INFINITE FLOCKS OF QUADRATIC CONES—II GENERALIZED FISHER FLOCKS

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ABSTRACT. This article discusses a new representation of the generalized Fisher flocks and shows that there is a unique flock for each full field $K$ of odd or zero characteristic that has a full field quadratic extension. It is also shown that partial flock extensions of 'critical linear subflocks' are completely determined.

1. Introduction

In Jha and Johnson [2], flocks of quadratic cones are considered within $PG(3, K)$, where $K$ is an arbitrary field. When $K$ is infinite, the authors develop a net replacement procedure that is called 'elation-nest replacement' or 'E-nest replacement'. The construction generalizes the $q$-nest construction given by Baker and Ebert [1], when $q$ is a prime and generalized by [6], for arbitrary odd order $q$. The translation planes corresponding to flocks of quadratic cones in $PG(3, K)$ admit an elation group $E$ with axis $\ell$ such that for any line $m$ of $PG(3, K)$ disjoint from $\ell$, $Em \cup \ell$ is a regulus. When $K$ is finite isomorphic to $GF(q)$, the order of $E$ is $q$. In general, such an elation group is said to be 'regulus-inducing'.

In the following, it is assumed that a 'Baer subplane' is always a 2-dimensional vector subspace over the kernel field $K$ that is not a 'line' of the spread in question.

The translation planes constructed by Payne and, by Baker and Ebert are constructed from a Desarguesian affine plane $\Sigma$ using a regulus-inducing group $E$ and a kernel homology group $H$ of order $(q + 1)$. Basically, a Baer subplane $\pi_o$ of $\Sigma$ is determined so that $EH\pi_o$ is a partial spread that covers a set of reguli of $\Sigma$ that are induced using $E$. If $R$ denotes the reguli sharing $x = 0$ of $\Sigma$ remaining that are not covered by the images of $\pi_o$, then there is a spread $EH\pi_o \cup R$. In this case, the
number of reguli in $\mathcal{R}$ is $(q - 1)/2$. Payne and Thas [7] have shown that the only finite flocks of quadratic cones that share a linear subflock of $(q - 1)/2$ conics are the Fisher flocks and the linear flocks (corresponding to Desarguesian affine plane). More generally, Johnson [4] has shown that, in fact, any non-linear partial flock in $PG(3, q)$ sharing a linear subflock of at least $(q - 1)/2$ conics may be uniquely extended to a Fisher flock.

Considering what might be a generalization of having such a maximum linear subflock, we define what we call a 'critical' linear subflock as follows:

**Definition 1.** Let $\mathcal{P}$ be a linear partial flock of a quadratic cone in $PG(3, K)$ where $K$ is a field. Assume that there is a flock $\mathcal{L}$ in $PG(3, K)$ containing $\mathcal{P}$. Let the partial spreads corresponding to $\mathcal{P}$ and $\mathcal{L}$ be denoted by $\Pi$ and $\Sigma$ respectively and note that $\Pi \subseteq \Sigma$. Then, there is a regulus-inducing elation group $E$ with axis $\ell$ such that $\Sigma$ is a union of reguli sharing $\ell$ and each regulus is induced from $E$. These reguli are called the 'base reguli'. Note that $\Pi$ is invariant under $E$ so is also a union of reguli sharing $\ell$.

We shall say that $\mathcal{P}$ is a 'critical partial flock' if and only if the following two conditions hold:

(i) Every Baer subplane within the affine plane defined by $\Sigma$ and disjoint from $\Pi$ intersects each base regulus of $\Sigma - \Pi$ in two components and there is some Baer subplane which is disjoint from $\Pi$,

(ii) if $\mathcal{C}$ is a set of distinct reguli sharing $\ell$, invariant under $E$, that covers $\Sigma - \Pi$, then every Baer subplane within $\Sigma$ that is disjoint from $\Pi$ and not in one of the reguli of $\mathcal{C}$ intersects exactly two components of each regulus of $\mathcal{C}$.

**Remark 1.** Any linear subset of $(q - 1)/2$ reguli in a spread of $PG(3, q)$ that is a union of reguli sharing a component $\ell$ is critical.

**Proof.** There are exactly $(q + 1)/2$ remaining reguli and a Baer subplane disjoint from $\Pi$ cannot be a Baer subplane of one of these reguli and therefore shares 0, 1 or 2 lines with each such regulus. However, this implies that there are exactly two shared lines with each regulus.

There are exactly $q(q + 1)/2$ components in the remaining reguli so if there is a covering of this set by a set $\mathcal{C}$ of reguli such that $\mathcal{C}$ is invariant under $E$ then there are exactly $(q + 1)/2$ reguli in $\mathcal{C}$. If $\pi_o$ is any Baer subplane of $\Sigma$ that lies within this set and is not within one of the reguli of $\mathcal{C}$ then $\pi_o$ has $q + 1$ components and cannot be an opposite line of any
of the reguli since \( \pi_\alpha \) is disjoint from \( \ell \). Hence, if \( \pi_\alpha \) is not a line of one of the reguli of \( \mathcal{C} \), then \( \pi_\alpha \) shares 0, 1, 2 components of each. However, since there are but \((q + 1)/2\) reguli, it follows that \( \pi_\alpha \) shares exactly two components with each regulus of \( \mathcal{C} \).

In this article, we consider the so-called 'generalized Fisher' planes, defined as those planes of possibly infinite order that may be obtained using infinite \( E \)-nest replacement.

In particular, in Jha and Johnson [2], there is an open question as to whether there could be two non-isomorphic generalized Fisher planes arising from different nest replacements using the same field \( K \) and quadratic field extension \( K[\theta] \), where both \( K \) and \( K[\theta] \) are full fields of characteristic odd or 0. (In this case, a full field is a field such that the non-zero squares form an index two subgroup of the multiplicative group.)

Furthermore, we consider non-linear partial flocks containing critical linear subflocks and ask whether there is an extension to a flock and whether such extensions are generalized Fisher flocks.

Assuming that critical linear subflocks exist, we are able to show that any partial flock containing a critical linear subflock may be uniquely extended either to a linear flock or to a generalized Fisher flock.

Furthermore, we develop a new representation of generalized Fisher flocks in \( PG(3, K) \) using the Galois group of \( K[\theta] \) over \( K \), which allows us to prove in general that there is a unique generalized Fisher flock over any full field \( K \) of characteristic odd or 0 that admits a quadratic extension full field \( K[\theta] \).

2. Representation of generalized Fisher flocks

In this section, we develop a new representation of generalized Fisher flocks and, show that, in fact, there is always a unique generalized Fisher flock when there is at least one in \( PG(3, K) \).

We assume that \( K \) is a full field of characteristic odd or 0 and that \( K[\theta] \) is a full field quadratic extension.

Let \( \sigma \in Gal_{K} K[\theta] \), \( \sigma \neq 1 \).

**Lemma 1.** All elements of \( K \) and of \( \{x^{\sigma^{-1}}; x \in K[\theta]\} \) are squares in \( K[\theta] \).

**Proof.** Let \( \{1, e\} \) be a \( K \)-basis for \( K[\theta] \) such that \( e^2 = \gamma \), for \( \gamma \) a non-square in \( K \) (since \( K \) has odd or 0 characteristic, this is possible).
Then \((e\alpha + \beta)^2 = \beta^2 + \gamma\alpha^2 + 2\alpha\beta e\). Hence, if \(\alpha\beta = 0\) then we obtain either \(\beta^2\) or \(\gamma\alpha^2\) and since we have an index two group of squares in \(K\), it follows that all elements of \(K\) are squares in \(K[\theta]\). Now \(x^{\sigma-1} = x^{\sigma+1}x^{-2}\), implying that \(x^{\sigma-1}\) is a square since \(x^{\sigma+1}\) is in \(K\) and a square in \(K[\theta]\) by the previous argument.

**Notation 1.** Since \(x^{\sigma-1} = z^2\), we write \(z = x^{(\sigma-1)/2}\), the ‘positive square root’.

**Lemma 2.** If \(\alpha\) is a non-zero square in \(K\) then \(\alpha^{(\sigma-1)/2} = 1\).

**Proof.** If \(\alpha = \delta^2\) then \(\delta^{(\sigma-1)/2} = \delta^{\sigma-1} = 1\) since \(\delta^\sigma = \delta\).

**Lemma 3.** Under the previous assumptions, let \(b\) be in the subgroup of squares in \(K[\theta]\). Then

\[(b^{1-\sigma} - 1)^{\sigma+1}\] is square in \(K\) if \(-1\) is a non-square in \(K\) and non-square in \(K\) if \(-1\) is a square in \(K\).

**Proof.** To see this, note that

\[(b^{1-\sigma} - 1)^{\sigma+1} = 2 - (b^{\sigma-1} + b^{1-\sigma}) = -(b^{(1-\sigma)/2} - b^{(\sigma-1)/2})^2.

We claim that

\[b^{\sigma(\sigma-1)/2} = b^{(1-\sigma)/2}.

This is true if and only if

\[b^{\sigma(\sigma-1)/2 - (1-\sigma)/2} = 1 = b^{((\sigma-1)/2)(\sigma+1)} = b^{(\sigma-1)/2},

which is valid since \(b\) is a square in \(K[\theta]\).

Then,

\[-(b^{(1-\sigma)/2} - b^{(\sigma-1)/2})^2\] is a square in \(K\),

implies that

\[-(1)^{(\sigma-1)/2}((b^{(1-\sigma)/2} - b^{(\sigma-1)/2})^2)^{(\sigma-1)/2} = (1)^{(\sigma-1)/2}(b^{(1-\sigma)/2} - b^{(\sigma-1)/2})^{\sigma-1}\]

\[= (1)^{(\sigma-1)/2}(b^{(1-\sigma)/2} - b^{(\sigma-1)/2})^{\sigma-1}\]

\[= (1)^{(\sigma-1)/2}(b^{(\sigma-1)/2} - b^{(1-\sigma)/2})/(b^{(1-\sigma)/2} - b^{(\sigma-1)/2})\]

\[= (1)^{(\sigma-1)/2}(b^{(\sigma-1)/2} - b^{(1-\sigma)/2})/(b^{(1-\sigma)/2} - b^{\sigma-1})\]

\[= (1)^{(\sigma-1)/2}(-1) = (-1)^{(\sigma+1)/2},

which is a contradiction if \(-1\) is a square in \(K\), since then \((-1)^{(\sigma-1)/2} = 1\). Hence, assume that \(-1\) is a non-square in \(K\). Let \(\gamma = -1\) so that \(e^2 = -1\) and \(e^{2(\sigma+1)/2} = e^{\sigma+1} = -e = 1\). Thus, we have completed the proof of the lemma.
Theorem 1. Let $K$ be a full field of odd or 0 characteristic and let $K[\theta]$ be a quadratic extension of $K$ that is also a full field. Let $\Sigma$ be the Pappian affine plane coordinatized by $K[\theta]$ and let $H$ be the kernel homology group of squares in $\Sigma$.

Let $s$ be any element of $K[\theta]$ such that $s^{\sigma+1}$ is nonsquare in $K$ if $-1$ is a non-square in $K$, and $s^{\sigma+1}$ is square in $K$ if $-1$ is a square in $K$. Let $E$ denote the regulus-inducing group and $H$ is the homology group of squares of kernel homologies in $\Sigma$. Then,

$$EH(y = x^\sigma s) \cup \{y = xm; (m + \beta)^{\sigma+1} \neq s^{\sigma+1} \forall \beta \in K\}$$

is a generalized Fisher conical spread in $PG(3, K)$.

Proof. We now take the group $H$ as the subgroup of squares of the kernel homology group of a Pappian plane $\Sigma$ coordinatized by $K[\theta]$, and $E$ the regulus-inducing elation group analogous to the finite case. By Johnson [5], any Baer subplane of $\Sigma$, the associated Pappian affine plane, disjoint from the axis $x = 0$ of $E$ has the form $y = x^\sigma m + xn$ for $m \neq 0$. That is,

$$EH(y = x^\sigma s) = \{(y = x^\sigma sb^{1-\sigma} + x\alpha); b \text{ is a square in } K[\theta], \alpha \in K\}.$$  

We first claim that this is a partial spread. Since we have an orbit under $EH$, we only need to check that $y = x^\sigma s$ is disjoint from all of the subspaces in the orbit.

Hence, assume that

$$x^\sigma_o s = x^\sigma_o sb^{1-\sigma} + x_o(\alpha), \text{ for some } x_o \in K[\theta].$$

Then,

$$x^\sigma_o s(1 - b^{1-\sigma}) = x_o \alpha.$$  

If $x_o \neq 0$ then we have

$$x^{\sigma-1}_o s(1 - b^{1-\sigma}) = \alpha,$$

implying that

$$(s(1 - b^{1-\sigma}))^{1+\sigma} = \alpha^{1+\sigma} = \alpha^2.$$  

First assume that $-1$ is a square in $K$, so that $s^{1+\sigma}$ is a square in $K$. Then, by lemma 3 we have $(b^{1-\sigma} - 1)^{\sigma+1}$ is a nonsquare. Hence, this is a contradiction so we have a partial spread. Similarly if $-1$ is a non-square in $K$ then $(b^{1-\sigma} - 1)^{\sigma+1}$ is a square in $K$ but since $s^{\sigma+1}$ is nonsquare, we have a contradiction and hence a partial spread.

It remains to show that we obtain a spread. Since we have an associated Desarguesian spread $\Sigma$, it remains to show that if an element of $EH(y = x^\sigma s)$ nontrivially intersects a component $y = xn$ of $\Sigma$, then this component is completely covered. Now an element of $EH(y = x^\sigma s)$
is a Baer subplane of $\Sigma$, $H$ is an index two subgroup of the full kernel homology group $H^+ \oplus H^+$ acting transitively on the non-zero points of any components. So, it follows that $y = xn$ is at least 'half' covered in the sense that the given subplane $\pi_o$ of $EH(y = x^\sigma s)$ intersects $y = xn$ in a 1-dimensional $K$-subspace $X$ and $XH$ is covered by images of intersections of the given subplane under $H$ as $y = xn$ is fixed by $H$. Now the component $y = xn$ is in a unique orbit $\Gamma$ of components under the group $E$. If $\pi_o$ intersects two components of $\Gamma$, say $y = xn$ and $y = x(n + \alpha_o)$ for $\alpha_o \in K$, then there is also a 1-dimensional $K$-subspace $X_{\alpha_o}$ in $\pi_o$ on $y = x(n + \alpha_o)$ and a corresponding orbit $X_{\alpha_o}H$ in $y = x(n + \alpha_o)$. Note that $E$ commutes with $H$. The elation $\tau : (x, y) \mapsto (x, -x\alpha_o + y)$ maps $X_{\alpha_o}H$ onto $X_{\alpha_o}\tau H$. Since $X_{\alpha_o}\tau$ is a 1-dimensional $K$-subspace on $y = xn$, it follows that either $XH$ and $X_{\alpha_o}\tau H$ define the same $H$-orbit on $y = xn$ or $XH \cup X_{\alpha_o}\tau H = \{(x, y); y = xn; x \neq 0\}$. But, if $XH = X_{\alpha_o}\tau H$, then we do not have a partial spread $EH(y = x^\sigma s)$.

Hence, it remains to show that when an element $\pi_o$ of $EH(y = x^\sigma s)$ intersects a component $y = xn$ then $\pi_o$ also intersects $y = x(n + \alpha_o)$ for some $\alpha_o \neq 0$.

Since we have an orbit under $EH$, we may assume that $\pi_o$ is $y = x^\sigma s$. Hence, $y = xn$ and $y = x^\sigma s$ intersect nontrivially if and only if

$$x_0n = x_0^\sigma s$$

for $x_0 \neq 0$. So,

$$n^{\sigma+1} = s^{\sigma+1}.$$ 

Now consider when $y = x^\sigma s$ will nontrivially intersect $y = x(n + \alpha)$ for some nonzero $\alpha \in K$. We claim that there is an intersection if and only if

$$s^{\sigma+1} = (n + \alpha)^{\sigma+1},$$

which is certainly necessary. To see that it is sufficient, we note, by Hilbert’s Theorem 90, that since $(s/(n + \alpha))^{\sigma+1} = 1$ then $s/(n + \alpha) = v^{1-\sigma}$, for some $v \in K[\theta] - \{0\}$. So,

$$v^\sigma s = v(n + \alpha),$$

which implies that $y = x^\sigma s$ and $y = x(n + \alpha)$ nontrivially intersect.

So, if

$$n^{\sigma+1} = s^{\sigma+1},$$

assume that

$$s^{\sigma+1} = (n + \alpha)^{\sigma+1},$$
but require that this equation implies that $\alpha = 0$. We see that the above equation is equivalent to

$$\alpha^2 + \alpha(n + n^\sigma) = 0.$$  

Hence, there are two distinct solutions, 0 and $-(n + n^\sigma)$ for $\alpha$ unless $n + n^\sigma = 0$. Let a basis for $K[\theta]$ be $\{1, e\}$ such that $e^2 = \gamma$, a nonsquare in $K$. Then $n = e\delta + \rho$ for $\delta, \rho \in K$ and $n^\sigma = -n$ if and only if $\rho = 0$. So, $n^{\sigma + 1} = -n^2 = -\gamma\delta^2$. Thus, we arrive at the equation:

$$s^{\sigma + 1} = -\gamma\delta^2.$$  

But, $s^{\sigma + 1}$ is nonsquare or square if and only if $-1$ is nonsquare or square respectively. If $s^{\sigma + 1}$ is nonsquare then $-\gamma$ is square so that $-\gamma\delta^2$ is square in $K$, a contradiction. Similarly if $s^{\sigma + 1}$ is square then $-\gamma$ is nonsquare and $-\gamma\delta^2$ is nonsquare, a contradiction.

Hence, we have that there are two intersections in an $E$-orbit of components of $\Sigma$ with an element of $EH(y = x^\sigma s)$ provided there is one. This completes the proof of the theorem.  

\[\square\]

3. Uniqueness of generalized Fisher flocks

We begin with a general result on André planes.

**Lemma 4.** Let $K$ be a field and $K[\theta]$ a quadratic field extension of $K$. Let $\Sigma$ denote the Pappian plane coordinatized by $K[\theta]$. Let $\sigma$ denote the involution in $Gal_KK[\theta]$.

Consider the following André partial spread: $A_\rho = \{y = xn; n^{\sigma + 1} = \rho\}$.

1. Then, $A_\rho$ is a regulus in $PG(3, K)$ with opposite regulus $A_0^\rho$, defined by $A_0^\rho = \{y = x^\sigma n; n^{\sigma + 1} = \rho\}$.
2. $A_\rho = \{y = x^\sigma n_o a^{1-\sigma}; n_o^{\sigma + 1} = \rho; \forall a \in K - \{0\}\}$.

**Proof.** We note that $y = x^\sigma m$ and $y = xn$ such that $m^{\sigma + 1} = n^{\sigma + 1}$ must intersect in a 1-dimensional $K$-space (a projective point). Furthermore, note that $(m/n)^{\sigma + 1} = 1$ if and only if $mn^{-1} = v^{1-\sigma}$ for some $v$ in $K[\theta]$ by Hilbert’s theorem 90, as we have a cyclic extension quadratic extension $K[\theta]$ of $K$ with Galois group over $K$ of order 2. Furthermore, $(v, v^\sigma m) = (v, vn)$ if and only if $v^{1-\sigma} = mn^{-1}$. If $y = xn_o$ is fixed in $A_\rho$, then $y = xn$ is in $A_\rho$ if and only if $y = xn_v v^{1-\sigma}$ for some $v$. Hence, every 1-dimensional subspace of $y = x^\sigma m$ lies uniquely on some element $y = xn$ of $A_\rho$ and $y = x^\sigma m$ must intersect each element of $A_\rho$. This proves part (1).
Now another application of Hilbert's theorem 90 gives the proof to part (2).

Now assume that we obtain a conical spread obtained via $E$-nest replacement.

Then, we must have a Baer subplane of the form $y = x^s m + xn$ acting in place of $y = x^s s$ above. The exact same argument will show that we only obtain a partial spread $EH\{y = x^s m + xn\}$ if and only if $m^{s+1}$ is non-square (respectively, square) in $K$ if and only if $-1$ is non-square (respectively, non-square) in $K$.

Now we consider the following mappings that normalize $E$:

$$
\tau_{a,b,\beta} : (x,y) \mapsto (xa, xb + ya\beta); a, b \in K[\theta]^*, \beta \in K^*.
$$

Note that $\tau_{a,0,\beta}$ maps $y = x^s m$ onto $y = x^s ma^{1-s} \beta$. Note that $(ma^{1-s} \beta)^{s+1} = m^{s+1} \beta^2$. Thus, since we have a full field, we apply Lemma 4 so show that for a fixed $m$:

$$
\{n; n^{s+1} \text{ is square in } K - \{0\}\} = \{ma^{1-s} \beta; m^{s+1} \text{ is square}; a \in K[\theta]^*, \beta \in K - \{0\}\},
$$

$$
\{n; n^{s+1} \text{ is nonsquare in } K - \{0\}\} = \{ma^{1-s} \beta; m^{s+1} \text{ is square}; a \in K[\theta]^* \beta \in K - \{0\}\}.
$$

It will now follow that we obtain an isomorphic plane whenever the basic conditions required for a partial spread above are met.

**Theorem 2.** Let $K$ be a full field of odd or 0 characteristic and let $K[\theta]$ be a quadratic extension of $K$ that is also a full field. $\Sigma$ be the Pappian affine plane coordinatized by $K[\theta]$.

Then, any two generalized Fisher conical spreads in $PG(3, K)$ are isomorphic.

**Proof.** The group $GL(2, K[\theta])$ is triply transitive on the components of the spread for $\Sigma$. This means that we may assume that in the construction of two generalized Fisher planes, we may assume that we use the same axis $x = 0$, regulus-inducing group $E$ and kernel homology group of squares of $\Sigma$ in the same form for both planes. The question therefore is merely the choice of the Baer subplane $\pi_0$ to use to form the partial spread $EH\pi_0$ that induces the spread. But, any two Baer subplanes have the form $y = x^s m_1 + xn_i$, for $i = 1, 2$ and $m_i \neq 0$. Clearly, we may apply an appropriate elation with axis $x = 0$ that normalizes $EH$ to allow $n_1 = 0$. Now a partial spread $EH\pi_0$ is obtained if and only if $m_1^{s+1}$ is square or non-square exactly when $-1$ is square or non-square,
respectively. We have shown above that we may apply mappings that normalize $EH$ and map $y = x^\sigma m_1$ onto $y = x^\sigma m_2$. But, then an appropriate elation with axis $x = 0$ will map $y = x^\sigma m_2$ onto $y = x^\sigma m_2 + xn_2$. Hence, any two generalized Fisher planes are isomorphic. \qed

4. Critical linear subflocks

Assume that $\mathcal{N}$ is a non-linear partial flock in $PG(3, K)$ containing a critical linear subflock $\mathcal{P}$. Let $\mathcal{L}$ denote a linear flock containing $\mathcal{P}$.

**Lemma 5.** There is a unique linear flock containing a critical linear subflock.

**Proof.** Suppose there are two such flocks and let $\Sigma$ and $\Sigma'$ denote the corresponding Pappian spreads defined by the linear flocks and containing the partial spread $\Pi$ defined by the critical linear subflock. Let $m$ be a line of $\Sigma' - \Sigma$, so that $m$ becomes a Baer subplane of $\Sigma$ disjoint from $\Pi$. Hence, $m$ intersects each base regulus of $\Sigma - \Pi$ in two components. We are finished unless possibly the critical linear subflock consists of exactly one regulus, which does not occur. Hence, $m$ intersects all but one base regulus of $\Sigma$ in two components, which cannot be the case. \qed

Now let $K[\theta]$ denote the quadratic extension field of $K$ coordinatizing the affine plane given by $\Sigma$. Assume that $K$ and $K[\theta]$ are full fields of odd or zero characteristic.

Let $\sigma$ denote the involution in $Gal_K K[\theta]$ and note by Johnson [5] that any Baer subplane disjoint from the elation axis $x = 0$ of $E$ has the form $y = x^\sigma m + xn$, for $m \neq 0$.

By assumption, we may assume that this Baer subplane $\tau_0$ intersects two components of each of the base reguli of $\Sigma - \Pi$, and this Baer subplane corresponds to a component of the partial spread given by $\mathcal{N} = \pm$.

We see by applying $(x, y) \mapsto (x, -xn + y)$, we may assume that $n = 0$.

Now $y = x^\sigma m$ intersects $y = xn$ if and only if $m^{\sigma + 1} = n^{\sigma + 1}$.

Since non-squares exist in $K$ we may choose a basis $\{1, e\}$ such that $e^2 = \gamma$, a non-square. Then, the base regulus defined by $y = xn$ is also defined by $y = x n_1$ for some $n_1$ in $K$.

Hence, we must have

$$m^{\sigma + 1} = \alpha^2 - \gamma n_1^2$$
has two solutions whenever it has one. Note that \((e\beta + \delta)^{\sigma+1} = \delta^2 - \gamma\beta^2\).
There is a solution \(\alpha\) if and only if \(-\alpha\) is also a solution. Moreover, if \(\alpha = 0\) then \(m^{\sigma+1}\) cannot be \(-\gamma n_1^2\).

Now consider \(EH(y = x^\sigma m)\), where \(H\) is the kernel subgroup of squares. This is the following set:

\[
\{y = x^\sigma b^{1-\sigma} m + xa; \alpha \in K \text{ and } b \text{ a square in } K[\theta]\}.
\]

We want to prove that this is a partial spread that covers the base reguli of intersection. Assume that \(-1\) is a square. We note that \(m^{\sigma+1}\) cannot be \(-\gamma n_1^2\), for any \(n_1^2\), so that in full fields, this implies that \(m^{\sigma+1}\) is square. Similarly, if \(-1\) is a square and \(m^{\sigma+1}\) cannot be \(-\gamma n_1^2\) for any \(n_1^2\), then, for full fields, this implies that \(m^{\sigma+1}\) is a square. In the following we show that we obtain a generalized Fisher spread; that \(N\) is a generalized Fisher spread.

Take two components \(m_1\) and \(m_2\) of \(N - P\) and extend each to two generalized Fisher spreads \(\pi_1\) and \(\pi_2\), respectively and note that this is guaranteed possible by the main theorem of Jha and Johnson [2]. Clearly as a set of vectors \(EHm_1 = EHm_2\). We wish to show that \(\pi_1 = \pi_2\) and contain \(N\); any non-linear extension of a critical partial flock may be uniquely extended to a generalized Fisher flock.

Hence, we may assume that \(m_2\) is not a component of \(\pi_1\). We note that \(m_1\) and \(m_2\) are both Baer subplanes of \(\Sigma\) and as such define reguli (regulus nets) of \(\Sigma\). Since \(N\) is a partial flock, it follows that \(Em_1\) and \(Em_2\) are either equal or disjoint (they share only the zero vector). If these two partial spreads are equal then \(\pi_1 = \pi_2\). Hence, \(Em_1\) and \(Em_2\) are disjoint partial spreads.

Since \(m_2\) is not in \(\pi_1\) as a component and since \(P\) is critical, the regulus \(R_2\) intersects two components of each of the reguli of \(\pi_1 - \Sigma\) defined by the \(E\)-orbits of components, which cannot occur since \(Em_2\) and \(Em_1\) are disjoint.

Note that by property (ii) in the definition of critical subflock, \(m_2\) intersects each regulus of \(\pi_1 - \Sigma\) in two components. However, \(Em_1\) union the axis of \(E\) is a regulus of \(\pi_1 - \Sigma\), implying that \(m_2\) non-trivially intersects \(Em_1\), contradicting the fact that \(Em_2\) and \(Em_1\) are disjoint. Hence, every component of \(N - P\) is a component of the generalized Fisher spread \(\pi_1\) obtained by use of a single component \(m_1\). This shows that the partial spread may be extended uniquely to a spread. So, we obtain the following result.

**Theorem 3.** Let \(K\) be a full field of characteristic 0 or odd and let \(K[\theta]\) be a full field quadratic extension of \(K\).
If there exists a linear critical partial flock $\mathcal{P}$ of a quadratic cone then any non-linear partial flock extension of $\mathcal{P}$ may be uniquely extended to a generalized Fisher flock.

Finally, we note some examples of full fields admitting quadratic extension full fields. Both of these also appear in Jha and Johnson [3].

**Example 1.** Let $P_o$ be isomorphic to $GF(p)$ where $p$ is an odd prime. Let $F$ be any algebraic field extension of $P_o$ which is not algebraically closed and which is not a series of quadratic extensions of extensions of $P_o$. Then $F$ is a full field.

**Example 2.** Let $F$ be an ordered field which admits an ordered quadratic extension $K$ such that the positive elements of each field have square roots in the field. Then both $F$ and $K$ are full fields.

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