STRUCTURAL AND SPECTRAL PROPERTIES OF
k-QUASI-∗-PARANORMAL OPERATORS

FEI ZUO AND HONGLIANG ZUO

Abstract. For a positive integer $k$, an operator $T$ is said to be
$k$-quasi-∗-paranormal if $||T^{k+2}x|||T^kx|| \geq ||T^*T^kx||^2$ for all $x \in H$,
which is a generalization of ∗-paranormal operator. In this paper,
we give a necessary and sufficient condition for $T$ to be a $k$-quasi-∗-
paranormal operator. We also prove that the spectrum is continuous
on the class of all $k$-quasi-∗-paranormal operators.

1. Introduction

Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators on
an infinite dimensional separable Hilbert space $H$. In paper [10] au-
thors introduced the class of $k$-quasi-∗-paranormal operators defined as
follows:

Definition 1.1. $T$ is a $k$-quasi-∗-paranormal operator if

$$||T^{k+2}x|||T^kx|| \geq ||T^*T^kx||^2$$

for every $x \in H$, where $k$ is a natural number.
A $k$-quasi-$\ast$-paranormal operator for a positive integer $k$ is an extension of $\ast$-paranormal operator, i.e., $||T^2x|| \geq ||T^*x||^2$ for unit vector $x$. A 1-quasi-$\ast$-paranormal operator is called a quasi-$\ast$-paranormal operator and it is normaloid \[10\], i.e., $||T^n|| = ||T||^n$, for $n \in \mathbb{N}$ (equivalently, $||T|| = r(T)$, the spectral radius of $T$). $\ast$-paranormal operator and quasi-$\ast$-paranormal operator have been studied by many authors and it is known that they have many interesting properties similar to those of hyponormal operators (see \[5, 9, 11, 14\]).

It is clear that 

\[ \ast \text{-paranormal} \Rightarrow \text{quasi-}\ast \text{-paranormal} \Rightarrow \text{normaloid} \]

and

\[ \text{quasi-}\ast \text{-paranormal} \Rightarrow k\text{-quasi-}\ast \text{-paranormal} \Rightarrow (k + 1)\text{-quasi-}\ast \text{-paranormal}. \]

In \[14\], the authors give an example to show that a quasi-$\ast$-paranormal operator need not be a $\ast$-paranormal operator.

**Example 1.2.** Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be operators on $\mathbb{R}^2$, and let $H_n = \mathbb{R}^2$ for all positive integers $n$. Consider the operator $T_{A,B}$ on $\oplus_{n=1}^{+\infty} H_n$ defined by

\[
T_{A,B} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
a & 0 & 0 & 0 & 0 & \cdots \\
0 & B & 0 & 0 & 0 & \cdots \\
0 & 0 & B & 0 & 0 & \cdots \\
0 & 0 & 0 & B & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Then $T_{A,B}$ is a quasi-$\ast$-paranormal operator, but not a $\ast$-paranormal operator.

We give the following example to show that there also exists a $(k + 1)$-quasi-$\ast$-paranormal operator, but not a $k$-quasi-$\ast$-paranormal operator.

**Example 1.3.** Given a bounded sequence of positive numbers $\alpha : \alpha_1, \alpha_2, \alpha_3, \ldots$ (called weights), the unilateral weighted shift $W_{\alpha}$ associated with $\alpha$ is the operator on $l_2$ defined by $W_{\alpha}e_n = \alpha_ne_{n+1}$ for all $n \geq 1$, where $\{e_n\}_{n=1}^\infty$ is the canonical orthogonal basis for $l_2$. Straightforward
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Calculations show that $W_\alpha$ is a $k$-quasi-∗-paranormal operator if and only if

$$W_\alpha = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_1 & 0 & 0 & 0 & \cdots \\
0 & \alpha_2 & 0 & 0 & \cdots \\
0 & 0 & \alpha_3 & 0 & \cdots \\
0 & 0 & 0 & \alpha_4 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},$$

where

$$\alpha_{i+1}\alpha_{i+2} \geq \alpha_i^2 (i = k, k + 1, k + 2, \cdots).$$

So, if $\alpha_{k+1} \leq \alpha_{k+2} \leq \alpha_{k+3} \leq \cdots$ and $\alpha_k > \alpha_{k+2}$, then $W_\alpha$ is a $(k + 1)$-quasi-∗-paranormal operator, but not a $k$-quasi-∗-paranormal operator.

Now it is natural to ask whether $k$-quasi-∗-paranormal operators are normaloid or not. For $k > 1$, an answer has been given: there exists a nilpotent operator which is a $k$-quasi-∗-paranormal operator. But it need not be normaloid.

In section 2, we give a necessary and sufficient condition for $T$ to be a $k$-quasi-∗-paranormal operator. In section 3, we prove that the spectrum is continuous on the class of all $k$-quasi-∗-paranormal operators.

2. $k$-quasi-∗-paranormal operators

In the sequel, we shall write $N(T)$ and $R(T)$ for the null space and range space of $T$, respectively.

**Lemma 2.1.** [10] $T$ is a $k$-quasi-∗-paranormal operator $\iff T^k(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$ for all $\lambda > 0$.

**Theorem 2.2.** If $T$ does not have a dense range, then the following statements are equivalent:

1. $T$ is a $k$-quasi-∗-paranormal operator;
2. $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = R(T^k) \oplus N(T^{*k})$, where $T_1^{*2}T_1^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2 \geq 0$ for all $\lambda > 0$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$. 
Proof. (1) ⇒ (2) Consider the matrix representation of $T$ with respect to the decomposition $H = \overline{R(T^k)} \oplus N(T^*k)$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$ 

Let $P$ be the projection onto $\overline{R(T^k)}$. Since $T$ is a $k$-quasi-∗-paranormal operator, we have

$$P(T^*TT^2 - 2\lambda TT^* + \lambda^2)P \geq 0 \text{ for all } \lambda > 0.$$

Therefore

$$T^*TT^2 - 2\lambda(T_1T_1^t + T_2T_2^t) + \lambda^2 \geq 0 \text{ for all } \lambda > 0.$$

On the other hand, for any $x = (x_1, x_2) \in H$, we have

$$(T^k_3 x_2, x_2) = (T^k(I - P)x, (I - P)x) = ((I - P)x, T^*k(I - P)x) = 0,$$

which implies $T^k_3 = 0$.

Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$, where $M$ is the union of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ by Corollary 7 of [8], and $\sigma(T_1) \cap \sigma(T_3)$ has no interior point and $T_3$ is nilpotent, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) ⇒ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^k)} \oplus N(T^*k)$, where $T^*TT^2 - 2\lambda(T_1T_1^t + T_2T_2^t) + \lambda^2 \geq 0$ for all $\lambda > 0$ and $T_3^k = 0$. Since

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix},$$

we have

$$T^k T^*k = \begin{pmatrix} T_1^k T_1^*k + \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j})^* \ & 0 \\ 0 \ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where $A = A^* = T_1^k T_1^*k + \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j})^*$. Hence, for all $\lambda > 0$,

$$T^k T^*k(T^*TT^2 - 2\lambda TT^* + \lambda^2)T^k T^*k.$$
\begin{equation}
\begin{pmatrix}
A(T_1^*T_1^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2) & 0 \\
0 & 0
\end{pmatrix} \geq 0.
\end{equation}

It follows that \( T^k(T^*T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0 \) on \( H = \overline{R(T^k)} \oplus N(T^k) \). Thus \( T \) is a \( k \)-quasi-\( * \)-paranormal operator.  \( \square \)

**Corollary 2.3.** [10] Let \( T \) be a \( k \)-quasi-\( * \)-paranormal operator, the range of \( T^k \) be not dense and

\[
T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^k).
\]

Then \( T_1 \) is a \( * \)-paranormal operator, \( T_3^k = 0 \) and \( \sigma(T) = \sigma(T_1) \cup \{0\} \).

**Corollary 2.4.** [11] If \( T \) is a quasi-\( * \)-paranormal operator and \( R(T) \) is not dense, then \( T \) has the following matrix representation:

\[
T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*)
\]

where \( T_1 \) is a \( * \)-paranormal operator on \( \overline{R(T)} \).

**Corollary 2.5.** Let \( T \) be a \( k \)-quasi-\( * \)-paranormal operator and \( 0 \neq \mu \in \sigma_p(T) \). If \( T \) is of the form \( T = \begin{pmatrix} \mu & B \\ 0 & C \end{pmatrix} \) on \( H = N(T - \mu) \oplus N(T - \mu)^\perp \), then \( B = 0 \).

**Proof.** Let \( P \) be the projection onto \( N(T - \mu) \) and \( x \in N(T - \mu) \). Since \( T \) is a \( k \)-quasi-\( * \)-paranormal operator and \( x = \frac{1}{\mu^4} T^k x \in \overline{R(T^k)} \), we have

\[
P(T^*T^2 - 2\lambda TT^* + \lambda^2)P \geq 0 \text{ for all } \lambda > 0,
\]

then

\[
\mu^4 - 2\lambda(\mu^2 + BB^*) + \lambda^2 \geq 0 \text{ for all } \lambda > 0,
\]

which yields that

\[
\mu^4 - 2\lambda\mu^2 + \lambda^2 \geq 2\lambda BB^* \text{ for all } \lambda > 0.
\]

Hence \( B = 0 \).  \( \square \)
3. Spectral properties of \( k \)-quasi-\( * \)-paranormal operators

For every \( T \in B(H) \), \( \sigma(T) \) is a compact subset of \( \mathbb{C} \). The function \( \sigma \) viewed as a function from \( B(H) \) into the set of all compact subsets of \( \mathbb{C} \), equipped with the Hausdorff metric, is well known to be upper semi-continuous, but fails to be continuous in general. Conway and Morrel [2] have carried out a detailed study of spectral continuity in \( B(H) \). Recently, the continuity of spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators in [6, 12]. It has been proved that is continuous in the set of normal operators and hyponormal operators in [7]. And this result has been extended to quasi-hyponormal operators by Djordjević in [3], to \( p \)-hyponormal operators by Hwang and Lee in [13], and to \( (p, k) \)-quasihyponormal, \( * \)-paranormal and paranormal operators by Duggal, Jeon and Kim in [4]. In this section we extend this result to \( k \)-quasi-\( * \)-paranormal operators.

**Lemma 3.1.** Let \( T \) be a \( k \)-quasi-\( * \)-paranormal operator. Then the following assertions hold:

1. If \( T \) is quasinilpotent, then \( T^{k+1} = 0 \).
2. For every non-zero \( \lambda \in \sigma_p(T) \), the matrix representation of \( T \) with respect to the decomposition \( H = N(T - \lambda) \oplus (N(T - \lambda))^\perp \) is: \( T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} \) for some operator \( B \) satisfying \( \lambda \notin \sigma_p(B) \) and \( \sigma(T) = \{\lambda\} \cup \sigma(B) \).

**Proof.** (1) Suppose \( T \) is a \( k \)-quasi-\( * \)-paranormal operator. If the range of \( T^k \) is dense, then \( T \) is a \( * \)-paranormal operator, which leads to that \( T \) is normaloid, hence \( T = 0 \). If the range of \( T^k \) is not dense, then

\[
T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}
\] on \( H = \overline{R(T^k)} \oplus N(T^k) \)

where \( T_1 \) is a \( * \)-paranormal operator, \( T_3^k = 0 \) and \( \sigma(T) = \sigma(T_1) \cup \{0\} \) by Theorem 2.2. Since \( \sigma(T_1) = \{0\} \), \( T_1 = 0 \). Thus

\[
T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2 T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.
\]

(2) If \( \lambda \neq 0 \) and \( \lambda \in \sigma_p(T) \), we have that \( N(T - \lambda) \) reduces \( T \) by Corollary 2.5. So we have that \( T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} \) on \( H = N(T - \lambda) \oplus \)
(N(T - λ))⊥ for some operator B satisfying λ /∈ σ_p(B) and σ(T) = {λ} ∪ σ(B).

**Lemma 3.2.** [1] Let H be a complex Hilbert space. Then there exists a Hilbert space K such that H ⊂ K and a map ϕ : B(H) → B(K) such that

1. ϕ is a faithful ∗-representation of the algebra B(H) on K;
2. ϕ(A) ≥ 0 for any A ≥ 0 in B(H);
3. σ_a(T) = σ_a(ϕ(T)) = σ_p(ϕ(T)) for any T ∈ B(H).

**Theorem 3.3.** The spectrum σ is continuous on the set of k-quasi-∗-paranormal operators.

**Proof.** Suppose T is a k-quasi-∗-paranormal operator. Let ϕ : B(H) → B(K) be Berberian's faithful ∗-representation of Lemma 3.2. In the following, we shall show that ϕ(T) is also a k-quasi-∗-paranormal operator. In fact, since T is a k-quasi-∗-paranormal operator, we have

\[ T^k(T^{*2}T^2 - 2λTT^* + λ^2)T^k ≥ 0 \text{ for all } λ > 0. \]

Hence we have

\[ (ϕ(T))^k((ϕ(T))^{*2}(ϕ(T))^2 - 2λϕ(T)(ϕ(T))^* + λ^2)(ϕ(T))^k \]

\[ = ϕ(T)^k(T^{*2}T^2 - 2λTT^* + λ^2)T^k \text{ by Lemma 3.2} \]

\[ ≥ 0 \text{ by Lemma 3.2,} \]

so ϕ(T) is also a k-quasi-∗-paranormal operator. By Lemma 3.1, we have T belongs to the set C(i) (see definition in [4]). Therefore, we have that the spectrum σ is continuous on the set of k-quasi-∗-paranormal operators by [4, Theorem 1.1].

A complex number λ is said to be in the point spectrum σ_p(T) of T if there is a nonzero x ∈ H such that (T - λ)x = 0. If in addition, (T^* - λ)x = 0, then λ is said to be in the joint point spectrum σ_jp(T) of T. If T is hyponormal, then σ_jp(T) = σ_p(T). Here we show that if T is a k-quasi-∗-paranormal operator, then σ_jp(T) \ {0} = σ_p(T) \ {0}.

**Lemma 3.4.** Let T be a k-quasi-∗-paranormal operator and λ ≠ 0. Then Tx = λx implies T^*x = ⃗λx.

**Proof.** It is obvious from Corollary 2.5.

The following example provides an operator T which is a k-quasi-∗-paranormal operator, however, the relation N(T) ⊆ N(T^*) does not hold.
Example 3.5. [14] Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be operators on $\mathbb{R}^2$, and let $H_n = \mathbb{R}^2$ for all positive integers $n$. Consider the operator $T_{A,B}$ on $\bigoplus_{n=1}^{+\infty} H_n$ defined by

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$ 

Then $T_{A,B}$ is a quasi-$\ast$-paranormal operator, hence $T_{A,B}$ is a $k$-quasi-$\ast$-paranormal operator, however for the vector $x = (0, 0, 1, -1, 0, 0, \cdots)$, $T_{A,B}^\ast(x) = 0$, but $T_{A,B}(x) \neq 0$. Therefore, the relation $N(T_{A,B}) \subseteq N(T_{A,B}^\ast)$ does not always hold.

Theorem 3.6. Let $T$ be a $k$-quasi-$\ast$-paranormal operator. Then $\sigma_{jp}(T) \{0\} = \sigma_p(T) \{0\}$.

Proof. It is clearly by Lemma 3.4. \qed

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References

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