FIXED POINT THEORY FOR MULTIMAPS
IN EXTENSION TYPE SPACES

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ABSTRACT. New fixed point results for the $\mathcal{A}^{c}_{e}$ selfmaps are given. The analysis relies on a factorization idea. The notion of an essential map is also introduced for a wide class of maps. Finally, from a new fixed point theorem of ours, we deduce some equilibrium theorems.

1. Introduction

This paper presents new fixed point results for multivalued selfmaps, in particular the $\mathcal{A}^{c}_{e}(X,X)$ maps. The most general result in the literature [12] assumes $X$ is convex and admissible (in the sense of Klee), but here we will show that it is enough to assume $X$ is an extension space (so it could be an absolute retract), or an approximate extension space, or indeed a neighborhood extension space under some restrictions. In Section 3 we present the notion of an essential map and discuss some of its properties. Section 4 presents some quasi-equilibrium theorems.

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. We will follow mainly [1, 2, 3, 12].

Let $Y$ be a convex subset of a Hausdorff topological vector space $E$. Recall a polytope $P$ in $Y$ is any convex hull of a nonempty finite subset of $Y$. A nonempty subset $X$ of $E$ is said to be admissible (in the sense of Klee) if for every compact subset $K$ of $X$ and every neighborhood $V$ of 0, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of $E$. For example, every convex subset of a Hausdorff locally convex topological vector space is admissible. For other examples, see [12] and references therein.

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Of particular importance in this paper will be the class \( \mathfrak{A}_c^\mathcal{X} \) due to Park. Suppose \( X \) and \( Y \) are topological spaces. Given a class \( \mathcal{X} \) of maps, \( \mathcal{X}(X,Y) \) denotes the set of maps \( F : X \to 2^Y \) (the set of nonempty subsets of \( Y \)) belonging to \( \mathcal{X} \), and \( \mathcal{X}_c \) the set of finite compositions of maps in \( \mathcal{X} \). We let

\[
\mathcal{F}(\mathcal{X}) = \{ X : \text{Fix } F \neq \emptyset \text{ for all } F \in \mathcal{X}(X,X) \},
\]

where \( \text{Fix } F \) denotes the set of fixed points of \( F : X \to 2^X \).

A class \( \mathfrak{A} \) of maps is defined by the following properties:

(i) \( \mathfrak{A} \) contains the class \( \mathfrak{C} \) of single-valued continuous functions;

(ii) each \( F \in \mathfrak{A}_c \) is upper semicontinuous (u.s.c.) and compact-valued; and

(iii) for any polytope \( P \), \( F \in \mathfrak{A}_c(P,P) \) has a fixed point, where the intermediate spaces of compositions are suitably chosen for each \( \mathfrak{A} \).

An \textit{admissible} class \( \mathfrak{A}_c^\mathcal{X}(X,Y) \) of maps \( F : X \to 2^Y \) is one such that, for each \( F \) and each nonempty compact subset \( K \) of \( X \) there exists a map \( G \in \mathfrak{A}_c(X,Y) \) satisfying \( G(x) \subseteq F(x) \) for all \( x \in K \).

Examples of \( \mathfrak{A}_c^\mathcal{X} \) are classes of continuous functions \( \mathfrak{C} \), the Kakutani maps \( \mathfrak{K} \) (u.s.c. with nonempty compact convex values and codomains are convex spaces), the Aronszajn maps \( \mathfrak{M} \) (u.s.c. with \( R_\delta \) values), the acyclic maps \( \mathfrak{V} \) (u.s.c. with compact acyclic values), the Powers maps \( \mathfrak{V}_c \) (finite compositions of acyclic maps), the O’Neill maps \( \mathfrak{N} \) (continuous with values of one or \( m \) acyclic components, where \( m \) is fixed), the approachable maps \( \mathfrak{A} \) (whose domains and codomains are uniform spaces), admissible maps of Göńiewicz, \( \sigma \)-selectional maps of Haddad and Lasry, permissible maps of Dzedzej, the class \( \mathfrak{K}_c^+ \) of Lassonde, the class \( \mathfrak{V}_c^+ \) of Park \textit{et al.}, and approximable maps of Ben-El-Mechaiekh and Idzik, and others. For details on the admissible classes, see [12].

In [12] Park gave an elementary proof of the following result.

\textbf{Theorem 1.1.} \textit{Let} \( E \) \textit{be a Hausdorff topological vector space and} \( X \) \textit{an admissible, convex, compact subset of} \( E \). \textit{Then any map} \( F \in \mathfrak{A}_c^\mathcal{X}(X,X) \) \textit{has a fixed point.}

A class of maps \( \mathcal{R}(X,Y) \) is said to be \textit{admissible} (in the sense of Ben-El-Mechaiekh and Deguire [3]) if

(i) \( \mathcal{R} \) contains the class \( \mathfrak{C} \); and

(ii) each \( F \in \mathcal{R}_c \) is upper semicontinuous and closed-valued.
The following result is given in [3, Proposition 2.2].

**Theorem 1.2.** Let \( R \) be an admissible class of maps. Then the Hilbert cube \( I^\infty \) (subset of \( l^2 \) consisting of points \((x_1, x_2, \ldots)\) with \(|x_i| \leq 1/i \) for all \( i \)) and the Tychonoff cube \( T \) (cartesian product of copies of the unit interval imbedded in a normed space) are in \( F(R_c) \) provided the closed unit ball \( B^n = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \} \) is in \( F(R_c) \) for all \( n \geq 1 \).

From Theorem 1.1 or 1.2, we immediately have the following.

**Theorem 1.3.** \( I^\infty \) and \( T \) are in \( F(A^\infty_c) \).

**Remark 1.1.** It is worth remarking that we do not need to introduce the class \( R \) (we did so to give credit to the authors in [1, 3]) since if we assume \( B^n \in F(R_c) \), then since \( B^n \) is a homeomorphic image of a polytope, we have for any polytope \( P \), that \( F \in R_c(P, P) \) has a fixed point, where the intermediate spaces of compositions are suitably chosen for each \( R \). Thus Theorem 1.3 follows immediately from [3] and the fact that \( B^n \in F(A^\infty_c) \).

**Remark 1.2.** Since \( I^\infty \) and \( T \) are compact, Theorem 1.3 is actually equivalent to

1. \( I^\infty \) and \( T \) are in \( F(A^\infty_c) \).

However, considering a Browder type map \( F \) (having nonempty convex values and open fibers), we notice that \( F \notin A^\infty_c \) but \( F \in A^\infty_c \).

For a subset \( K \) of a topological space \( X \), we denote by \( Cov_X(K) \) the directed set of all coverings of \( K \) by open sets in \( X \) (usually we write \( Cov(K) = Cov_X(K) \)). Given a map \( F : X \to 2^X \) and \( \alpha \in Cov(X) \), a point \( x \in X \) is said to be an \( \alpha \)-fixed point of \( F \) if there exists a member \( U \in \alpha \) such that \( x \in U \) and \( F(x) \cap U \neq \emptyset \).

Given two maps \( F, G : X \to 2^Y \) and \( \alpha \in Cov(Y) \), \( F \) and \( G \) are said to be \( \alpha \)-close, if for any \( x \in X \) there exist \( U_x \in \alpha \), \( y \in F(x) \cap U_x \), and \( w \in G(x) \cap U_x \).

### 2. Extension type spaces and fixed points

In this section, we show that various extension type spaces have the fixed point property with respect to the \( A^\infty_c \) selfmaps. For details and examples of such extension type spaces, see [1, 3] and references therein.

In the definitions in this section by a space we mean a Hausdorff topological space.
Let $Q$ be a class of topological spaces. A space $Y$ is an extension space for $Q$ (written $Y \in ES(Q)$) if for any pair $(X,K)$ in $Q$ with $K \subset X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

We now present a new fixed point result for the $\mathfrak{A}_c^\infty$ maps.

**Theorem 2.1.** Let $X \in ES(\text{compact})$ and $F \in \mathfrak{A}_c^\infty(X,X)$ a compact map. Then $F$ has a fixed point.

**Proof.** It is known [8] that every compact space is homeomorphic to a closed subset of the Tychonoff cube $T$, so as a result $K = F(X)$ can be embedded as a closed subset $K^*$ of $T$; let $s : K \to K^*$ be a homeomorphism. Also let $i : K \hookrightarrow X$ and $j : K^* \hookrightarrow T$ be inclusions. Now since $X \in ES(\text{compact})$ and $i s^{-1} : K^* \to X$, then $i s^{-1}$ extends to a continuous function $h : T \to X$. Let $G = j s F h$ and notice $G \in \mathfrak{A}_c^\infty(T,T)$. Hence, Theorem 1.3 guarantees that there exists $x \in T$ with $x \in G(x)$. Let $y = h(x)$, so

$$y \in h j s F(y) \quad \text{i.e.} \quad y = h j s(q) \quad \text{for some} \quad q \in F(y).$$

Since $h j (z) = i s^{-1}(z)$ for $z \in K^*$, we have $h j s(q) = (h j) s(q) = i(q) = q$, and so $y \in F(y)$. \qed

**Remark 2.1.** If $X \in AR$ (an absolute retract as defined in [5]) then of course $X \in ES(\text{compact})$ [We know from the Arens–Eells theorem that $X$ is $r$–dominated by a normed space $E$ so there exist maps $r : E \to X$ and $s : X \to E$ with $rs = 1$. Now since any normed space is $ES(\text{compact})$, it follows immediately that $X \in ES(\text{compact})$]. So a special case of Theorem 2.1 occurs if $X \in AR$.

A space $Y$ is an approximate extension space for $Q$ (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair $(X,K)$ in $Q$ with $K \subset X$ closed, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|_K$ is $\alpha$–close to $f_0$.

We now extend Theorem 2.1 to approximate extension spaces. To prove this we need the following elementary result for $\alpha$–fixed points (see [1, Lemma 1.2]).

**Lemma 2.2.** Let $X$ be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subset Cov(X)$ such that $F$ has an $\alpha$–fixed point for every $\alpha \in \theta$. Then $F$ has a fixed point.

**Theorem 2.3.** Let $X \in AES(\text{compact})$ and $F \in \mathfrak{A}_c^\infty(X,X)$ a compact map. Then $F$ has a fixed point.
Proof. Let $K$, $K^*$, $s$, $i$ and $j$ be as in the proof of Theorem 2.1. Let $\alpha \in Cov(X)$ and let $h : T \rightarrow X$ be such that $h$ and $i s^{-1}$ are $\alpha$-close on $K^*$ (guaranteed since $X \in AES(\text{compact})$). Let $G \in \mathfrak{A}_n^*(T, T)$. Now Theorem 1.3 guarantees that there exists $x \in T$ with $x \in G(x)$. Let $y = h(x)$, so

$$ y \in h j s F(y) \quad \text{i.e.} \quad y = h j s(q) \quad \text{for some} \quad q \in F(y). $$

Since $i s^{-1}$ and $h$ are $\alpha$-close on $K^*$ there exists $U \in \alpha$ with $i s^{-1}(s(q)) \in U$ and $h j s(q) \in U$ i.e. $q \in U$ and $y \in U$. Thus

$$ y \in U \quad \text{and} \quad F(y) \cap U \neq \emptyset \quad \text{since} \quad q \in F(y). $$

As a result $F$ has an $\alpha$-fixed point. Since $\alpha$ is arbitrary, Lemma 2.2 guarantees that $F$ has a fixed point. \qed

**Definition 2.1.** Let $V$ be a subset of a Hausdorff topological vector space $E$. Then we say $V$ is Schauder admissible if for every compact subset $K$ of $V$ and every covering $\alpha \in Cov(V)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \rightarrow V$ such that

(i) $\pi_\alpha$ and $i : K \hookrightarrow V$ are $\alpha$-close;

(ii) $\pi_\alpha(K)$ is contained in a subset $C \subset V$ with $C \in AES(\text{compact})$.

If $V \in AES(\text{compact})$ then $V$ is trivially Schauder admissible. If $V$ is an open convex subset of a Hausdorff locally convex topological vector space $E$, then it is well known [1, Lemma 4.8] that $V$ is Schauder admissible.

We next present a result of Himmelberg type [9].

**Theorem 2.4.** Let $V$ be a Schauder admissible subset of a Hausdorff topological vector space $E$ and $F \in \mathfrak{A}_n^*(V, V)$ a compact map. Then $F$ has a fixed point.

Proof. Since $F(V) \subset K$, $K$ compact, for each $\alpha \in Cov(V)$ there exist $\pi_\alpha : K \rightarrow V$ (as described in Definition 2.1) and a subset $C \subset V$ with $C \in AES(\text{compact})$ such that, by putting $F_\alpha \equiv \pi_\alpha F$,

$$ F_\alpha(V) = \pi_\alpha F(V) \subset C. $$

Notice $F_\alpha \in \mathfrak{A}_n^*(C, C)$ so Theorem 2.3 guarantees that there exists $x \in C$ with $x \in \pi_\alpha F(x)$ i.e. $x = \pi_\alpha(q)$ for some $q \in F(x)$. Now Definition 2.1 (i) guarantees that there exists $U \in \alpha$ with $\pi_\alpha(q) \in U$ and $i(q) \in U$ i.e. $q \in U$ and $q \in U$. Thus

$$ x \in U \quad \text{and} \quad F(x) \cap U \neq \emptyset \quad \text{since} \quad q \in F(x). $$
As a result $F$ has an $\alpha$-fixed point. Since $\alpha$ is arbitrary, Lemma 2.2 guarantees that $F$ has a fixed point. □

A space $Y$ is a neighborhood extension space for $Q$ (written $Y \in NES(Q)$) if for any pair $(X, K)$ in $Q$ with $K \subset X$ closed and any continuous function $f_0 : K \to Y$ there is a continuous extension $f : U \to Y$ of $f_0$ over a neighborhood $U$ of $K$ in $X$.

We would like to extend Theorem 2.3 to neighborhood extension spaces. However even in the case when $F$ is admissible in the sense of Görniewicz [6] extra conditions need to be added (recall that maps admissible in the sense of Görniewicz are in the class $\mathcal{A}_c^s$).

Recall the following well known result [1, Lemma 4.7].

**Lemma 2.5.** Let $T$ be a Tychonoff cube contained in a Hausdorff topological vector space. Then $T$ is a retract of $span(T)$.

Let $X \in NES(\text{compact})$ and $F \in \mathcal{A}_c^s(X, X)$ a compact map.

Let $K, K^*$, s and $i$ be as in the proof of Theorem 2.1. Let $U$ be an open neighborhood of $K^*$ in $T$ and $h : U \to X$ be a continuous extension of $i s^{-1} : K^* \to X$ on $U$ (guaranteed since $X \in NES(\text{compact})$).

Let $j : K^* \hookrightarrow U$ be the natural embedding so $h j = i s^{-1}$. Now consider $span(T)$ in a Hausdorff locally convex topological vector space containing $T$. Now Lemma 2.5 guarantees that there exists a retraction $r : span(T) \to T$. Let $i^* : U \hookrightarrow r^{-1}(U)$ be an inclusion and consider $G = i^* j s F h r$. Notice $G \in \mathcal{A}_c^s(r^{-1}(U), r^{-1}(U))$. Assume

\begin{equation}
G \in \mathcal{A}_c^s(r^{-1}(U), r^{-1}(U)) \text{ has a fixed point.}
\end{equation}

If (2.1) is true then there exists $x \in r^{-1}(U)$ with $x \in G x$. Let $y = h r (x)$, so

\[ y = h r i^* j s F (y) \quad \text{i.e.} \quad y = h r i^* j s (q) \quad \text{for some} \quad q \in F(y). \]

Since $h(z) = i s^{-1}(z)$ for $z \in K^*$, we have $h r i^* j s(q) = (h r i^*) s(q) = i(q)$, and so $y \in F(y)$.

Thus existence of a fixed point of $F$ is guaranteed if (2.1) is satisfied; recall $G = i^* j s F h r$ and $r^{-1}(U)$ is an open subset of a Hausdorff locally convex topological vector space.

For specific classes of maps (2.1) is known to be true. For example, if $F$ is admissible in the sense of Görniewicz [6] and the Lefschetz set $\Lambda(F) \neq \{0\}$ then we know [6] that (2.1) holds. More generally, if $F$ is permissible in the sense of Dzedzej [7] and $\Lambda(F) \neq \{0\}$ then (2.1) holds. It would be of interest to know other examples.
3. Essential maps

Throughout this section $Y$ will be a completely regular topological space with $Y \in AES(\text{compact})$, so in particular the results in this section will hold if $Y \in ES(\text{compact})$ or $Y \in AR$. [Of course $Y \in AES(\text{compact})$ could be replaced by $Y$ Schauder admissible in this section]. Also $U$ will be an open subset of $Y$. In this section we consider a subclass $\mathcal{A}$ of $\mathcal{X}$. The subclass must have the following property: for subsets $X_1$, $X_2$ and $X_3$ of Hausdorff topological vector spaces

if $F \in \mathcal{A}(X_2, X_3)$ and $f \in \mathcal{C}(X_1, X_2)$, then $F f \in \mathcal{A}(X_1, X_3)$.

The theory in this section will work for any class of maps $\mathcal{A}$ which satisfy a normalization property. In particular one can view the class $\mathcal{A}$ as any class where we can get a Leray–Schauder type result. For example we could take $\mathcal{A}$ to be $\mathcal{V}$ since clearly (3.3) (and (3.4), (3.5)) hold.

**Definition 3.1.** $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ if $F \in \mathcal{A}(\bar{U}, Y)$ with $F$ compact and $x \notin F(x)$ for $x \in \partial U$.

**Definition 3.2.** A map $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ is essential if for every $G \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

**Theorem 3.1.** (Homotopy Invariance) Let $Y$ and $U$ be as above. Suppose $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ is an essential map and $H \in \mathcal{A}(\bar{U} \times [0,1], Y)$ is a compact map. Also assume the following two properties hold:

(3.1) $H(x, 0) = F(x)$ for $x \in \bar{U}$

and

(3.2) $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0,1]$ (here $H_t(x) = H(x,t)$).

Then $H_1$ has a fixed point in $U$.

**Proof.** Let

$$B = \{ x \in \bar{U} : x \in H_t(x) \text{ for some } t \in [0,1] \}.$$ 

When $t = 0$, $H_t = F$ and since $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$ is essential there exists an $x \in U$ with $x \in F(x)$. Thus $B \neq \emptyset$. Since $H$ is upper semicontinuous and compact, it is immediate that $B$ is closed and compact. In addition (3.2) (together with $F \in \mathcal{A}_{\partial U}(\bar{U}, Y)$) implies $B \cap \partial U = \emptyset$. Thus (since $Y$ is completely regular) there exists a continuous function $\mu : \bar{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map $R$ by $R(x) = H(x, \mu(x))$ for $x \in \bar{U}$. Let $j : \bar{U} \to \bar{U} \times [0,1]$ be given
by \( j(x) = (x, \mu(x)) \). Note \( j \) is continuous so \( R = H \subseteq \mathcal{A}(\overline{U}, Y) \). In addition, \( R \) is compact since \( H \) is. Also notice for \( x \in \partial U \) that \( R(x) = H_0(x) = F(x) \), and so \( R \in \mathcal{A}_{\partial U}(\overline{U}, Y) \). Now \( R|_{\partial U} = F|_{\partial U} \) and \( F \in \mathcal{A}_{\partial U}(\overline{U}, Y) \) essential implies that there exists \( x \in U \) with \( x \in R(x) \) (i.e. \( x \in H_{\mu(x)}(x) \)). Thus \( x \in B \) and so \( \mu(x) = 1 \). Consequently \( x \in H_1(x) \). \( \square \)

Next we give an example of an essential map.

**Theorem 3.2.** (Normalization) Let \( Y \) and \( U \) be as above with \( 0 \in U \). Suppose the following condition is satisfied:

\[
\begin{cases}
\text{for any map } \theta \in \mathcal{A}_{\partial U}(\overline{U}, Y) \text{ with } \theta|_{\partial U} = \{0\}, \\
\text{the map } J \text{ is in } \mathcal{A}_E^c(Y, Y); \text{ here} \\
J(x) = \begin{cases} \\
\theta(x), & x \in \overline{U} \\
\{0\}, & x \in Y \setminus \overline{U}.
\end{cases}
\end{cases}
\] (3.3)

Then the zero map is essential in \( \mathcal{A}_{\partial U}(\overline{U}, Y) \).

**Proof.** Let \( \theta \in \mathcal{A}_{\partial U}(\overline{U}, Y) \) with \( \theta|_{\partial U} = \{0\} \). We must show that there exists \( x \in U \) with \( x \in \theta(x) \). Define a map \( J \) by

\[
J(x) = \begin{cases} \\
\theta(x), & x \in \overline{U} \\
\{0\}, & x \in Y \setminus \overline{U}.
\end{cases}
\]

From (3.3) we know \( J \in \mathcal{A}_E^c(Y, Y) \). Clearly \( J \) is compact since \( \theta \) is. Hence, Theorem 2.3 implies that there exists \( x \in Y \) with \( x \in J(x) \). Now if \( x \notin U \) we have \( x \in J(x) = \{0\} \), which is a contradiction since \( 0 \in U \). Thus \( x \in U \) so \( x \in J(x) = \theta(x) \). \( \square \)

Next we present another version of the normalization property when we are in the topological vector space setting. Let \( Y \in AES(\text{compact}) \) be a convex subset of a topological vector space \( E \) and let \( U \) be an open subset of \( Y \) with \( 0 \in U \). In addition assume there exists a continuous retraction \( r : Y \to \overline{U} \).

**Theorem 3.3.** (Normalization) Let \( E, Y, U \) and \( r \) be as above and suppose the following condition is satisfied:

\[
\begin{cases}
\text{for any continuous function } \mu : Y \to [0,1] \text{ and} \\
\text{any map } G \in \mathcal{A}(Y, Y) \text{ we have } \mu G \in \mathcal{A}_E^c(Y, Y).
\end{cases}
\] (3.4)

Then the zero map is essential in \( \mathcal{A}_{\partial U}(\overline{U}, Y) \).

**Proof.** Let \( \theta \in \mathcal{A}_{\partial U}(\overline{U}, Y) \) with \( \theta|_{\partial U} = \{0\} \). Let

\[
A = \{ x \in \overline{U} : x \in \lambda \theta(x) \text{ for some } \lambda \in [0,1] \}.
\]
Now $A \neq \emptyset$ is compact and $A \subset U$ (this is clear since $0 \in U$ and $\theta|_{\partial U} = \{0\}$). Thus there exists a continuous function $\mu : Y \to [0,1]$ with $\mu(A) = 1$ and $\mu(Y \setminus U) = 0$. Define a map $J_0$ by

$$J_0(x) = \mu(x) \theta(\tau(x)) \quad \text{for} \quad x \in Y.$$  

Note $\theta \tau \in A(Y, Y)$ so $J_0 \in A_0^c(Y, Y)$ from (3.4). Theorem 2.3 implies that there exists $x \in Y$ with $x \in \mu(x) \theta(\tau(x))$. If $x \in Y \setminus U$ then $\mu(x) = 0$, a contradiction since $0 \in U$. Thus $x \in U$ and so $x \in \mu(x) \theta(x)$. As a result $x \in A$, so $\mu(x) = 1$. Consequently $x \in \theta(x)$. 

Of course we can obtain a nonlinear alternative of Leray–Schauder type by combining Theorems 3.1 and 3.3. In fact, we can obtain a more general result.

**Theorem 3.4.** Let $E$, $Y$, $U$ and $r$ be as above. Suppose $F \in A(\overline{U}, Y)$ satisfies (3.4) and assume the following conditions hold:

$$\begin{align*}
&\text{for any continuous function } \mu : \overline{U} \to [0,1] \text{ and } \\
&\text{any map } G \in A(\overline{U}, Y) \text{ we have } \mu G \in A(\overline{U}, Y)
\end{align*}$$

and

$$x \notin \lambda F(x) \quad \text{for every } x \in \partial U \text{ and } \lambda \in (0,1].$$

Then $F$ is essential in $A_{\partial U}(\overline{U}, Y)$.

**Proof.** Let $\Phi \in A_{\partial U}(\overline{U}, Y)$ with $\Phi|_{\partial U} = F|_{\partial U}$. We must show $\Phi$ has a fixed point in $U$. Let

$$D = \{x \in \overline{U} : x \in \lambda \Phi(x) \text{ for some } \lambda \in [0,1]\}.$$  

Now $D \neq \emptyset$ is compact and $D \cap \partial U = \emptyset$ (note (3.6) with $\Phi|_{\partial U} = F|_{\partial U}$ and that $0 \in U$). Thus there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R$ by $R(x) = \mu(x) \Phi(x)$. Now (3.5) guarantees that $R \in A(\overline{U}, Y)$. Also $R$ is compact with $R|_{\partial U} = \{0\}$. Now since $R \in A_{\partial U}(\overline{U}, Y)$ and since the zero map is essential in $A_{\partial U}(\overline{U}, Y)$ (Theorem 3.3) there exists $x \in U$ with $x \in R(x)$. Thus $x \in D$ and so $\mu(x) = 1$, i.e., $x \in \Phi(x)$. 

**4. Quasi-equilibrium theorem**

We begin this section by expressing Theorem 2.4 as an equilibrium theorem. Then a general result will be deduced from our main theorem.

**Theorem 4.1.** Let $E$ and $Y$ be Hausdorff topological vector spaces, $Q$ a subset of $E$, $G : Q \to k(Q)$ (nonempty compact subsets of $Q$) and
\[ T : Q \to 2^C \text{ where } C \text{ is a subset of } Y \text{. In addition assume the following conditions hold:} \]

(4.1) \[ f : Q \times C \times Q \to \mathbb{R} \text{ is a upper semicontinuous function,} \]

(4.2) \[ G \text{ and } T \text{ are compact maps,} \]

(4.3) \[ Q \times C \text{ is an Schauder admissible subset of } E \times Y, \]

and

(4.4) \[ F \in \mathfrak{A}_c^c(Q \times C, Q \times C); \]

here \( F(x, y) = \Phi(x, y) \times T(x) \) for \((x, y) \in Q \times C\) with

\[ \Phi(x, y) = \{ w \in G(x) : f(x, y, w) = M(x, y) \} \]

and \( M(x, y) = \max_{w \in G(x)} f(x, y, w) \). Then there exist \((x_0, y_0) \in Q \times C, x_0 \in G(x_0), \) and \( y_0 \in T(x_0) \) with

\[ f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0). \]

If in addition

(4.5) \[ f(x, y, x) \leq 0 \text{ for all } (x, y) \in Q \times C, \]

then there exists \((x_0, y_0) \in Q \times C \) such that \( x_0 \in G(x_0), y_0 \in T(x_0), \) and

\[ f(x_0, y_0, z) \leq 0 \text{ for all } z \in G(x_0). \]

**Proof.** Notice \( \Phi(x, y) \) is nonempty (and compact) for each \((x, y) \in Q \times C\). As a result \( F : Q \times C \to 2^{Q \times C} \) and also \( F \) is compact since \( F(Q \times C) \subseteq G(Q) \times T(Q) \). Now Theorem 2.4 guarantees that there exists \((x_0, y_0) \in Q \times C \) with \((x_0, y_0) \in \Phi(x_0, y_0) \times T(x_0) \). That is, there exists \((x_0, y_0) \in Q \times C \) with \( x_0 \in G(x_0), y_0 \in T(x_0) \) and \( f(x_0, y_0, x_0) = M(x_0, y_0) \) (i.e., \( f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \) for all \( z \in G(x_0) \)), so we are finished the first part. For the second part assume (4.5) holds, and so the result is immediate from the first part. \( \square \)

Next we consider a subclass \( D \) of \( \mathfrak{A}_c^c \). If \( X \) and \( Y \) are subsets of Hausdorff topological vector spaces then we say \( F \in D(X, Y) \) if \( F \in \mathfrak{A}_c^c(X, Y) \) and is upper semicontinuous with nonempty compact values and satisfies Property (C) (to be specified in the examples considered). Also we assume for subsets \( X_1 \) and \( X_2 \) of Hausdorff topological vector spaces

(4.6) \[
\begin{cases}
  \text{if } F_1 \in D(X_1 \times X_2, X_1) \text{ and } F_2 \in D(X_1, X_2) \\
  \text{then } F_3 \in \mathfrak{A}_c^c(X_1 \times X_2, X_1 \times X_2);
\end{cases}
\]
here $F_3(x, y) = F_1(x, y) \times F_2(x)$. A typical example of a class $\mathcal{D}$ is the acyclic maps $\mathcal{V}$ (i.e., Property (C) means the map is acyclic valued).

**Theorem 4.2.** Let $E$ and $Y$ be Hausdorff topological vector spaces, $Q$ a subset of $E$, $G : Q \to k(Q)$ and $T : Q \to k(C)$ where $C$ is a subset of $Y$. Suppose (4.1), (4.2), (4.3), (4.6) hold and in addition assume the following conditions are satisfied:

(4.7) \hspace{1cm} G : Q \to 2^Q \text{ is upper semicontinuous}

\begin{align*}
\{ & M : Q \times C \to Q \text{ is lower semicontinuous} \\
& \text{(here } M(x, y) = \max_{w \in G(x)} f(x, y, w)) \}
\end{align*}

(4.9) \hspace{1cm} T \in \mathcal{A}^c_c(Q, C) \text{ is upper semicontinuous and satisfies Property (C)}

and

(4.10) \hspace{1cm} \Phi \in \mathcal{A}^c_c(Q \times C, Q) \text{ and satisfies Property (C)};

here

$$\Phi(x, y) = \{ w \in G(x) : f(x, y, w) = M(x, y) \}.$$  

Then there exists $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with

$$f(x_0, y_0, z) \leq f(x_0, y_0, x_0) \text{ for all } z \in G(x_0).$$

If in addition (4.5) holds, then there exists $(x_0, y_0) \in Q \times C$, $x_0 \in G(x_0)$ and $y_0 \in T(x_0)$ with $f(x_0, y_0, z) \leq 0$ for all $z \in G(x_0)$.

**Proof.** The result follows from Theorem 4.1 once we show (4.4) holds. First we show $\Phi$ is upper semicontinuous. To show this it suffices (note $\Phi$ is compact) to show $\Phi$ is closed. Let $\{ (x_\alpha, y_\alpha, w_\alpha) \}$ be a net in $\text{graph}(\Phi)$ with $(x_\alpha, y_\alpha, w_\alpha) \to (x, y, w)$. From (4.8) it follows that

$$f(x, y, w) \geq \lim \sup f(x_\alpha, y_\alpha, w_\alpha) \geq \lim \inf M(x_\alpha, y_\alpha) \geq M(x, y).$$

Also $w_\alpha \in G(x_\alpha)$ together with $x_\alpha \to x$, $w_\alpha \to w$ and $G$ upper semicontinuous (so $G$ is closed) implies $w \in G(x)$ and $f(x, y, w) \geq M(x, y)$. Consequently $f(x, y, w) = M(x, y)$, so $(x, y, w) \in \text{graph}(\Phi)$. Thus $\Phi$ is upper semicontinuous with nonempty, compact values, so this together with (4.10) implies $\Phi \in \mathcal{D}(Q \times C, Q)$. Also (4.2) and (4.9) guarantees that $T \in \mathcal{D}(Q, C)$. As a result $F \in \mathcal{A}^c_c(Q \times C, Q \times C)$ from (4.6); here $F(x, y) = \Phi(x, y) \times T(x)$. Thus (4.4) holds.

For the motivation and some related results in this section, the reader can refer to [4, 10, 11, 13, 14, 16].
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References


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