FINITUDE OF INFINITESIMAL DEFORMATIONS
OF CR MAPPINGS OF CR MANIFOLDS
OF NONDEGENERATE LEVI FORM

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ABSTRACT. Let $M$ and $N$ be CR manifolds with nondegenerate
Levi forms of hypersurface type of dimension $2m+1$ and $2n+1$,
respectively, where $1 \leq m \leq n$. Let $f : M \rightarrow N$ be a CR mapping.
Under a generic assumption we construct a complete system of
finite order for the infinitesimal deformations of $f$. In particular,
we prove the space of infinitesimal deformations of $f$ forms a finite
dimensional Lie algebra.

0. Introduction

In [9] the second author proved under generic assumptions that the
equations for CR embeddings admit a complete prolongation of finite
order. The present paper is its infinitesimal version: Let $M$ and $N$ be
CR manifolds of hypersurface type of nondegenerate Levi forms with
dim $M \leq$ dim $N$. Let $f : M \rightarrow N$ be a CR embedding. We construct a
complete system for the infinitesimal deformations of $f$, which implies
that the space of infinitesimal deformations is finite dimensional. In
particular, if $M = N$ is a real analytic real hypersurface in $\mathbb{C}^N$ our
result implies that a sufficiently smooth infinitesimal CR automorphism
is real analytic and extends to a holomorphic vector field of the ambient
space and that the set of all the infinitesimal CR automorphisms of $M$
is a finite dimensional Lie algebra. This is an alternate approach to the
problem of deciding holomorphic extendability and finite dimensionality
of CR automorphisms as treated in [17] and [18].
The finite dimensionality of infinitesimal deformations of \( f \) may be regarded as rigidity of embedding. Let \( d \) be the dimension of the linear space of the infinitesimal deformations of \( f \) and \( d' \) the dimension of CR automorphism group of \( N \). Every 1-parameter group of local CR automorphisms of \( N \) gives rise to an infinitesimal deformation of \( f \), thus \( d \geq d' \). If \( d = d' \) the embedding \( f \) is infinitesimally rigid. The following example shows that the infinitesimal deformations of embeddings of a Levi flat CR manifold can be infinite dimensional.

**Example 0.1.** Let \( M := \mathbb{C} \times \mathbb{R} \) with the usual complex structure on \( \mathbb{C} \) and \( N = \mathbb{C}^2 \times \mathbb{R} \). Let \( f(z,t) = (f_1(z), f_2(z), t) \), where \( f_j, j = 1, 2, \) are holomorphic and the complex derivative of at least one of \( f_j \) is nonvanishing. Then for any smooth real valued function \( \phi(t) \) the real vector field \( V := \phi(t) \frac{\partial}{\partial t} \) is an infinitesimal deformation of CR embedding \( f \). The set of such vector fields is infinite dimensional.

Generically, an overdetermined system admits prolongation to a complete system of finite order, that is, we can solve for all the partial derivatives of the unknown functions of certain order, say \( k \), as functions of derivatives of the unknown functions of order less than \( k \) after differentiating the original equations sufficiently many times. This occurs when the coefficients to the \( k \)-th order partial derivatives satisfy the nondegeneracy condition of the implicit function theorem.

In this paper we shall show that the linearized system (2.5)–(2.6) at a CR embedding \( f \) admits prolongation to a complete system of finite order under certain generic conditions on the embedding, which implies the finite-dimensionality of the space of infinitesimal deformations.

1. **Prolongation and complete system**

Let \( m, n \in \mathbb{N} \). Let \( X \) be an open subset of \( \mathbb{R}^n \) and let \( \mathbb{R}^{(q)} \) be a Euclidean space whose coordinates represent all the partial derivatives of \( \mathbb{R}^m \)-valued smooth maps defined on \( X \) of all orders from 0 to \( q \). A multi-index of order \( r \) is an unordered \( r \)-tuple of integers \( J = (j_1, \ldots, j_r) \), with \( 1 \leq j_s \leq n \). The order of a multi-index \( J \) is denoted by \( |J| \). By \( u_J \) we denote the \( |J| \)-th order partial derivative of \( u^\alpha \) with respect to \( x^{j_1}, \ldots, x^{j_1|J|} \), and we often drop the parentheses and commas in writing multi-indices, thus \( u^\alpha_J = u_J^\alpha = \partial u^\alpha / \partial x^J \),
\( u^0_{j,k} = u^0_{(j,k)} = (\partial^2 u^\alpha)/(\partial x^j \partial x^k), \) and so forth. A point in \( \mathbb{R}^{(q)} \) will be denoted by \( u^{(q)} \), so that \( u^{(q)} = (u^0_j)_{0 \leq j \leq m, 0 \leq |J| \leq q} \).

The product space \( J^q(X, \mathbb{R}^m) = X \times \mathbb{R}^{(q)} \) is called the \( q \)-th order jet space of the space \( X \times \mathbb{R}^m \). If \( f = (f^1, \ldots, f^m) : X \rightarrow \mathbb{R}^m \) is smooth, let \( (j^q f)(x) = (x, \partial f^\alpha(x)) : 1 \leq \alpha \leq m, |J| \leq q \), then \( j^q f \), called the \( q \)-graph of \( f \), is a smooth section of \( J^q(X, \mathbb{R}^m) \).

Consider a system of partial differential equations of order \( q, q \geq 1 \), for unknown functions \( u = (u^1, \ldots, u^m) \) of independent variables \( x = (x^1, \ldots, x^n) \):

\[
(1.1) \quad \Delta_\nu(x, u^{(q)}) = 0, \quad \nu = 1, \ldots, \ell,
\]

where each \( \Delta_\nu(x, u^{(q)}) \) is a smooth function in its arguments. Then \( \Delta = (\Delta_1, \ldots, \Delta_\ell) \) is a smooth map from \( J^q(X, \mathbb{R}^m) \) into \( \mathbb{R}^\ell \).

Then the subset \( S_\Delta \) of \( J^q(X, \mathbb{R}^m) \) defined by \( \Delta = 0 \) is called the solution subvariety of (1.1). Thus, a smooth solution of (1.1) is a smooth map \( f : X \rightarrow \mathbb{R}^m \) whose \( q \)-graph is contained in \( S_\Delta \).

A differential function \( P(x, u^{(q)}) \) of order \( q \) is a smooth function defined on an open subset of \( J^q(X, \mathbb{R}^m) \). The total derivatives of \( P(x, u^{(q)}) \) with respect to \( x^i \) is the differential function of order \( q + 1 \) defined by

\[
D_i P(x, u^{(q+1)}) := \frac{\partial P}{\partial x^i} + \sum_{a=1}^m \sum_{|J| \leq q} u^a_{j,i} \frac{\partial P}{\partial u^a_j},
\]

where \( J, i \) denotes the multi-index \( (j_1, \ldots, j_i, j_1, i) \). For each nonnegative integer \( r \), the \( r \)-th-prolongation \( \Delta^{(r)} \) of the system (1.1) is the system consisting of all the total derivatives of (1.1) of order up to \( r \). Let \( (\Delta^{(r)}) \) be the ideal generated by \( \Delta^{(r)} \) of the ring of differential functions on \( J^{q+r}(X, \mathbb{R}^m) \). If \( \tilde{\Delta} \in (\Delta^{(r)}) \) for some \( r \), the equation

\[
(1.2) \quad \tilde{\Delta}(x, u^{(q+r)}) = 0
\]

is called a prolongation of (1.1). Note that any smooth solution of (1.1) must satisfy (1.2). If \( k \) is the order of the highest derivative involved in \( \tilde{\Delta} \), we call (1.2) a prolongation of order \( k \). We now define the complete system.
**Definition 1.1.** We say that (1.1) admits prolongation to a complete system of order \( k \) if there exist prolongations of (1.1) of order \( k \)

\[
(1.3) \quad \tilde{\Delta}_\nu (x, u^{(k)}) = 0, \quad \nu = 1, \cdots, N
\]

which can be solved for all the \( k \)-th order partial derivatives as smooth functions of lower order derivatives of \( u \), namely, for each \( a = 1, \cdots, m \) and for each multi-index \( J \) with \( |J| = k \),

\[
(1.4) \quad u^a_J = H^a_J(x, u^{(k-1)})
\]

for some function \( H^a_J \) which is smooth in its arguments. (1.4) is called a complete system of order \( k \).

The complete system (1.4) is obtained from (1.3) when the coefficients to \( u^a_J \), \( |J| = k \), in (1.3) satisfies the nondegeneracy condition of the implicit function theorem, therefore, generically an overdetermined system admits prolongation to a complete system of order \( k \) for some sufficiently large \( k \). Prolongation to a complete system for overdetermined systems seems to have wide application. Very recently A. Hayashimoto, D. Zaitsev and S.Y. Kim use the method of prolongation or consequences of the complete system ([10], [11], [20], and [15]).

We find the idea of the complete system in the equivalence problem of E. Cartan ([5], [12]): Let \( G \) be a Lie-subgroup of \( \text{GL}(n; \mathbb{R}) \). Suppose that a manifold \( E \) of dimension \( n \) has a \( G \)-structure and \( \pi : Y \to E \) is the associated principal bundle of \( G \)-frames. The equivalence problem is finding canonically a system of differential 1-forms

\[
(1.5) \quad \omega^1, \cdots, \omega^N, \quad \text{where} \quad N = n + \dim G
\]
on \( Y \) so that a mapping \( f : E \to \tilde{E} \) preserves the \( G \)-structure if and only if there exists a mapping \( F : Y \to \tilde{Y} \), which satisfies that \( \tilde{\pi} \circ F = f \circ \pi \), and that

\[
(1.6) \quad F^* \tilde{\omega}^i = \omega^i, \quad i = 1, \cdots, N,
\]

where \( \tilde{E} \) is a manifold of dimension \( n \) with a \( G \)-structure and \( \tilde{\pi} : \tilde{Y} \to \tilde{E} \) is the associated principal bundle and \( \tilde{\omega}^i \) are the corresponding 1-forms on \( \tilde{Y} \). (1.5) is called a complete system of invariants of the \( G \)-structure and (1.6) is a complete system of order 1 for \( F \) in the sense of Definition
1.1. It turns out that (1.6) is equivalent to a complete system of order 2 for \( f \).

Now we recall that solving the given system of partial differential equations (1.1) is equivalent to finding an integral manifold of the corresponding exterior differential system

\[
du^a_j - \sum_{i=1}^n u^a_{j,i} dx^i = 0
\]

for all multi-index \( I \) with \( |I| < q \) and \( a = 1, \ldots, m \), with an independence condition \( dx_1 \wedge \cdots \wedge dx_n \neq 0 \) on \( S_\Delta ([4]) \). If (1.1) admits prolongation to a complete system of order \( k \) then we have the following Pfaffian system on \( J^{k-1}(X, \mathbb{R}^m) \):

\[
\begin{align*}
du^a - \sum_{j=1}^n u^a_j dx^j &= 0, \\
&\vdots \\
du^a_I - \sum_{j=1}^n u^a_{I,j} dx^j &= 0, \quad |I| = k - 2, \\
du^a_I - \sum_{i=1}^n H^a_{I,i} dx^j &= 0, \quad |I| = k - 1
\end{align*}
\]

with an independence condition \( dx^1 \wedge \cdots \wedge dx^n \neq 0 \), where \( H^a_{I,i} \) are as in (1.4). Therefore, if (1.1) admits a prolongation to a complete system (1.4) a \( C^k \)-function \( u = f(x) \) is a solution of (1.1) if and only if

\[
(x) \mapsto (x, \partial_j f(x) : |J| \leq k - 1)
\]

is an integral manifold of the Pfaffian system (1.7). In particular, we have

**Proposition 1.2.** Suppose that (1.1) admits a complete system (1.4), then a solution is uniquely determined by its \((k - 1)\) jet at a point and is \( C^\infty \) provided that it is \( C^k \). Furthermore, if (1.1) is real analytic in its arguments then each \( H^a_I \) is real analytic and every \( C^k \) solution of (1.1) is real analytic.

Now let \( \omega^1, \ldots, \omega^N \) be the differential 1-form on \( J^{k-1}(X, \mathbb{R}^m) \) given in the left hand side of (1.7) and let \( \omega = (\omega^1, \ldots, \omega^N)^T \). Then

\[
d\omega = \Theta \wedge \omega + \Omega
\]
for some $N \times N$ matrix-valued 1-form $\Theta$ and a 2-form $\Omega$ which is determined uniquely modulo the ideal generated by $\omega^1, \ldots, \omega^N$. $\Omega$ is the obstruction to the existence of solutions of (1.7). If $\Omega$ is identically zero then (1.7) is involutive and by the Frobenius theorem there exists a unique solution for any initial condition. Little further is known about the existence of solutions.

2. Embedding of Cauchy-Riemann (CR) manifolds

Let $M$ be a differentiable manifold of dimension $2m + 1$. A CR structure on $M$ is a subbundle $\mathcal{V}$ of the complexified tangent bundle $T_CM$ having the following properties:

(a) each fiber is of complex dimension $m$,
(b) $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$,
(c) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ (integrability).

Given a CR structure $\mathcal{V}$ we define the Levi form $\mathcal{L}$ by

$$\mathcal{L}(L_1, L_2) := \sqrt{-1}[L_1, \overline{L}_2], \mod (\mathcal{V} + \overline{\mathcal{V}}).$$

$\mathcal{L}$ is a hermitian form on $\mathcal{V}$ with values in $T_CM/(\mathcal{V} + \overline{\mathcal{V}})$. $M$ is said to be strictly pseudoconvex if $\mathcal{L}$ is positive or negative definite. A real hypersurface in a complex manifold has natural CR structure induced from the complex structure of the ambient space.

A complex valued function $f$ is called a CR function if $f$ is annihilated by $\mathcal{V}$. Let $\{L_1, \ldots, L_m\}$ be a set of complex vector fields that generates $\mathcal{V}$. Then $f$ is a CR function if and only if

$$\overline{L}_i f = 0, \quad i = 1, \ldots, m \quad (\text{tangential Cauchy-Riemann equations}).$$

Let $(N, \mathcal{V'})$ be a CR manifold of dimension $2n + 1$, $n \geq m$, with the CR structure bundle $\mathcal{V'}$. A mapping $f : M \to N$ is called a CR mapping if $f$ preserves the CR structure, that is,

$$f_* \mathcal{V} \subset \mathcal{V'}.$$
holomorphic change of coordinates \( r(z, \bar{z}) \) takes the normal form as in [6].

\[
(2.1) \quad r(z, \bar{z}) = z^{n+1} + \bar{z}^{n+1} + \sum_{j=1}^{n} \lambda_j z^j \bar{z}^j + \sum_{A,B} c_{AB} z^A \bar{z}^B,
\]

where each \( \lambda_j \) is either 1 or \(-1\) and each term in the last summand is of weight \( \geq 4 \). Weight of a term \( c_{AB} z^A \bar{z}^B \) is \( \sum_{j=1}^{n} (a_j + b_j) + 2(a_{n+1} + b_{n+1}) \).

We denote by \( C \) the last summand of (2.1) and write \( C \) in Moser’s normal form ([6]) as

\[
(2.2) \quad C := \sum_{A,B} c_{AB} z^A \bar{z}^B
\]

\[
= F_{11}(z', \bar{z}', z^{n+1}, \bar{z}^{n+1}) + \sum_{\min(k,\ell) \geq 2} F_{k\ell}(z', \bar{z}', z^{n+1}, \bar{z}^{n+1}),
\]

where \( z' = (z^1, \cdots, z^n) \) and \( F_{k\ell} \) satisfies

\[
F_{k\ell}(tz', s\bar{z}', z^{n+1}, \bar{z}^{n+1}) = t^k s^\ell F_{k\ell}(z', \bar{z}', z^{n+1}, \bar{z}^{n+1})
\]

and

\[
F_{11}(z', \bar{z}', 0, 0) = 0.
\]

A system of complex valued functions \( f = (f^1, \cdots, f^{n+1}) \) on \( M \) is a CR mapping of \( M \) into \( N \) if and only if

\[
(2.3) \quad \bar{L}_i f^j = 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n + 1
\]

(tangential Cauchy-Riemann equations), and

\[
(2.4) \quad r \circ f = 0.
\]

A mapping \( g = (g^1, \cdots, g^{n+1}) : M \to \mathbb{C}^{n+1} \) is an infinitesimal deformation of \( f \) if and only if \( g \) satisfies the linearization of (2.3)–(2.4), namely,

\[
(2.5) \quad \bar{L}_i g^j = 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n + 1
\]

(tangential Cauchy-Riemann equations), and

\[
(2.6) \quad \sum_{j=1}^{n+1} \left\{ \left( \frac{\partial r}{\partial z^j} \circ f \right) g^j + \left( \frac{\partial r}{\partial \bar{z}^j} \circ f \right) \bar{g}^j \right\} = 0.
\]

For an \( m \)-tuple of non-negative integers \( \alpha = (\alpha_1, \cdots, \alpha_m) \) let \( L^\alpha = L_1^{\alpha_1} \cdots L_m^{\alpha_m} \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_m \). We have
THEOREM 2.1. Let $M^{2m+1}$, $m \geq 1$, be a germ of $C^\omega$ CR manifold of nondegenerate Levi form. Let $\{L_1, \cdots, L_m\}$ be $C^\omega$ independent sections of the CR structure bundle $\mathcal{V}$. Let $N$ be a germ of $C^\omega$ real hypersurface at the origin of $\mathbb{C}^{n+1}$, $n \geq m$, defined by $r(z, \bar{z}) = 0$, where $r(z, \bar{z})$ is normalized as in (2.1) and (2.2). Let $f : M \to N$ be an analytic $(C^\omega)$ CR mapping. Suppose that for some positive integer $k$ the vectors $\{L^\alpha f : \lvert \alpha \rvert \leq k\}$ evaluated at the reference point together with $(0, \cdots, 0,1)$ span $\mathbb{C}^{n+1}$ over $\mathbb{C}$. Then the system (2.5)–(2.6) admits prolongation to a complete system of order $2k+1$. Thus, the space of infinitesimal deformations of $f$ is finite dimensional and an infinitesimal deformation is uniquely determined by its $2k$-jet at a point.

In [9] it is shown that the CR embeddings of $M$ into $N$ satisfies a complete system of finite order and that if a CR embedding is sufficiently smooth then it is real analytic. The integer $k$ in the hypotheses of Theorem 2.1 is independent of the choice of basis $\{L_1, \cdots, L_m\}$.

Now we consider the special case of $m = n$, and $f$ is the identity map. In this case an infinitesimal deformation is an infinitesimal CR automorphism of $M$. Since $\{L_1 f, \cdots, L_m f\}$ together with $(0, \cdots, 0,1)$ span $\mathbb{C}^{m+1}$, $k = 1$, and thus we have

COROLLARY 2.2. Let $M^{2m+1}$, $m \geq 1$, be a $C^\omega$ CR manifold of nondegenerate Levi form. Then the infinitesimal CR automorphisms satisfy a complete system of order 3. Thus, the space of infinitesimal CR automorphisms forms a finite dimensional Lie algebra and an infinitesimal CR automorphism is uniquely determined by its 2-jet at a point.

Corollary 2.2 is a well known basic fact of CR geometry. Recently N. Stanton studied holomorphic vector fields tangent to $M$: let $\text{hol}(M)$ be the set of holomorphic vector fields on an open subset of $\mathbb{C}^{n+1}$ that are tangent to an analytic real hypersurface $M$. $M$ is said to be holomorphically nondegenerate at $O$ if there are no such vector fields. In [18] it is shown that $\text{hol}(M)$ is finite dimensional if and only if $M$ is holomorphically nondegenerate. Relevant results are found in [1] and [2].

Proof of Theorem 2.1. We shall construct a complete system by differentiating (2.6) repeatedly and by reducing the order of derivatives.
using (2.5). First we obtain $\frac{\partial r}{\partial z^j}$ and $\frac{\partial r}{\partial \bar{z}^j}$ from (2.1) and substitute in (2.6), to get

\begin{equation}
(2.7) \quad g^{n+1} + \bar{g}^{n+1} + \sum_{j=1}^{n} \lambda_j (\bar{f}^j g^j + f^j \bar{g}^j) + \sum_{j=1}^{n+1} (\frac{\partial C}{\partial z^j} \circ f) g^j + (\frac{\partial C}{\partial \bar{z}^j} \circ f) \bar{g}^j = 0.
\end{equation}

Apply $\bar{L}^\alpha$ to (2.7) for each $\alpha$ with $|\alpha| \leq k$. Then by (2.5) we have

\begin{equation}
(2.8) \quad \bar{L}^\alpha \bar{g}^{n+1} + \sum_{j=1}^{n} \lambda_j (\bar{L}^\alpha \bar{f}^j g^j + f^j \bar{L}^\alpha \bar{g}^j) + \sum_{j=1}^{n+1} a_j (x, \bar{L}^\beta \bar{g} : |\beta| \leq k) g^j = 0,
\end{equation}

where $a_j$ is analytic in its arguments, $x$ is the local coordinates of $M$. We observe that (2.2) implies that each $a_j$ vanishes at the reference point $0 \in M$. Then by the hypothesis of the theorem we can solve (2.7)–(2.8) for $g^j$, $j = 1, \ldots, n+1$, to get

\begin{equation}
(2.9) \quad g^j = H^j(x, \bar{L}^\alpha \bar{g} : |\alpha| \leq k), \quad j = 1, \ldots, n+1,
\end{equation}

where each $H^j$ is an analytic function of the arguments in the parenthesis.

Let $\beta = (\beta_1, \ldots, \beta_m)$ be any multi-index. Apply $L^\beta$ to (2.9). Then we have

\begin{equation}
(2.10) \quad L^\beta g^j = L^\beta \left( H^j(x, \bar{L}^\alpha \bar{g} : |\alpha| \leq k) \right).
\end{equation}

Now let $T$ be a $C^0$ real vector field on $M$ which is transversal to the $\mathcal{V} \oplus \bar{\mathcal{V}}$, so that the set $\{T, L_j, \bar{L}_j, j = 1, \ldots, m\}$ forms a basis of the complexified tangent space of $M$. Let $[L_j, \bar{L}_k] = \sqrt{-1} \rho_{j\bar{k}} T \mod (\mathcal{V}, \bar{\mathcal{V}})$. Then $(\rho_{j\bar{k}})$, $j, k = 1, \ldots, m$, is a non-degenerate hermitian matrix. We may assume that $[\rho_{j\bar{k}}(0)]$ is diagonal at the reference point $0 \in M$. In the right hand side of (2.10), each time we apply $L_i$ to $H^j(x, \bar{L}^\alpha \bar{g} : |\alpha| \leq k)$, computations by chain rule show that $T$-directional derivatives occurs when commuting $L$ and $\bar{L}$, and by (2.5) the total order of the derivatives remains $\leq k$, for example,

\begin{equation}
(2.11) \quad \bar{L}_1 L_1 g^j = (L_1 \bar{L}_1 - [L_1, \bar{L}_1]) g^j
= \{L_1 \bar{L}_1 - (\sqrt{-1} \rho_{1\bar{1}} T + \sum_{i=1}^{m} (a_i L_i + b_i \bar{L}_i))\} g^j
\end{equation}

for some functions $a_i$ and $b_i$
Now we introduce notations: for each pair of non-negative integers $(p, q)$ with $p \geq q$, let $C_p$ be the set of $C^\omega$ functions in the arguments

$$T^t L^\alpha g^j : t + |\alpha| \leq p, \ j = 1, \cdots, n + 1$$

and $C_{p, q}$ be the subset of $C_p$ of all the $C^\omega$ functions in the arguments

$$T^t L^\alpha g^j : t + |\alpha| \leq p, \ t \leq q, \ j = 1, \cdots, n + 1,$$

and let $\bar{C}_p, \bar{C}_{p, q}$ be the complex conjugate of $C_p$ and $C_{p, q}$, respectively. Then (2.10) implies that $L^\beta g^j \in \bar{C}_k$, for any multi-index $\beta = (\beta_1, \cdots, \beta_m)$.

In particular, for each $i = 1, \cdots, m$

(2.12) \hspace{1cm} L_i g^j \in \bar{C}_k.

Apply $\bar{L}_i$ to (2.12), then by the same calculation as in (2.11) we have

(2.13) \hspace{1cm} T g^j \in \bar{C}_{k+1, k}.

Similarly, for each $i, k = 1, \cdots, m$, we have

(2.14) \hspace{1cm} L_k L_i g^j \in \bar{C}_k.

Apply $\bar{L}_k$ to (2.14), then by (2.12), (2.13), and (2.14) we have

(2.15) \hspace{1cm} T L_i g^j \in \bar{C}_{k+1, k}.

Then by induction on $|\alpha|$, we have

(2.16) \hspace{1cm} T L^\alpha g^j \in \bar{C}_{k+1, k}.

Now apply $\bar{L}_i \bar{L}_k$ to (2.14), then by (2.12)–(2.16) we have

(2.17) \hspace{1cm} T^2 g^j \in \bar{C}_{k+2, k},

and by induction on $|\alpha|$, we have

(2.18) \hspace{1cm} T^2 L^\alpha g^j \in \bar{C}_{k+2, k}.

Then by induction on $t$, we have

(2.19) \hspace{1cm} T^t L^\alpha g^j \in \bar{C}_{k+t, k} \quad \text{for each} \quad j = 1, \cdots, n + 1,
which shows that

\[(2.20) \quad C_{p,q} \subset C_{k+q,k} \quad \text{for any pair } (p, q) \text{ with } p \geq q.\]

Taking the complex conjugate of (2.20), we have

\[(2.21) \quad \overline{C}_{p,q} \subset C_{k+q,k} \quad \text{for any pair } (p, q) \text{ with } p \geq q.\]

In particular, if \(q = k\)

\[(2.22) \quad \overline{C}_{p,k} \subset C_{2k,k} \quad \text{for all } p \geq k.\]

Substitute (2.22) in (2.20), to get

\[(2.23) \quad C_{p,q} \subset C_{2k,k} \quad \text{for any pair } (p, q) \text{ with } p \geq q.\]

In particular, we have

\[(2.24) \quad C_{2k+1} \subset C_{2k}.\]

Now consider the derivatives \(T^tL^{\alpha}L^{\beta}g^j\), where \(t + |\alpha| + |\beta| = 2k + 1\).

If \(|\beta| \neq 0\), this is zero by (2.3). If \(|\beta| = 0\), then (2.24) shows that \(T^tL^{\alpha}g^j, t + |\alpha| = 2k + 1\), can be expressed as a \(C^0\) function in the arguments \(T^tL^{\beta}g^j : t + |\beta| \leq 2k\), thus, \(g\) satisfies a complete system of order \(2k + 1\), which completes the proof. \(\square\)

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