ABSOLUTE CONTINUITY OF THE REPRESENTING MEASURES OF THE HYPERGEOMETRIC TRANSLATION OPERATORS ATTACHED TO THE ROOT SYSTEM OF TYPE $B_2$ AND $C_2$

KHALIFA TRIMÈCHE

Abstract. We prove in this paper the absolute continuity of the representing measures of the hypergeometric translation operators $T_x$ and $T^W_x$ associated respectively to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type $B_2$ and $C_2$ which are studied in [9].

1. Introduction

We consider the differential-difference operators $T_j$, $j = 1, 2, \ldots, d$ associated with a root system $\mathcal{R}$, a Weyl group $W$ and a multiplicity function $k$, introduced by I. Cherednik in [2], and called the Cherednik operators in the literature. These operators are helpful for the extension and simplification of the theory of Heckman-Opdam, which is a generalization of the harmonic analysis on the symmetric spaces $G/K$ (see [3, 4, 5, 7]).
The notion of hypergeometric translation operators introduced in [8] is basic in the harmonic analysis associated to the Cherednik operators and the Heckman-Opdam theory. We have considered in [9] the hypergeometric translation operators $T_x$, and $T^W_x$, $x \in \mathbb{R}^2$, associated respectively to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type $B_2$ and $C_2$ we have proved that these operators are integral transforms, more precisely, for all function $f$ in $\mathcal{E}(\mathbb{R}^2)$ (the space of $C^\infty$-functions on $\mathbb{R}^2$) we have

$$\forall \ t \in \mathbb{R}^2, T_x(f)(t) = \int_{\mathbb{R}^2} f(z) dm_{x,t}(z),$$

(1.1)

where $m_{x,t}$ is a positive measure with compact support contained in the set $\{z \in \mathbb{R}^2; ||x|| - ||t|| \leq ||z|| \leq ||x|| + ||t||\}$, and of norm equal to 1. From this result we have deduced that for all function $f$ in $\mathcal{E}(\mathbb{R}^2)^W$ (the subspace of $\mathcal{E}(\mathbb{R}^2)$ of $W$-invariant functions), we have

$$\forall \ t \in \mathbb{R}^2, T^W_x(f)(t) = \int_{\mathbb{R}^2} f(z) dm^W_{x,t}(z),$$

(1.2)

where

$$m^W_{x,t} = \frac{1}{|W|^2} \sum_{w,w'}^W m_{wx,w't}.$$  

(1.3)

In this paper we prove that for all $x,t \in \mathbb{R}^2_{\text{reg}}$ (the regular part of $\mathbb{R}^2$) the measures $m_{x,t}$ and $m^W_{x,t}$ are absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. More precisely there exist positive functions $W(x,t,.)$ and $W^W(x,t,.)$ such that

$$dm_{x,t}(z) = W(x,t,z)A_k(z)dz,$$

(1.4)

and

$$dm^W_{x,t}(z) = W^W(x,t,z)A_k(z)dz,$$

(1.5)

where $A_k$ is a weight function on $\mathbb{R}^2$ which will be given in the following section (see (2.8)). The functions $z \rightarrow W(x,t,z)$ and $z \rightarrow W^W(x,t,z)$ have their support contained in the set $\{z \in \mathbb{R}^2; ||x|| - ||t|| \leq ||z|| \leq ||x|| + ||t||\}$ and satisfy

$$\int_{\mathbb{R}^2} W(x,t,z)A_k(z)dz = 1,$$

(1.6)

and

$$\int_{\mathbb{R}^2} W^W(x,t,z)A_k(z)dz = 1.$$  

(1.7)
As applications of the previous results, we prove that for all \( \lambda \in \mathbb{C}^2 \), the Opdam-Cherednik kernel \( G_\lambda \) and the Heckmann-Opdam hypergeometric function \( F_\lambda \) possess the following product formulas

\[
\forall x, t \in \mathbb{R}^2_{\text{reg}}, G_\lambda(x)G_\lambda(t) = \int_{\mathbb{R}^2} G_\lambda(z)W(x, t, z)\mathcal{A}_k(z)dz, \quad (1.8)
\]

and

\[
\forall x, t \in \mathbb{R}^2_{\text{reg}}, F_\lambda(x)F_\lambda(t) = \int_{\mathbb{R}^2} F_\lambda(z)W^W(x, t, z)\mathcal{A}_k(z)dz. \quad (1.9)
\]

2. The Cherednik operators and their eigenfunctions

We consider \( \mathbb{R}^2 \) with the standard basis \( \{e_1, e_2\} \) and inner product \( \langle.,. \rangle \) for which this basis is orthonormal. We extend this inner product to a complex bilinear form on \( \mathbb{C}^2 \).

2.1. The root systems of type \( B_2 \) and \( C_2 \) and the multiplicity functions.

The root system of type \( B_2 \) can be identified with the set \( \mathcal{R} \) given by

\[
\mathcal{R} = \{\pm e_1, \pm e_2\} \cup \{\pm e_1 \pm e_2\}, \quad (2.1)
\]

which can also be written in the form

\[
\mathcal{R} = \{\pm \alpha_1, \pm \alpha_2, \pm \alpha_3 \pm \alpha_4\},
\]

with

\[
\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = (e_1 - e_2), \alpha_4 = (e_1 + e_2). \quad (2.2)
\]

We denote by \( \mathcal{R}_+ \) the set of positive roots

\[
\mathcal{R}_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},
\]

and by \( \mathcal{R}_+^0 \) the set of positive indivisible roots i.e, the roots \( \alpha \in \mathcal{R}_+ \) such that \( \frac{\alpha}{2} \notin \mathcal{R}_+ \). Then we have

\[
\mathcal{R}_+^0 = \mathcal{R}_+. \quad (2.4)
\]

For \( \alpha \in \mathcal{R} \), we consider

\[
r_\alpha(x) = x - \langle \bar{\alpha}, x \rangle \alpha, \text{ with } \bar{\alpha} = \frac{2\alpha}{\|\alpha\|^2}, \quad (2.5)
\]

the reflection in the hyperplan \( H_\alpha \subset \mathbb{R}^2 \) orthogonal to \( \alpha \). The reflections \( r_\alpha, \alpha \in \mathcal{R} \), generate a finite group \( W \subset O(2) \), called the Weyl group associated with \( \mathcal{R} \). In this case \( W \) is isomorphic to the hyperoctahedral
group which is generated by permutations and sign changes of the \( e_i, i = 1, 2, \ldots \).

The multiplicity function \( k : \mathcal{R} \to ]0, +\infty[ \) can be written in the form \( k = (k_1, k_2) \) where \( k_1 \) is the value on the roots \( \alpha_1, \alpha_2 \), and \( k_2 \) is the value on the roots \( \alpha_3, \alpha_4 \).

The positive Weyl chamber denoted by \( a^+ \) is given by
\[
a^+ = \{ x \in \mathbb{R}^2 ; \forall \alpha \in \mathcal{R}_+, \langle \alpha, x \rangle > 0 \},
\]
(2.6)
it can also be written in the form
\[
a^+ = \{ (x_1, x_2) \in \mathbb{R}^2 ; x_1 > x_2 > 0 \}.
\]
(2.7)

Let also \( \mathbb{R}^2_{\text{reg}} \) be the subset of regular elements in \( \mathbb{R}^2 \), i.e., those elements which belong to no hyperplane \( H_\alpha = \{ x \in \mathbb{R}^2 ; \langle \alpha, x \rangle = 0 \}, \alpha \in \mathcal{R} \).

Let \( \mathcal{A}_k \) denote the weight function
\[
\forall x \in \mathbb{R}^2, \mathcal{A}_k(x) = \prod_{\alpha \in \mathcal{R}_+} |\sinh(\frac{\alpha}{2}, x)|^{2k(\alpha)}.
\]
(2.8)

**Remark 2.1.** The root system of type \( C_2 \) can be identified with the set \( \mathcal{R} \) given by
\[
\mathcal{R} = \{ \pm 2e_1, \pm 2e_2 \} \cup \{ \pm e_1 \pm e_2 \},
\]
which can also be written in the form
\[
\mathcal{R} = \{ \pm \alpha_1, \pm \alpha_2, \pm \alpha_3, \pm \alpha_4 \},
\]
with
\[
\alpha_1 = 2e_1, \alpha_2 = 2e_2, \alpha_3 = (e_1 - e_2), \alpha_4 = (e_1 + e_2).
\]

The set of positive roots is the following
\[
\mathcal{R}_+ = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}.
\]
If we denote by \( W(C_2) \) the Weyl group associated to the root system \( \mathcal{R} \) of type \( C_2 \), then we have
\[
W(C_2) = W(B_2).
\]
We denote also by \( k = (k_1, k_2) \) the multiplicity function of the root system \( \mathcal{R} \) of \( C_2 \), where \( k_1 \) is the value on the roots \( \alpha_1, \alpha_2 \), and \( k_2 \) is the value on the roots \( \alpha_3, \alpha_4 \).

In the remainder of the paper we shall give the results and their proofs only for the root system of type \( B_2 \). It is easy to obtain the analogous of these results in the case of the root system of type \( C_2 \).
2.2. The Cherednik operators attached to the root system of type $B_2$.

The Cherednik operators $T_j, j = 1, 2$, on $\mathbb{R}^2$ associated with the Weyl group $W$ and the multiplicity function $k$ are defined for $f$ of class $C^1$ on $\mathbb{R}^2$ and $x \in \mathbb{R}_{reg} = \mathbb{R}^2 \setminus \bigcup_{\alpha \in \mathcal{R}} H_\alpha$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \rho_j f(x), \quad (2.9)$$

with

$$\rho_j = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j, \quad \text{and} \quad \alpha^j = \langle \alpha, e_j \rangle. \quad (2.10)$$

These operators can also be written in the following form

$$T_1 f(x) = \frac{\partial}{\partial x_1} f(x) + k_1 \left\{ f(x) - f(r_{\alpha_1} x) \right\} + k_2 \left[ \frac{f(x) - f(r_{\alpha_3} x)}{1 - e^{-\langle \alpha_2, x \rangle}} \right] + \frac{f(x) - f(r_{\alpha_4} x)}{1 - e^{-\langle \alpha_4, x \rangle}} - \frac{1}{2} (k_1 + k_2) f(x). \quad (2.11)$$

$$T_2 f(x) = \frac{\partial}{\partial x_2} f(x) + k_1 \left\{ f(x) - f(r_{\alpha_2} x) \right\} + k_2 \left[ - \frac{f(x) - f(r_{\alpha_3} x)}{1 - e^{-\langle \alpha_3, x \rangle}} + \frac{f(x) - f(r_{\alpha_4} x)}{1 - e^{-\langle \alpha_4, x \rangle}} \right] - \frac{1}{2} k_1 f(x). \quad (2.12)$$

2.3. The eigenfunctions of the Cherednik operators attached to the root system of type $B_2$.

We denote by $G_\lambda, \lambda \in \mathbb{C}^2$, the eigenfunction of the operators $T_j, j = 1, 2$. It is the unique analytic function on $\mathbb{R}^2$ which satisfies the differential difference system

$$\begin{cases} T_j G_\lambda(x) = -i\lambda_j G_\lambda(x), & x \in \mathbb{R}^2, j = 1, 2, \\ G_\lambda(0) = 1 \end{cases} \quad (2.13)$$

It is called the Opdam-Cherednik kernel.

We consider the function $F_\lambda, \lambda \in \mathbb{C}^2$, defined by

$$\forall x \in \mathbb{R}^2, F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx). \quad (2.14)$$
This function is the unique analytic $W$-invariant function on $\mathbb{R}^2$, which satisfies the partial differential equation

\[
\begin{cases}
p(T)F_\lambda(x) = p(-i\lambda)F_\lambda(x), & x \in \mathbb{R}^2, \\
F_\lambda(0) = 1,
\end{cases}
\tag{2.15}
\]

for all $W$-invariant polynomials $p$ on $\mathbb{R}^2$ and $p(T) = p(T_1, T_2)$. It is called the Heckman-Opdam hypergeometric function.

The functions $G_\lambda$ and $F_\lambda$ possess the following properties

i) For all $x \in \mathbb{R}^2$ the function $\lambda \to G_\lambda(x)$ is entire on $\mathbb{C}^2$.

ii) We have

\[
\forall x \in \mathbb{R}^2, \forall \lambda \in \mathbb{C}^2, \quad G_\lambda(x) = G_{-\lambda}(x).
\tag{2.16}
\]

iii) We have

\[
\forall x \in \mathbb{R}^2, \forall \lambda \in \mathbb{C}^2, \quad |G_\lambda(x)| \leq G_{i\text{Im}(\lambda)}(x).
\tag{2.17}
\]

iv) We have

\[
\forall x \in \mathbb{R}^2, \forall \lambda \in \mathbb{R}^2, \quad |G_\lambda(x)| \leq 1.
\tag{2.18}
\]

\[
\forall x \in \mathbb{R}^2, \forall \lambda \in \mathbb{R}^2, \quad |F_\lambda(x)| \leq 1.
\tag{2.19}
\]

v) The function $G_\lambda, \lambda \in \mathbb{C}^2$, admits the following Laplace type representation

\[
\forall x \in \mathbb{R}^2, G_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle \lambda, y \rangle} d\mu_x(y), \tag{2.20}
\]

where $\mu_x$ is a positive measure on $\mathbb{R}^2$ with support in $\Gamma = \text{conv}\{wx, w \in W\}$ (the convex hull of the orbit of $x$ under $W$).

vi) From (2.14), (2.20) we deduce that the function $F_\lambda, \lambda \in \mathbb{C}^2$, possesses the Laplace type representation

\[
\forall x \in \mathbb{R}^2, F_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle \lambda, y \rangle} d\mu^W_x(y), \tag{2.21}
\]

where $\mu^W_x$ is the positive measure with support in $\Gamma$ given by

\[
\mu^W_x = \frac{1}{|W|} \sum_{w \in W} \mu_{wx}. \tag{2.22}
\]
3. The hypergeometric translation operator $\mathcal{T}_x$

We consider the hypergeometric translation operator $\mathcal{T}_x$, $x \in \mathbb{R}^2$, given by the relation (1.1). In the following we give some properties of this operator (see [9]).

i) For all $x \in \mathbb{R}^2$, the operator $\mathcal{T}_x$ is continuous from $\mathcal{E}(\mathbb{R}^2)$ (resp. $\mathcal{D}(\mathbb{R}^2)$ the space of $C^\infty$-functions on $\mathbb{R}^2$ with compact support) into itself, and for all $f$ in $\mathcal{D}(\mathbb{R}^2)$ with support in the closed ball $\overline{B}(0,a)$ of center 0 and radius $a > 0$, we have

$$\text{supp}\mathcal{T}_x(f) \subset \overline{B}(0,a + \|x\|).$$

(3.1)

ii) For all $f$ in $\mathcal{E}(\mathbb{R}^2)$ and $x,y \in \mathbb{R}^2$, we have

$$\mathcal{T}_x(f)(0) = f(x), \quad \text{and} \quad \mathcal{T}_x(f)(y) = \mathcal{T}_y(f)(x).$$

(3.2)

iii) For $x \in \mathbb{R}^2$, $g \in \mathcal{E}(\mathbb{R}^2)$ and $f$ in $\mathcal{D}(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} \mathcal{T}_x(g)(y)f(y)A_k(y)dy = \int_{\mathbb{R}^2} g(z)\mathcal{T}_x(\tilde{f})(-z)A_k(z)dz,$$

where $\tilde{f}$ is the function given by

$$\forall \ x \in \mathbb{R}^2, \ \tilde{f}(x) = f(-x).$$

Remark 3.1. The hypergeometric translation operator $\mathcal{T}_x^W$, $x \in \mathbb{R}^2$, given by the relation (1.2) satisfies the same properties as for the operator $\mathcal{T}_x$, $x \in \mathbb{R}^2$, by considering the spaces $\mathcal{E}(\mathbb{R}^2)^W$ and $\mathcal{D}(\mathbb{R}^2)^W$ (the subspace of $\mathcal{D}(\mathbb{R}^2)$ of $W$-invariant functions).

Notation. We denote by $B(c,a)$ the open ball of $\mathbb{R}^2$ of center $c$ in $\mathbb{R}^2$ and radius $a > 0$, and by $\overline{B}(c,a)$ its closure.

Proposition 3.2. Let $y_0 \in \mathbb{R}^2$ and $a > 0$. We consider the sequence $\{f_n\}_{n \in \mathbb{N}\backslash\{0\}}$ of functions in $\mathcal{D}(\mathbb{R}^2)$, positive, increasing such that:

$$\forall \ n \in \mathbb{N}\backslash\{0\}, \ \text{supp} f_n \subset \overline{B}(y_0,a), \ \forall \ t \in B(y_0,a - \frac{1}{n}), \ f_n(t) = 1,$$

and

$$\forall \ t \in \mathbb{R}^2, \ \lim_{n \to +\infty} f_n(t) = 1_{B(y_0,a)}(t),$$
where \(1_{B(y_0,a)}\) is the characteristic function of the ball \(B(y_0,a)\). We have
\[
\forall x, z \in \mathbb{R}^2, \lim_{n \to +\infty} T_x(f_n)(z) = \lim_{n \to +\infty} \int_{\mathbb{R}^2} f_n(t) dm_{x,z}(t) = \int_{\mathbb{R}^2} 1_{B(y_0,a)}(t) dm_{x,z}(t).
\]
The function \(z \to m_{x,z}(B(y_0,a)) = \int_{\mathbb{R}^2} 1_{B(y_0,a)}(t) dm_{x,z}(t)\), which can also be denoted by \(T_x(1_{B(y_0,a)})(z)\) is defined almost everywhere on \(\mathbb{R}^2\) (see [1] p. 17), measurable and for all function \(h \in D(\mathbb{R}^2)\) we have
\[
\int_{\mathbb{R}^2} m_{x,z}(B(y_0,a)) h(z) A_k(z) dz = \int_{B(y_0,a)} T_x(\tilde{h})(-t) A_k(t) dt. \tag{3.4}
\]

**Proof.** For all \(x \in \mathbb{R}^2\) and \(n \in \mathbb{N}\setminus\{0\}\), the function \(T_x(f_n)\) belongs to \(D(\mathbb{R}^2)\). Then we obtain the results of this proposition from the monotonic convergence theorem and the relation (3.3).

**Remark 3.3.** There exists a \(\sigma\)-algebra \(m\) in \(\mathbb{R}^2\) which contains all Borel sets in \(\mathbb{R}^2\). Then for all \(E \in m\), the function \(z \to m_{x,z}(E)\) is defined almost everywhere on \(\mathbb{R}^2\), measurable and we have the following relation
\[
\int_{\mathbb{R}^2} m_{x,z}(E) h(z) A_k(z) dz = \int_{E} T_x(\tilde{h})(-t) A_k(t) dt, \quad h \in D(\mathbb{R}^2). \tag{3.5}
\]

In this section we shall prove that for all \(x \in \mathbb{R}^2 \setminus \text{reg}\), \(t \in \mathbb{R}^2\), the measures \(m_{x,t}\) and \(m^{\text{W}}_{x,t}\) given by the relations (1.1) and (1.3) are absolute continuous with respect to the Lebesgue measure on \(\mathbb{R}^2\).

### 3.1. Absolute continuity of the measure \(m_{x,z}\).

**Notation.** We denote by \(\lambda\) the Lebesgue measure on \(\mathbb{R}^2\).

**Proposition 3.4.** For \(x \in \mathbb{R}^2 \setminus \text{reg}\), \(z \in \mathbb{R}^2\), there exists a unique positive function \(\ominus(x,z,t)\) integrable on \(\mathbb{R}^2\) with respect to the Lebesgue measure \(\lambda\), and a positive measure \(m^{*}_{x,z}\) on \(\mathbb{R}^2\) such that for every Borel set \(E\), we have
\[
m_{x,z}(E) = \int_E \ominus(x,z,t) dt + m^{*}_{x,z}(E). \tag{3.6}
\]

**Proof.** We deduce (3.6) from (1.1) and Theorem 6.9 of [6] p.129-130, and Theorem 8.6 and its Corollary of [6] p. 166.
Remark 3.5.
i) The supports of the function $t \to \ominus(x, z, t)$ and the measure $m_{x,z}^s$ are contained in the set \{ $t \in \mathbb{R}^2; ||x|| - ||z|| \leq ||t|| \leq ||x|| + ||z||$ \}.

ii) The measures $m_{x,z}^s$ and the Lebesgue mesure $\lambda$ are mutually singular.

iii) From Theorem 8.6, p.166 and Definition 8.3, p.164, of [6], we have

$$\ominus(x, z, t) = \lim_{a \to 0} \frac{m_{x,z}(B(t, a))}{\lambda(B(t, a))}. \quad (3.7)$$

Proposition 3.6. We consider $x \in \mathbb{R}_{reg}^2$ and a positive function $h$ in $\mathcal{D}(\mathbb{R}^2)$ with support contained in the ball $\bar{B}(0, R)$, $R > 0$.

i) For all Borel set $E$, we have

$$\int_E N_x^h(t) dt = \int_{B(0, R)} h(z)m_{x,z}^s(E)A_k(z)dz, \quad \text{ (3.8)}$$

where

$$N_x^h(t) = T_x(\hat{h})(-t)A_k(t) - \int_{B(0, R)} \ominus(x, z, t)h(z)A_k(z)dz. \quad (3.9)$$

ii) We have

$$\forall t \in \mathbb{R}^2, N_x^h(t) \geq 0. \quad (3.10)$$

Proof.

i) By using the relations (3.5), (3.6), we obtain

$$\int_E T_x(\hat{h})(-t)A_k(t)dt = \int_{B(0, R)} m_{x,z}(E)h(z)A_k(z)dz$$

$$= \int_{B(0, R)} \left[ \int_{E} \ominus(x, z, t)dt + m_{x,z}^s(E) \right] h(z)A_k(z)dz. \quad (3.8)$$

We deduce (3.8) by applying Fubini-Tonelli’s theorem to the second member.

ii) From the relation (3.8), the positivity of the measure $m_{x,z}^s$ implies that for all Borel set $E$, we have

$$\int_E N_x^h(t) dt \geq 0.$$ 

Thus

$$\forall t \in \mathbb{R}^2, N_x^h(t) \geq 0.$$

$\square$
Proposition 3.7. The measure $\Lambda^h_x$ on $\mathbb{R}^2$ given for all Borel set $E$ by

$$\Lambda^h_x(E) = \int_E K^h_x(t)dt,$$  \hspace{1cm} (3.11)

is positive and bounded.

Proof.
- The relation (3.10) gives the positivity of the measure $\Lambda^h_x$.
- From the relation (3.11) (3.8), for all Borel set $E$ we have

$$\Lambda^h_x(E) \leq \int_{B(0,R)} \|m^s_{x,z}\| h(z) A_k(z) dz.$$  \hspace{1cm} (3.12)

On the other hand by using (3.6), we obtain for all $z \in \mathbb{R}^2_{\text{reg}}$,

$$m^s_{x,z}(E) \leq m_{x,z}(E),$$

thus

$$\|m^s_{x,z}\| \leq \|m_{x,z}\| = 1.$$  \hspace{1cm} (3.13)

By using this result, the relation (3.12) implies that for all Borel set $E$, we have

$$\Lambda^h_x(E) \leq M_h,$$

where

$$M_h = \int_{B(0,R)} h(z) A_k(z) dz.$$  \hspace{1cm} (3.14)

Then the measure $\Lambda^h_x$ is bounded.

Proposition 3.8. Let $x \in \mathbb{R}^2_{\text{reg}}$ and $h$ be a positive function in $D(\mathbb{R}^2)$ with support contained in the ball $B(0, R), R > 0$.

i) For all Borel set $E$ we have

$$\Lambda^h_x(E) = 0$$  \hspace{1cm} (3.15)

ii) For $x, t \in \mathbb{R}^2_{\text{reg}}$, we have

$$T_x(h)(t) = \int_{B(0,R)} h(z) W(x, t, z) A_k(z) dz,$$  \hspace{1cm} (3.16)

with

$$W(x, t, z) = \frac{\Theta(x, z, t)}{A_k(t)}$$  \hspace{1cm} (3.17)

Proof.
i) From the relations (3.11), (3.8), for all Borel set $E$ the measure $\Lambda^h_x$ possesses also the following form

$$\Lambda^h_x(E) = \int_{B(0,R)} m^s_{x,z}(E)h(z)A_k(z)dz. \quad (3.16)$$

On the other hand from Proposition 3.7 the measure $\Lambda^h_x$ is absolute continuous with respect to the Lebesgue measure $\lambda$ and from Remark 3.5 ii) the measure $m^s_{x,z}, z \in \bar{B}(0,R)$ and the Lebesgue measure $\lambda$ are mutually singular. Then from Proposition 6.8,(f), p. 129, of [6], the measure $\Lambda^h_x$ and $m^s_{x,z}, z \in \bar{B}(0,R)$, are mutually singular. By using the definition of measures mutually singular (see p. 128 of [6]), we deduce (3.13) from (3.16).

ii) By using the i) and (3.11), (3.9), we get

$$T_x(\tilde{h})(-t)A_k(t) = \int_{B(0,R)} \Theta(x,z,t)h(z)A_k(z)dz \quad (3.17)$$

As

$$A_k(t) \neq 0 \Leftrightarrow t \in \mathbb{R}^2_{reg},$$

then for $t \in \mathbb{R}^2_{reg}$, we deduce (3.14), (3.15) from (3.17).

\[ \square \]

**Theorem 3.9.** For all $f$ in $\mathcal{E}(\mathbb{R}^2)$ and $x, t \in \mathbb{R}^2_{reg}$, we have

$$T_x(f)(t) = \int_{\mathbb{R}^2} f(z)W(x,t,z)A_k(z)dz, \quad (3.18)$$

with

$$\forall \ z \in \mathbb{R}^2, \ W(x,t,z) = W(t,x,z). \quad (3.19)$$

**Proof.** We obtain (3.18), (3.19) by writing $f = f^+ - f^-$ and by using Proposition 3.8, and the properties i), ii) of the operator $T_x$. \[ \square \]

**Remark 3.10.** Theorem 3.9 shows that for all $x \in \mathbb{R}^2_{reg}$, $t \in \mathbb{R}^2$ the measure $m_{x,t}$ is absolute continuous with respect to the measure $A_k(z)dz$. More precisely for all $z \in \mathbb{R}^2$, we have

$$dm_{x,t}(z) = W(x,t,z)A_k(z)dz. \quad (3.20)$$

**Corollary 3.11.**

i) For all $\lambda \in \mathbb{C}^2$ and $x, t \in \mathbb{R}^2_{reg}$, we have

$$G_\lambda(x)G_\lambda(t) = \int_{\mathbb{R}^2} G_\lambda(z)W(x,t,z)A_k(z)dz. \quad (3.21)$$
ii) For all $x, t \in \mathbb{R}^2_{\text{reg}}$, we have
\[
\int_{\mathbb{R}^2} W(x, t, z) A_k(z) dz = 1. \tag{3.22}
\]

iii) For all $x, t \in \mathbb{R}^2_{\text{reg}}$, the support of the function $z \to W(x, t, z)$ is contained in the set \{ $z \in \mathbb{R}^d ; ||x|| - ||t|| \leq ||z|| \leq ||x|| + ||t||$ \}.

Proof. We deduce the results of this Corollary from (1.1), (3.20), Theorem 3.9 and the product formula for the Opdam-Cherednik kernel $G_{\lambda, \lambda} \in \mathbb{C}^2$, (see [9] p. 24).

\[\square\]

3.2. Absolute continuity of the measure $m^W_{x,t}$.

**Proposition 3.12.** For all $x, t \in \mathbb{R}^2_{\text{reg}}$ the measure $m^W_{x,t}$ is absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. More precisely for all $z \in \mathbb{R}^2$, we have
\[
dm^W_{x,t}(z) = W^W(x, t, z) A_k(z) dz, \tag{3.23}
\]
where $W^W(x, t, z)$ is the function given by
\[
W^W(x, t, z) = \frac{1}{|W|^2} \sum_{w, w' \in W} W(wx, w't, z). \tag{3.24}
\]

Proof. The relation (1.3) and Theorem 3.9 imply (3.23), (3.24).

\[\square\]

**Corollary 3.13.**

i) For all $\lambda \in \mathbb{C}^2$ and $x, t \in \mathbb{R}^2_{\text{reg}}$, we have
\[
F_{\lambda}(x)F_{\lambda}(t) = \int_{\mathbb{R}^2} F_{\lambda}(z) W^W(x, t, z) A_k(z) dz. \tag{3.25}
\]

ii) For all $x, t \in \mathbb{R}^2_{\text{reg}}$, we have
\[
\int_{\mathbb{R}^2} W^W(x, t, z) A_k(z) dz = 1. \tag{3.26}
\]

iii) For all $x, t \in \mathbb{R}^2_{\text{reg}}$, the support of the function $z \to W^W(x, t, z)$ is contained in the set \{ $z \in \mathbb{R}^2; ||x|| - ||t|| \leq ||z|| \leq ||x|| + ||t||$ \}.

Proof. We obtain the results of this Corollary from the relation (1.2), Proposition 3.12, and the product formula for the Heckman-Opdam hypergeometric function $F_{\lambda}, \lambda \in \mathbb{C}^2$, (see [9] p. 27).

\[\square\]
References


Khalifa Trimèche
Department of Mathematics
Faculty of sciences of Tunis
University Tunis El-Manar
CAMPUS, 2092 Tunis, Tunisia
E-mail: khlifa.trimeche@fst.rnu.tn