7 are due to Samhan ([6]).

\textbf{Proposition 2.5.} Let $\mu$ be a fuzzy relation on a set $X$. Then $\bigcup_{n=1}^{\infty} \mu^n$ is the smallest transitive fuzzy relation on $X$ containing $\mu$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.

\textit{Proof.} See Proposition 2.3 of [6].

\textbf{Proposition 2.6.} Let $\mu$ be a fuzzy relation on a set $X$. If $\mu$ is symmetric, then so is $\bigcup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.

\textit{Proof.} See Proposition 2.4 of [6].

\textbf{Proposition 2.7.} If $\mu$ is a fuzzy relation on a semigroup $S$ that is fuzzy left and right compatible, then so is $\bigcup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.

\textit{Proof.} See Proposition 3.6 of [6].

\textbf{Proposition 2.8.} Let $\mu$ and each $\nu_i$ be fuzzy relations in a set $X$ for all $i \in I$. Then $\mu \circ (\bigcap_{i \in I} \nu_i) \subseteq \bigcap_{i \in I} (\mu \circ \nu_i)$ and $(\bigcap_{i \in I} \nu_i) \circ \mu \subseteq \bigcap_{i \in I} (\nu_i \circ \mu)$.

\textit{Proof.} Straightforward.
PROPOSITION 2.9. If $\mu$ is a reflexive fuzzy relation on a set $X$, then $\mu^{n+1}(x, y) \geq \mu^n(x, y)$ for all natural numbers $n$ and all $x, y \in X$.

Proof. Straightforward.

3. Redefined fuzzy congruences on semigroups

In this section we develop some basic properties of the fuzzy congruences, find the fuzzy congruence generated by a fuzzy relation on a semigroup, and give some lattice theoretic properties of fuzzy congruences.

PROPOSITION 3.1. Let $\mu$ be a fuzzy relation on a set $S$. If $\mu$ is reflexive, then so is $\bigcup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.

Proof. Clearly $\mu^1 = \mu$ is reflexive. Suppose that $\mu^k$ is reflexive. Then $\mu^{k+1}(x, x) \geq \mu^k(x, x) \geq \epsilon > 0$ for all $x \in S$ by Proposition 2.9. The remaining part of the proof is exactly same as that of Proposition 3.1 in [1].

PROPOSITION 3.2. Let $\mu$ and $\nu$ be fuzzy congruences in a set $X$. Then $\mu \cap \nu$ is a fuzzy congruence.

Proof. It is clear from Proposition 2.8.

It is easy to see that even though $\mu$ and $\nu$ are fuzzy congruences, $\mu \cup \nu$ is not necessarily a fuzzy congruence. We find the fuzzy congruence generated by $\mu \cup \nu$ in the following proposition.

PROPOSITION 3.3. Let $\mu$ and $\nu$ be fuzzy congruences on a semigroup $S$. Then the fuzzy congruence generated by $\mu \cup \nu$ in $S$ is $\bigcup_{n=1}^{\infty} (\mu \cup \nu)^n = (\mu \cup \nu) \cup [(\mu \cup \nu) \circ (\mu \cup \nu)] \cup \ldots$.

Proof. Clearly $(\mu \cup \nu)(x, x) \geq \epsilon > 0$ for all $x \in S$. The remaining part of the proof is exactly same as that of Proposition 3.3 in [1].

We now turn to the characterization of the fuzzy congruence generated by a fuzzy relation on a semigroup.

DEFINITION 3.4. Let $\mu$ be a fuzzy relation on a semigroup $S$ and let $S^1 = S \cup \{e\}$, where $e$ is the identity of $S$. We define the fuzzy relation
\( \mu^* \) on \( S \) as
\[
\mu^*(x, y) = \bigcup_{c, d \in S^1, \ c a d = x, \ c b d = y} \mu(a, b) \quad \text{for all } x, y \in S.
\]

**Proposition 3.5.** Proposition 3.5 Let \( \mu \) and \( \nu \) be two fuzzy relations on a semigroup \( S \). Then

1. \( \mu \subseteq \mu^* \)
2. \( (\mu^*)^{-1} = (\mu^{-1})^* \)
3. If \( \mu \subseteq \nu \), then \( \mu^* \subseteq \nu^* \)
4. \( (\mu \cup \nu)^* = \mu^* \cup \nu^* \)
5. \( \mu = \mu^* \) if and only if \( \mu \) is fuzzy left and right compatible
6. \( (\mu^*)^* = \mu^* \)

**Proof.** See Proposition 3.5 of [6].

The generalized fuzzy congruence in a semigroup is not always generated by a fuzzy relation (see Theorem 3.6 of [1]). We show that the fuzzy congruence on a semigroup, which is newly defined in this note, is always generated by a fuzzy relation.

**Theorem 3.6.** Let \( \mu \) be a fuzzy relation on a semigroup \( S \). Then the fuzzy congruence generated by \( \mu \) is
\[
\begin{cases}
\bigcup_{n=1}^\infty [\mu^* \cup (\mu^*)^{-1} \cup \theta^*]n, & \text{if } \mu(x, y) > 0 \text{ for some } x \neq y \in S \\
\bigcup_{n=1}^\infty (\mu^* \cup \zeta^*)n, & \text{if } \mu(x, y) = 0 \text{ for all } x \neq y \in S 
\end{cases}
\]
where \( \theta(z, z) = \max \left[ \sup_{x \neq y \in S} \mu(x, y), \epsilon \right] \) for all \( z \in S \), \( \theta = \theta^{-1} \), \( \theta(x, y) \leq \mu(x, y) \) for all \( x, y \in S \) with \( x \neq y \), \( \zeta(z, z) = \epsilon \) for all \( z \in S \), \( \zeta(x, y) = 0 \) for all \( x \neq y \in S \), and \( \mu^*, \theta^*, \text{ and } \zeta^* \) are fuzzy relation on \( S \) defined in Definition 3.4.

**Proof.** We consider the case that \( \mu(x, y) > 0 \) for some \( x \neq y \in S \). Let \( \mu_1 = \mu^* \cup (\mu^*)^{-1} \cup \theta^* \). Then \( \mu_1(z, z) \geq \theta^*(z, z) \geq \theta(z, z) \geq \epsilon > 0 \) for all \( z \in S \). Let \( S^1 = S \cup \{e\} \), where \( e \) is the identity of \( S \). Since \( x \neq y \) implies \( a \neq b \) in Definition 3.4, \( \mu^*(x, y) \leq \sup_{x \neq y \in S} \mu(x, y) \leq \theta(t, t) \) for all \( t \in S \). Since \( \theta(x, y) \leq \mu(x, y) \), \( \theta^*(x, y) \leq \mu^*(x, y) \) by (3) of Proposition 3.5. That is,
\[
\inf_{t \in S} \mu_1(t, t) \geq \inf_{t \in S} \theta^*(t, t) \geq \theta(t, t) \geq \mu^*(x, y) \geq \theta^*(x, y).
\]
Since \( \inf_{t \in S} \mu_1(t, t) \geq \theta(t, t) \geq \mu^*(y, x) \), \( \inf_{t \in S} \mu_1(t, t) \geq (\mu^*)^{-1}(x, y) \). Thus

\[
\inf_{t \in S} \mu_1(t, t) \geq \max[\mu^*(x, y), (\mu^*)^{-1}(x, y), \theta^*(x, y)] = \mu_1(x, y).
\]

That is, \( \mu_1 \) is reflexive. By Proposition 3.1, \( \bigcup_{n=1}^{\infty} \mu_1^n \) is reflexive. Since \( \theta = \theta^{-1}, \theta^* = (\theta^{-1})^* = (\theta^*)^{-1} \) by (2) of Proposition 3.5, and hence

\[
\mu_1(x, y) = \max [(\mu^*)^{-1}(y, x), \mu^*(y, x), (\theta^*)^{-1}(x, y)] = \mu_1(y, x).
\]

Thus \( \mu_1 \) is symmetric. By Proposition 2.6, \( \bigcup_{n=1}^{\infty} \mu_1^n \) is symmetric. By Proposition 2.5, \( \bigcup_{n=1}^{\infty} \mu_1^n \) is transitive. Hence \( \bigcup_{n=1}^{\infty} \mu_1^n \) is a fuzzy equivalence relation containing \( \mu \). By (2), (4), and (6) of Proposition 3.5,

\[
\mu_1 = (\mu^* \cup (\mu^*)^{-1} \cup \theta^*)^* = (\mu^* \cup (\mu^{-1})^* \cup \theta^*)^* = (\mu^* \cup ((\mu^{-1})^* \cup (\theta^*)^* = \mu^* \cup (\mu^{-1})^* \cup \theta^* = \mu^* \cup (\mu^*)^{-1} \cup \theta^* = \mu_1.
\]

Thus \( \mu_1 \) is fuzzy left and right compatible by (5) of Proposition 3.5. By Proposition 2.7, \( \bigcup_{n=1}^{\infty} \mu_1^n \) is fuzzy left and right compatible. Thus \( \bigcup_{n=1}^{\infty} \mu_1^n \) is a fuzzy congruence containing \( \mu \). Let \( \nu \) be a fuzzy congruence containing \( \mu \). Then \( (\mu \cup \mu^{-1} \cup \theta)(x, y) \leq \nu(x, y) \) for all \( x, y \in S \) such that \( x \neq y \). Since \( \theta(a, a) = \max \{ \sup \mu(x, y), \epsilon \} \leq \nu(a, a) \) for all \( a \in S \), max \( \{ \mu(a, a), \mu^{-1}(a, a), \theta(a, a) \} \leq \nu(a, a) \) for all \( a \in S \). Thus \( \mu \cup \mu^{-1} \cup \theta \subseteq \nu \). By (2), (3), and (4) of Proposition 3.5,

\[
\mu_1 = \mu^* \cup (\mu^*)^{-1} \cup \theta^* = \mu^* \cup (\mu^{-1})^* \cup \theta^* = (\mu \cup \mu^{-1} \cup \theta)^* \subseteq \nu^*.
\]

Since \( \nu \) is fuzzy left and right compatible, \( \nu = \nu^* \) by (5) of Proposition 3.5. Thus \( \mu_1 \subseteq \nu \). Suppose \( \mu_1^k \subseteq \nu \). Then

\[
\mu_1^{k+1}(b, c) = (\mu_1 \circ \mu_1)(b, c) = \sup_{d \in S} \min[\mu_1^k(b, d), \mu_1(d, c)] = \sup_{d \in S} \min[\nu(b, d), \nu(d, c)] = (\nu \circ \nu)(b, c)
\]

for all \( b, c \in S \). That is, \( \mu_1^{k+1} \subseteq (\nu \circ \nu) \). Since \( \nu \) is transitive, \( \mu_1^{k+1} \subseteq \nu \). By the mathematical induction, \( \mu_1^n \subseteq \nu \) for every natural number \( n \). Thus

\[
\bigcup_{n=1}^{\infty} [\mu^* \cup (\mu^*)^{-1} \cup \theta]^n = \bigcup_{n=1}^{\infty} \mu_1^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1) \cup \cdots \subseteq \nu.
\]

We consider the case that \( \mu(x, y) = 0 \) for all \( x \neq y \in S \). Let \( \mu_2 = \mu^* \cup \zeta^* \). Then \( \mu_2(a, a) \geq \epsilon > 0 \) for all \( a \in S \). Let \( S^1 = S \cup \{ e \} \), where \( e \) is the identity of \( S \). Since \( x \neq y \) implies \( a \neq b \) in Definition 3.4,
\(\mu^*(x, y) = 0\) and \(\zeta^*(x, y) = 0\) from \(\mu(x, y) = 0\) and \(\zeta(x, y) = 0\). That is, 
\((\mu^* \cup \zeta^*)(x, y) < \zeta(t, t)\) for all \(t \in S\). Thus 
\[\inf_{t \in S} \mu_2(t, t) \geq \inf_{t \in S} \zeta^*(t, t) \geq \zeta(t, t) > \max[\mu^*(x, y), \zeta^*(x, y)] = \mu_2(x, y).\]

Hence \(\mu_2\) is reflexive. By Proposition 3.1, \(\cup_{n=1}^\infty \mu_2^n\) is reflexive. Since \(\mu^*(x, y) = 0\) and \(\zeta^*(x, y) = 0\), \(\mu_2\) is symmetric. By Proposition 2.6, \(\cup_{n=1}^\infty \mu_2^n\) is symmetric. By Proposition 2.5, \(\cup_{n=1}^\infty \mu_2^n\) is transitive. Hence \(\cup_{n=1}^\infty \mu_2^n\) is a fuzzy equivalence relation containing \(\mu\). The proof of the remaining parts is similar to that of the above case.

We now turn to the lattice theoretic properties of fuzzy congruences. For the collection \(\{\mu_j : j \in J\}\) of all generalized fuzzy congruences on a semigroup \(S\) with a relation \(\leq\) defined in Proposition 3.7, it is easy to see that \(\{\mu_j : j \in J\}, \leq\) is not a complete lattice since \(\inf_{j \in J} \mu_j\) does not exist (see [1]). In next proposition, we show that the collection of the redefined fuzzy congruences is a complete lattice.

**Proposition 3.7.** Let \(C(S)\) be the collection of all fuzzy congruences on a semigroup \(S\). Then \((C(S), \leq)\) is a complete lattice, where \(\leq\) is a relation on the set of all fuzzy congruences on \(S\) defined by \(\mu \leq \nu\) iff \(\mu(x, y) \leq \nu(x, y)\) for all \(x, y \in S\).

**Proof.** Clearly \(\leq\) is a partial order relation. It is easy to check that the relation \(\sigma\) defined by \(\sigma(x, y) = 1\) for all \(x, y \in S\) is in \(C(S)\) and the relation \(\lambda\) defined by \(\lambda(x, y) = \epsilon\) for \(x = y\) and \(\lambda(x, y) = 0\) for \(x \neq y\) is in \(C(S)\). Also \(\sigma\) is the greatest element and \(\lambda\) is the least element of \(C(S)\) with respect to the ordering \(\leq\). Let \(\{\mu_j\}_{j \in J}\) be a non-empty collection of fuzzy congruences in \(C(S)\). Let \(\mu(x, y) = \inf_{j \in J} \mu_j(x, y)\) for all \(x, y \in S\). Clearly \(\mu(x, x) \geq \epsilon\) for all \(x \in S\), \(\inf_{t \in X} \mu(t, t) \geq \mu(y, z)\) for all \(y \neq z \in X\), \(\mu = \mu^{-1}\), \(\mu(x, y) \leq \mu(zx, zy)\), and \(\mu(x, y) \leq \mu(xz, yz)\) for all \(x, y, z \in S\). It is easy to see that \(\mu \circ \mu \leq \mu\) (see Proposition 6.1 of [4]). That is, \(\mu \in C(S)\). Since \(\mu\) is the greatest lower bound of \(\{\mu_j\}_{j \in J}\), \((C(S), \leq)\) is a complete lattice.

We define a join \(\vee\) and a meet \(\wedge\) on \(C(S)\) by \(\mu \vee \nu = < \mu \cup \nu >_\epsilon\) and \(\mu \wedge \nu = \mu \cap \nu\), where \(< \mu \cup \nu >_\epsilon\) is the fuzzy congruence generated by \(\mu \cup \nu\). It is clear that if \(\mu, \nu \in C(S)\), then \(\mu \wedge \nu \in C(S)\) and \(\mu \vee \nu \in C(S)\) from Proposition 3.2 and Proposition 3.3, respectively. Let
\[ C_k(S) = \{ \mu \in C(S) : \mu(c,c) = k \text{ for all } c \in S \}. \] Then it is easy to see that \((C_k(S), \lor, \land)\) is a sublattice of \(C(S)\) for \(0 < \epsilon \leq k \leq 1\).

**Definition 3.8.** A lattice \((L, \lor, \land)\) is called modular if \((x \lor y) \land z \leq x \lor (y \land z)\) for all \(x, y, z \in L\) with \(x \leq z\).

**Lemma 3.9.** Let \(\mu\) and \(\nu\) be fuzzy congruences on a semigroup \(S\) such that \(\mu(c,c) = \nu(c,c)\) for all \(c \in S\). If \(\mu \circ \nu = \nu \circ \mu\), then \(\mu \circ \nu\) is the fuzzy congruence on \(S\) generated by \(\mu \cup \nu\).

**Proof.** \((\mu \circ \nu)(a,a) = \sup \min [\mu(a,z), \nu(z,a)] \geq \min [\mu(a,a), \nu(a,a)] \geq \epsilon > 0\) for all \(a \in S\). The remaining part of the proof is same as that of Lemma 4.3 in [1]. \(\square\)

**Theorem 3.10.** Let \(S\) be a semigroup and let \(H\) be a sublattice of \((C_k(S), \lor, \land)\) such that \(\mu \circ \nu = \nu \circ \mu\) for all \(\mu, \nu \in H\). Then \(H\) is a modular lattice for \(0 < \epsilon \leq k \leq 1\).

**Proof.** Let \(\mu, \nu, \rho \in H\) with \(\mu \leq \rho\). Let \(x, y \in S\). Then it is straightforward (see Theorem 4.5 of [6]) that \((\mu \circ \nu) \land \rho \leq \mu \circ (\nu \land \rho)\). Since \(\mu, \nu \in C_k(S)\), \(\mu(c,c) = \nu(c,c) = k\) for all \(c \in S\). By Lemma 3.9, \(\mu \circ \nu\) is the fuzzy congruence generated by \(\mu \cup \nu\). That is, \(\nu \land \nu = \mu \circ \nu\). Thus \((\mu \lor \nu) \land \rho \leq \mu \circ (\nu \land \rho)\). Since \(H\) is a sublattice and \(\rho, \nu \in H\), \(\nu \land \rho \in H\). Since \(\mu \in H\) and \(\nu \land \rho \in H\), \((\nu \land \rho) \circ \mu\). Also \((\nu \land \rho)(c,c) = k\) and \(\mu(c,c) = k\) for all \(c \in S\). By Lemma 3.9, \(\mu \circ (\nu \land \rho)\) is the fuzzy congruence generated by \(\mu \cup (\nu \land \rho)\). That is, \(\mu \circ (\nu \land \rho) = \mu \lor (\nu \land \rho)\). Thus \((\mu \lor \nu) \land \rho \leq \mu \lor (\nu \land \rho)\). Hence \(H\) is modular. \(\square\)

**Corollary 3.11.** If \(S\) is a group and \(0 < \epsilon \leq k \leq 1\), then \((C_k(S), \lor, \land)\) is a modular lattice.

**Proof.** It is easy to see that if \(S\) is a group, then \(\mu \circ \nu = \nu \circ \mu\) for all \(\mu, \nu \in C_k(S)\) (see Proposition 4.3 of [6]). By Theorem 3.10, \((C_k(S), \lor, \land)\) is modular. \(\square\)

**References**


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