THE CLASSIFICATION OF (3,3,4) TRILINEAR FORMS

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Abstract. Let $U$, $V$ and $W$ be complex vector spaces of dimensions 3, 3 and 4 respectively. The reductive algebraic group $G = \text{PGL}(U) \times \text{PGL}(V) \times \text{PGL}(W)$ acts linearly on the projective tensor product space $\mathbb{P}(U \otimes V \otimes W)$. In this paper, we show that the $G$-equivalence classes of the projective tensors are in one-to-one correspondence with the $\text{PGL}(3)$-equivalence classes of unordered configurations of six points on the projective plane.

1. Introduction

Let $U$, $V$ and $W$ be complex vector spaces (or vector spaces over an algebraically closed field of characteristic 0) of dimensions 3, 3 and 4 respectively. We consider the variety of projective trilinear forms $\mathcal{P} := \mathbb{P}(U \otimes V \otimes W)$. $\mathcal{P}$ can be regarded as the projective space $\mathbb{P}(\text{PHom}(U^*, \text{Hom}(W^*, V)))$, i.e., as the variety of $(3 \times 4)$ matrices of linear forms on $\mathbb{P}(U^*)$. The reductive algebraic group $G := \text{PGL}(U) \times \text{PGL}(V) \times \text{PGL}(W)$ acts naturally on $\mathcal{P} = \text{PHom}(U^*, \text{Hom}(W^*, V))$ via row and column operations, and linear transformations on $U^*$. Given a $\phi \in \mathcal{P}(U \otimes V \otimes W)$, regarded as a linear map $\phi_u : W^* \to V$, we can define the determinantal variety $X_\phi \subset \mathcal{P}(U^*)$ as the rank $(\phi_u) \leq 2$ degeneracy locus, it is generically a zero dimensional length six scheme. We can also define $Y_\phi \subset \mathcal{P}(V^*)$ and $Z_\phi \subset \mathcal{P}(W^*)$ in a similar manner. $Y_\phi$ is similar to $X_\phi$ but $Z_\phi$ is a cubic surface. Trilinear forms of dimension $(p, q, r)$ have been studied via the degeneracy loci $X_\phi$, $Y_\phi$ and $Z_\phi$ since the 30’s by [27], [10], [33], [7] and [34] etc. [27] contains a very detailed account of the geometric properties of determinantal varieties in general. [10] deals specifically with the relation among the three determinantal varieties in special cases. On the other hand, [33] and [7] study the case when $U$, $V$ and $W$ are all three dimensional and they
classify such trilinear forms. In [23], the author classifies and derives explicit matrix representations for the reduced \((3, 3, 3)\) trilinear forms. In [24], the author uses the matrix expressions to investigate the closure relations among the semi-stable orbits of the \((3, 3, 3)\) trilinear forms and constructs a Geometric Invariant Theory quotient explicitly.

In two separate papers, the author will treat the subject of \((3, 3, 4)\) trilinear forms in a similar manner. In this paper, the author classifies and derives matrix representations for all \((3, 3, 4)\) trilinear forms. In a subsequent paper [25], the author constructs the Geometric Invariant Theory quotient for \((3, 3, 4)\) trilinear forms. The matrix expressions for semi-stable \((3, 3, 4)\) trilinear forms are necessary for the construction of the G.I.T. quotient \(\mathbb{P}^3/G\).

In [34], \((3, 3, 4)\) trilinear forms are classified in a similar manner as in [33]. It is proven that a “general non-degenerate” trilinear form \(\phi\), i.e., one for which \(Z_{\phi}\) is a smooth cubic surface, is uniquely determined by \(X_{\phi}\). The same statement is also an immediate consequence of the Hilbert-Burch Theorem ([2], [14] and [26]) which we will use to derive explicit matrix expressions for smooth trilinear forms. In his paper, Thrall also describes how to obtain the “approximately a hundred different types” of “non-degenerate but special” trilinear forms \(\phi\) (i.e., when \(X_{\phi}\) is not in general position and \(Z_{\phi}\) has only isolated singularities, or in [13]’s terminology “\(X_{\phi}\) is not in unnodal position”). To date, there has been no published work on the detailed description of these trilinear forms in terms of \(X_{\phi}\). We adopt the Hilbert-Burch approach to derive the matrix expressions of all these trilinear forms (which we call Duval since they are precisely the ones for which \(Z_{\phi}\) has only rational double points). Finally, [34] provides a table for the “degenerate” trilinear forms (i.e., when \(X_{\phi}\) contains a non-curvilinear triple point \(\text{Spec}(\mathbb{C}[x, y]/(x, y)^2)\) or when \(Z_{\phi}\) has non-isolated singularities or an \(E_6\) singularity). We derive these trilinear forms using the Hilbert-Burch approach and we call them non-Duval. We provide an ideal theoretic description of the \(X_{\phi}\) for these non-Duval trilinear forms.

The quotient \(Q\) of smooth \((3, 3, 4)\) trilinear forms is closely related to the moduli space of smooth cubic surfaces \(S\) and the configuration space \(P\) of six points in general position on the plane. There is a vast amount of literature on \(S\) and \(P\). Among the more recent ones, we have [21], [22], [4], [6], [1], [13], [29], [30], [31] and [32] etc. It is well known that the G.I.T. stable surfaces are precisely the smooth surfaces and the surfaces with only \(A_1\) singularities. The \(PGL(4)\)-orbits of these surfaces form a geometric quotient. The strictly semi-stable surfaces are
those with at least one $A_2$ singularity but no $A_{23}$, $D_4$, $D_6$, $E_6$, $E_8$ or non-isolated singularities. However, there is only one such orbit which is closed, i.e., the unique $A_2^3$ orbit (See [22] and [1], the latter contains a proof which seems to be the first one to appear in print). Hence, the G.I.T. compactification $\mathcal{S}$ has only one point $[A_2^3]$ representing all the strictly semi-stable orbits. The moduli space $Q$ clearly admits a finite morphism onto $S$. This is due to the well known fact that $X_\phi$ and $Y_\phi$ give rise to a Schl"{a}fli's double-six on $Z_\phi$ and there are exactly thirty six double-sixes on any smooth cubic surface (See [12], [13] and Proposition 1 in Section 2). The remaining semi-stable trilinear forms necessarily map onto the point $[A_2^3]$ in $\mathcal{S}$. In [25], we will find the fibre over $[A_2^3]$.

The other way to view $Q$ is via the ordered configuration space $\mathcal{P}$ of six points in general position, i.e., the set of $PGL(3)$-equivalence classes of point sets in $(\mathbb{P}^2)^6$ ([13], [29] and [31]). It is easy to see that $Q$ is a $S_6$-quotient of $\mathcal{P}$. The Weyl group $W(E_6)$ acts on $\mathcal{P}$ and it is generated by $S_6$ and an involution $\sigma$ (which corresponds to an elementary quadratic transformation of the plane). Moreover, $\mathcal{P}$ admits an involution $\ast \not\in W(E_6)$ which commutes with the $S_6$-action (See [29] and [32]). On the other hand, $Q$ has the involution $\tau$ interchanging the vector space $U$ and $V$. The four spaces fit into a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{S_6} & Q \\
\downarrow \mathbb{Z}/2 & & \downarrow \mathbb{Z}/2 \\
\mathcal{P} / / < \ast > & \xrightarrow{S_6} & Q / / < \tau >
\end{array}
$$

We organize our paper as follows.

In Sections 2 and 3, we summarize the classical results on smooth cubic surfaces and determinantal varieties in Proposition 1 and in the Hilbert-Burch Theorem. In Proposition 1, we have the relations among $X_\phi$, $Y_\phi$ and $Z_\phi$ for a smooth trilinear form. The Hilbert-Burch Theorem applied to a set $X$ of six distinct points in general position says that the matrix $M$ in the minimal free resolution of $X$:

$$
0 \longrightarrow S(-4)^3 \xrightarrow{M} S(-3)^4 \longrightarrow S \longrightarrow S(X) \longrightarrow 0
$$

corresponds to the trilinear form $\phi$ such that $X_\phi = X$. In the above complex, $S$ denotes the homogeneous coordinate ring of $\mathbb{P}^2$, $S(X)$ denotes that of $X$ and $S(-n)$ denotes the $n^{th}$-twist of $S$. Hence, $M$ can be computed by solving for the syzygy relations among the generators of the defining ideal of $X$.

Finally, we state our theorem on smooth trilinear forms as Theorem 2. Essentially, we formulate a one-to-one correspondence between
the $G$-orbits of smooth trilinear forms and the $PGL(3) \times S_6$-orbits of six points in general position on the plane. In other words, we have an isomorphism between the moduli space of smooth trilinear forms $Q$ and the unordered configuration space of six points $P//S_6$. Our computation results in the explicit matrix representation $\phi_\lambda$ of a general smooth trilinear form $\phi$ (such that $X_\phi$ is a preassigned set $\{(1,0,0),[0,1,0],[0,0,1],[1,1,1],[1,\lambda,\mu],[1,\alpha,\beta]\}$). The expression is used to demonstrate that $Y_\phi$, as an associated point set of $X_\phi$, is projectively equivalent to $\{(1,0,0),[0,1,0],[0,0,1],[1,1,1],[1,\lambda,\alpha],[1,\mu,\beta]\}$ which is $\ast X_\phi$ (See [32]).

In Section 4, we discuss what happens when a zero dimensional length six subscheme $X$ in general position degenerates to one with l-configurations, i.e., collections of three points in $X$ which are collinear, with curvilinear multiple points or with non-curvilinear multiple points. The last possibility leads to the “degenerate” cases of [34], where $Z_\phi$ has non-Duval singularities. The defining ideals of multiple points are listed in this section too.

The main contribution of this paper is in Sections 5 and 6, where we provide a comprehensive list of zero dimensional length six subschemes $X$ which give rise to the Duval trilinear forms in Theorems 3 and 5. We also derive their matrix representations with the Hilbert-Burch Theorem as the latter is applicable as long as $X$ does not lie on a conic. These trilinear forms are precisely the ones for which $Z_\phi$ is a singular cubic surface with only rational double point(s). Those with only $A_{\leq2}$ singularities are listed in Theorem 3. These trilinear forms are obviously semi-stable in the sense of Geometric Invariant Theory. The relations among these $G$-orbits will be investigated in [25] as part of the construction of the G.I.T. quotient. The rest of the trilinear forms with $A_{\geq3}$, $D_4$ or $D_5$ singularities are in Theorem 5. Their instability will be verified in [25] as well. At the same time, we also single out all the subschemes $X$ which lie on some conic in the plane. These subschemes are not determinantal in the sense that $X$ is not equal to $X_\phi$ for any $(3,3,4)$ trilinear form $\phi$. However, the syzygy relations among the generators of any linear system through $X$ always lead to a unique trilinear form called the complete intersection trilinear form $\phi_{c.i}$. which is easily seen to be G.I.T. unstable. These subschemes are listed in Theorems 4 and 6.

Finally, we derive the non-Duval trilinear forms in Section 7. These are the trilinear forms for which $Z_\phi$ is either $P^3$, contains non-isolated
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singularities or an $\overline{E}_6$ singularity. This section is included for completeness sake as the same matrix expressions have already been provided by [34]. In our paper, we use the Hilbert-Burch approach to achieve the same results. A byproduct is the ideal theoretic description of the multiple points in $X_\phi$'s. As far as the construction of the G.I.T. quotient $P//G$ is concerned, Section 7 is almost redundant since non-Duval trilinear forms are not expected to appear in the compactification of $Q$. Indeed, all the explicit matrix representations in the theorems of Section 7 are already written in the destabilizing coordinate systems.

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2. Preliminaries on $(3, 3, 4)$ trilinear forms

In this section, we introduce some terminology and notations for trilinear forms and recall some well known results on their associated determinantal varieties.

We usually regard a trilinear form $\phi$ in $\mathbb{P}(U \otimes V \otimes W)$ as a linear map $\phi_u : U^* \rightarrow \text{Hom}(W^*, V)$ or $\phi_w : W^* \rightarrow \text{Hom}(V^*, U)$. The map $\phi_u$ can be represented by a $3 \times 4$ matrix of linear forms $\langle x_1, x_2, x_3 \rangle$ on $\mathbb{P}(U^*)$ and is a family of linear maps from $W^*$ to $V$ parametrized by the vector space $U^*$. Its evaluation at $[u^*] \in \mathbb{P}(U^*)$ is denoted by $\phi_u(u^*)$. The other representation $\phi_v : V^* \rightarrow \text{Hom}(W^*, U)$ is similar to $\phi_u$. Finally, $\phi_w$ can be written as a $3 \times 3$ matrix of linear forms $\langle z_1, z_2, z_3, z_4 \rangle$ on $\mathbb{P}(W^*)$.

We define the following determinantal varieties associated with $\phi$:

$$X_\phi := \{ [u^*] \in \mathbb{P}(U^*) \mid \text{rank } \phi(u^*) \leq 2 \},$$

$$Y_\phi := \{ [v^*] \in \mathbb{P}(V^*) \mid \text{rank } \phi(v^*) \leq 2 \} \quad \text{and}$$

$$Z_\phi := \{ [w^*] \in \mathbb{P}(W^*) \mid \text{rank } \phi(w^*) \leq 2 \}.$$

$X_\phi$ (and $Y_\phi$) is generically a set of six distinct points which is in general position, i.e., $X_\phi$ does not lie on any conic and no three points in $X_\phi$ lie on a line whereas $Z_\phi$ is a smooth cubic surface. We can give the following definitions for trilinear forms based on their associated determinantal varieties:

DEFINITIONS.

1. A trilinear form $\phi$ is non-degenerate if $\phi_u$, $\phi_v$, and $\phi_w$ are all injective. It is equivalent to saying that $\phi_u$, $\phi_v$, and $\phi_w$ do not have any row or column consisting entirely of zeros for any bases we
use for $U$, $V$ and $W$. This implies that $X_\phi \neq \mathbb{P}U^*$, $Y_\phi \neq \mathbb{P}V^*$ and $Z_\phi \neq \mathbb{P}W^*$ (but not vice versa).

2. A trilinear form $\phi$ is pure dimensional if $X_\phi$ and $Y_\phi$ are both zero dimensional.

3. A trilinear form $\phi$ is smooth (singular, irreducible or reducible) if $Z_\phi$ is a smooth (respectively singular, irreducible or reducible) cubic surface. In general, we say that a trilinear form $\phi$ has property $T$ if $Z_\phi$ is a cubic surface with property $T$.

4. A trilinear form $\phi$ is Duval of type $A_i^nA_j^n$ or $A_i^nA_j^n$, $D_4$ or $D_5$ if $Z_\phi$ has singularities of type $A_i^nA_j^n$, $D_4$ or $D_5$ respectively and it is the image of $\mathbb{P}(U^*)$ under $\phi_{uw}$ (see Proposition 1). In other words, $\phi$ corresponds to a birational modification of a singular cubic surface with rational double point(s). The non-Duval trilinear forms are those for which $Z_\phi$ has a triple point, non-isolated singularities or is simply $\mathbb{P}(W^*)$ itself.

It is well known that the only cubic surfaces which are not determinantal are those which contain only one line, equivalently those with a unique singularity of type $E_6$, and they are projectively equivalent to $xy^2 + yt^2 + z^3 = 0$. A proof is also provided by [34] and [5]. So we have excluded $E_6$ from the last definition. Indeed, a cubic surface $Z$ with an $E_6$ singularity can only result from blowing up the plane at a multiple six point which contains a linear triple point. Such a subscheme is an inadmissible configuration (see Section 4.1) and is never determinantal.

The relations among $X_\phi$, $Y_\phi$ and $Z_\phi$ in the case of smooth trilinear forms are well known and summarized in the following proposition:

**Proposition 1.** Let $f_k$ be $(-1)^k$ times the minor determinant of the submatrix obtained from $\phi(u^*)$ by deleting the $k^{th}$ column and $I$ be the ideal generated by $f_k, 1 \leq k \leq 4$, then $X_\phi$ is the zero locus of $I$.

(i) For $[w^*] \in Z_\phi$, the morphisms

$$\pi : Z_\phi \to \mathbb{P}(U^*)$$

defined by $\pi([w^*]) = [\ker (\phi(w^*)^t)] = ([\im (\phi(w^*))]^t)$

and

$$\pi' : Z_\phi \to \mathbb{P}(V^*)$$

defined by $\pi'([w^*]) = [\ker (\phi(w^*))]$ are blow-downs of the smooth cubic surface $Z_\phi$.

(ii) The birational inverse of $\pi$, $\phi_{uw} : \mathbb{P}(U^*) \cdots \to Z_\phi$ is defined by

$$\phi_{uw}([u^*]) = [\ker (\phi(u^*))] = [f_1(u^*), \cdots, f_4(u^*)]$$

and is a blowup with centre $X_\phi$. 
The birational inverse of $\pi'$, $\phi_{uvw} : \mathbb{P}(V^*) \cdots \to Z_\phi$, is defined by

$$\phi_{uvw}(u^*) = [\ker \phi(u^*)] = [f'_1(u^*), \cdots, f'_4(u^*)],$$

where $(f'_1, f'_2, f'_3, f'_4)$ is the linear system of cubics through $Y_\phi$. It is a blowup with centre $Y_\phi$.

Moreover, the exceptional lines of $\phi_{uw}$ and $\phi_{vw}$ (which are $\pi^{-1}(X_\phi)$ and $(\pi')^{-1}(Y_\phi)$ respectively) form a Schläfi double six on $Z_\phi$, i.e., if $\pi^{-1}(X_\phi) = \{l_1, \cdots, l_6\}$ and $(\pi')^{-1}(Y_\phi) = \{l'_1, \cdots, l'_6\}$, then we have $l_i \cap l_j = \emptyset$, $l'_i \cap l'_j = \emptyset$ and $l_i \cap l'_j \neq \emptyset$ if and only if $i \neq j$.

(iii) $X_\phi$ and $Y_\phi$ are associated in the following sense of Coble ([8], [13] and [32]): if we represent $X_\phi$ by a $(3 \times 6)$ matrix $X$ such that the columns of $X$ correspond to points in $X_\phi$, then $X$ has maximal rank. The kernel (in $\mathbb{C}^6$) can be represented by some $(6 \times 3)$ matrix $Y$. The points in $Y_\phi$ are, up to projective equivalence, represented by the columns of $Y^t$.

The assertions in (i) and (ii) are classical (See for instance [12] and [16]). The theory of associated point sets in projective spaces was developed by A. Coble in a series of papers ([9], [8] and its sequels). A modern treatment for associated point sets can be found in [13]. In the next section, we shall use the explicit $\phi_{uv}$ expression for a smooth trilinear form to obtain simple coordinate systems for both the associated sets $X_\phi$ and $Y_\phi$.

3. Smooth trilinear forms

In this section, we use Hilbert-Burch Theorem to derive an explicit matrix representation $\phi_{uw}$ for a smooth trilinear form $\phi$ such that $X_\phi$ is a preassigned zero dimensional length six subscheme $X$ on the plane.

3.1. The Hilbert-Burch theorem

We have seen that a smooth $(3, 3, 4)$ trilinear form $\phi \in \mathbb{P}(U \otimes V \otimes W)$ determines an unordered set $X_\phi = \{p_1, \cdots, p_6\} \subset \mathbb{P}(U^*)$ in general position, i.e., $X_\phi$ does not lie on any conic and there is no line on $\mathbb{P}(U^*)$ which contains more than two points in $X_\phi$. Moreover, $\phi_{uw} : \mathbb{P}(U^*) \cdots \to Z_\phi$ is the blowup at $X_\phi$.

The converse is the content of the Hilbert-Burch Theorem ([2] Theorem 5, [14]):

**Theorem 1** (Hilbert-Burch Theorem). Suppose $R$ is a local ring and $I$ is an ideal, such that there exists an exact complex:

$$0 \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} R \to R/I \to 0,$$
and $F_1 \cong \mathbb{R}^n$, then $F_2 \cong \mathbb{R}^{n-1}$ and there exists a non-zero divisor $a$ such that $I = aI_{n-1}(\varphi_2)$, where $I_{n-1}(\varphi_2)$ is the ideal generated by the size $(n-1)$ minor determinants of the $(n \times (n-1))$ matrix $\varphi_2$.

Indeed, if $X = \{p_1, \ldots, p_6\} \subset \mathbb{P}(U^*)$ is an unordered set of six distinct points in general position, its defining ideal $I$ is generated by any four linearly independent homogeneous cubic polynomials $f_1, \ldots, f_4$ vanishing on $X$ and its minimal free resolution is

$$0 \rightarrow S(-4)^3 \xrightarrow{M} S(-3)^4 \rightarrow S \rightarrow S(X) \rightarrow 0,$$

where $S = \mathbb{C}[x_1, x_2, x_3]$ denotes the homogeneous coordinate ring of $\mathbb{P}^2(U^*)$, $S(X)$ denotes that of $X$ and $S(-n)$ denotes the $n^{th}$-twist of $S$.

The Hilbert-Burch Theorem asserts that $I$ is generated by $f_k$, $1 \leq k \leq 4$, where $f_k$ is $(-1)^k$-times the maximal minor determinants of the $(4 \times 3)$ matrix $M$. It is easy to see that $M$ is uniquely determined by $X$ up to row operations, column operations and linear transformations on $U$, i.e., a $G$-action on $M$. Hence, a zero dimensional length six subscheme of the plane determines a $(3, 3, 4)$ trilinear form (given by the syzygy matrix $M$, or more precisely its transpose) up to a $G$-action.

The Hilbert-Burch Theorem is all we need to derive all the smooth and Duval trilinear forms. It is even applicable for non-Duval trilinear forms whenever either $X_\varphi$ or $Y_\varphi$ is zero dimensional. The following proposition is a formulation of the consequences of Proposition 1 and Hilbert-Burch Theorem:

**Theorem 2 (Smooth Trilinear Forms).** Let $\mathbb{P}^{\text{sm}}$ denote the set of smooth trilinear forms in $\mathbb{P}(U \otimes V \otimes W) = \mathbb{P}(\text{Hom}(U^* , \text{Hom}(W^*, V)))$, where $U, V$ and $W$ are complex vector spaces of dimensions 3, 3, 4 respectively. Let $G = \text{PGL}(U) \times \text{PGL}(V) \times \text{PGL}(W)$ act on $\mathbb{P}^{\text{sm}}$ via row and column operations and linear transformations on $U$. Let $\mathbb{C}^4_{\lambda, \mu, \alpha, \beta}$ be the affine space represented by the set of "standard" matrices:

$$X(\lambda, \mu, \alpha, \beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & \lambda & \alpha \\ 0 & 0 & 1 & 1 & \mu & \beta \end{pmatrix}.$$ 

Let $D$ be the union of 15 hypersurfaces of $\mathbb{C}^4$ defined by the following polynomials:

\begin{align*}
l_{125} &= \lambda, \quad l_{126} = \beta, \quad l_{135} = \lambda, \quad l_{136} = \alpha, \\
l_{145} &= \lambda - \mu, \quad l_{146} = \alpha - \beta, \quad l_{156} = \lambda \beta - \alpha \mu, \quad l_{245} = \mu - 1, \quad l_{246} = \beta - 1, \\
l_{256} &= \mu - \beta, \quad l_{345} = \lambda - 1, \quad l_{346} = \alpha - 1, \quad l_{356} = \lambda - \alpha, \\
l_{456} &= (\lambda - 1)(\beta - 1) - (\alpha - 1)(\mu - 1) = (\lambda \beta - \lambda - \beta) - (\alpha \mu - \alpha - \mu), \\
c &= \lambda \beta (1 - \alpha)(1 - \mu) - \alpha \mu (1 - \lambda)(1 - \beta) = \lambda \beta (1 - \alpha - \mu) - \alpha \mu (1 - \lambda - \beta).
\end{align*}
There is a one-to-one correspondence between the two sets of equivalence classes:

\[
P^{nm} / G \leftrightarrow PGL(3) \backslash \{ \mathbb{C}^4 - D \} / (S_6 \times (\mathbb{C}^*)^6),
\]

where \( PGL(3) \) acts on the left by matrix multiplication, \( S_6 \) acts on \( \mathbb{C}^4 \) by permutation of the six columns of \( X(\lambda, \mu, \alpha, \beta) \) and \((\mathbb{C}^*)^6 \) acts via multiplication by \((6 \times 6)\) diagonal matrices.

Proof. The correspondences are defined as follows: given a smooth trilinear form \( \phi \), we have a zero dimensional length six subscheme \( X_\phi = \{ p_1, \cdots, p_6 \} \) in general position on \( P(U^*) \) represented by a \((3 \times 6)\) matrix. Multiplying this matrix on the left by a suitable element in \( PGL(U) \) and on the right by a diagonal \((6 \times 6)\) matrix puts it in the “standard” form \( X(\lambda, \mu, \alpha, \beta) \). However, distinct orderings of the points \( p_i \)’s lead to different \( \lambda \)’s and \( \mu \)’s. Hence, \( \phi \) only determines \( X(\lambda, \mu, \alpha, \beta) \) up to an action of \( S_6 \). Furthermore, \( X_\phi \) is in general position, a condition which is equivalent to the non-vanishing of \( ijk \)’s (corresponding to the collinearity of \( p_i, p_j \) and \( p_k \)) and that of \( c \) (corresponding to \( X_\phi \) lying on a conic).

Conversely, suppose we have a subscheme \( X \) in a projective plane \( P(U^*) \) represented by \( X(\lambda, \mu, \alpha, \beta) \notin D \), then \( X \) is in general position. The fact that \( X \) imposes six linearly independent relations on the linear system of cubics is well-known (See for example [17]). Hence, we can pick a suitable basis \( \{ f_1, \cdots, f_4 \} \subset H^0(P(U^*), \mathcal{O}(3)) \) for the three dimensional linear system of cubics vanishing at \( X \). Define the following vector spaces:

\[
U := \{ x_1, x_2, x_3 \} \quad \text{\{linear forms on} \ P(U^*)\}\}
W := \{ f_1, \cdots, f_4 \} \quad \text{\{homogeneous cubic polynomials vanishing at} \ X\}\}
F := \{ \text{homogeneous quartic polynomials vanishing at} \ X\}
\]

\((W \text{ and } F \text{ are four and nine dimensional respectively})\), then we have a surjective linear map with a three dimensional kernel defined as \( V^* := \text{Ker}(\otimes)\):

\[
0 \rightarrow V^* \rightarrow U \otimes W \otimes F \rightarrow 0,
\]

where “\( \otimes \)” denotes the usual multiplication of polynomials. \( V^* \) is precisely the vector space of syzygies spanned by three linearly independent relations (over \( \mathbb{C} \)). It can be found by solving the following equation with unknown scalars \( \phi_{ijk} \)’s:

\[
(\phi_{11}x_1 + \phi_{21}x_2 + \phi_{31}x_3)f_1 + (\phi_{12}x_1 + \phi_{22}x_2 + \phi_{32}x_3)f_2 + (\phi_{13}x_1 + \phi_{23}x_2 + \phi_{33}x_3)f_3 + (\phi_{14}x_1 + \phi_{24}x_2 + \phi_{34}x_3)f_4 = 0.
\]
The three dimensional solution \((\phi_{ijk})\) is indexed by \(j = 1, 2, 3\). The injection of \(V^*\) into \(U \otimes W\) can be canonically identified with a \((3, 3, 4)\) trilinear form \(\phi\). The \((3 \times 4)\) matrix \(\phi_a = (\Sigma \phi_{ijk} x_i)\) is equal to the transpose of the Hilbert-Burch matrix \(M\) mentioned above. Clearly, \(\phi\) is determined up to a choice of the coordinates of \(X\) (action of \(PGL(U)\)), the choice of basis \(\{f_k\}\) (action of \(PGL(W)\)), and the choice of basis for the syzygies (action of \(PGL(V)\)). Hence, \(\phi\) is determined up to a \(G\)-action. By the Hilbert-Burch Theorem, \(f_k\)'s are equal (up to sign) to the maximal minor determinants of \(\phi_a\), i.e., \(X_\phi = X\). It is easy to see that the correspondences defined above are inverse to each other. 

**Remarks.**

(i) A point in \((\lambda, \mu, \alpha, \beta) \in \mathbb{C}^4 - D\) can be identified with the standard matrix \(X(\lambda, \mu, \alpha, \beta)\). Hence it determines an **ordered set** of six points on the plane. However, a smooth trilinear form only determines an **unordered set** of six points. Hence, we have an isomorphism between the moduli space of smooth trilinear forms and \(\mathcal{P}/\!/S_6\), where \(\mathcal{P}\) is the configuration space of six points in general position on the plane. The configuration space \(\mathcal{P}\) has been studied in numerous papers (for instance, [13], [29] and [31]). Our construction of the G.I.T. quotient of \((3, 3, 4)\) trilinear forms by \(G\) can thus be viewed as a compactification of the unordered configuration space of six points on the plane.

The simplest analogue of this situation is the unordered configuration space of four points on the projective line. Given four points \(0, 1, \infty, \lambda\) on the projective line, we can define the \(j\)-invariant

\[
   j = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1 - \lambda)^2}
\]

The permutation group \(S_3\) acts on \(0, 1, \infty\) to generate the six representations (except over \(j = 0\) and \(j = 12^3\) \(\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{\lambda}{\lambda - 1}, \lambda - 1, \frac{1}{1 - \lambda}\)) for the fourth point but \(j\) is invariant under \(S_3\). We can regard \(\Lambda_1^\lambda\) as a \(6:1\) cover of \(\mathbb{A}_x^1\) ramified over \(0\) and \(12^3\).

When we have a set \(X\) with multiple points, we have modifications of \(X(\lambda, \mu, \alpha, \beta)\) (See Sections 4, 5 and 6) to describe the points in the higher order neighborhoods of \(X\).

(ii) It is clear that the orbit of a smooth trilinear form corresponds to the equivalence class of a triplet \([(Z, X, \pi : Z \to X)]\), where \(Z\) is a smooth cubic surface and \(\pi\) is the blowup of \(\mathbb{P}^2\) at \(X\) (\(X\) is uniquely determined by \(\pi\) with the choice of a collection of six skew lines on \(Z\)). Two such triplets \((Z_1, X_1, \pi_1)\) and \((Z_2, X_2, \pi_2)\) are equivalent if and only if there
exist \( g \in PGL(4) \) and \( h \in PGL(3) \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
Z_1 & \xrightarrow{g} & Z_2 \\
\pi_1 & \downarrow & \downarrow \pi_2 \\
X_1 & \xrightarrow{h} & \mathbb{P}^2 \supset X_2
\end{array}
\]

and \( h(X_1) = X_2 \).

### 3.2. Derivation of \( \phi_u \) for a smooth trilinear form

The algorithm for deriving the \( \phi_u \) representations for smooth trilinear forms has been described in Theorem 2. We derive the trilinear form \( \phi \) which corresponds to \( X = \{ p_i \mid 1 \leq i \leq 6 \} \) represented by the matrix

\[
X(\lambda, \mu, \alpha, \beta) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & \lambda & \alpha \\
0 & 0 & 1 & 1 & \mu & \beta
\end{pmatrix}
\]

with \( p_i \) being the \( i \)-th-column.

We can choose the following basis elements \( f_k \)'s \( (k = 1, 2, 3, 4) \) for \( W \) defined in the theorem:

\[
\begin{align*}
f_1 &= x_3(\lambda x_1 - x_2)(\alpha - \beta)x_1 + (\beta - 1)x_2 + (1 - \alpha)x_3, \\
f_2 &= x_3(\alpha x_1 - x_2)(\lambda - \mu)x_1 + (\mu - 1)x_2 + (1 - \lambda)x_3, \\
f_3 &= (\lambda x_1 - x_2)(x_1 - x_3)(\beta x_2 - \alpha x_3), \\
f_4 &= x_2(\mu x_1 - x_3)(\alpha - \beta)x_1 + (\beta - 1)x_2 + (1 - \alpha)x_3.
\end{align*}
\]

Note that each \( f_k \) represents a union of three suitably chosen lines passing through \( X \). This choice simplifies the calculations considerably.

To derive \( \phi_u \), we solve the following equation to find all the \( U \)-linear relations among \( f_k \)'s:

\[
\begin{align*}
(\phi_{11}x_1 + \phi_{21}x_2 + \phi_{31}x_3)f_1 + (\phi_{12}x_1 + \phi_{22}x_2 + \phi_{32}x_3)f_2 \\
+ (\phi_{13}x_1 + \phi_{23}x_2 + \phi_{33}x_3)f_3 + (\phi_{14}x_1 + \phi_{24}x_2 + \phi_{34}x_3)f_4 &= 0.
\end{align*}
\]

The above equation is equivalent to a linear system of equations with twelve unknowns \( \{ \phi_{ik} \mid 1 \leq i \leq 3, 1 \leq k \leq 4 \} \) and twelve equations (corresponding to the vanishing of the coefficients of the twelve quartic monomials \( x_1^2x_2, x_1^2x_3, x_1^2x_2^2, x_2^2x_3, x_1^2x_3^2, x_1x_2^2x_3, x_1x_2x_3^2, x_1x_3^3, x_2^3x_3, x_2^2x_3^2, x_2x_3^3 \)). We always have a three dimensional solution which we index by \( j = 1, 2, 3 \), since the index \( j \) is reserved for the vector space \( V^* \).

With the aid of the Maple with(linalg) package, the system can be solved very easily. We write down the \( (4 \times 3) \) syzygy matrix \( M_u \) instead
of its transpose $\phi_u$ due to space constraint:

$$
M_u = \begin{pmatrix}
\alpha(\beta-1)(\mu-\lambda)x_1 & \alpha(\lambda-\mu)[(\lambda(\beta-1)+(\alpha-\beta)]x_1 & \alpha(\mu-\lambda\beta)+\beta(\lambda-\mu)x_1 \\
+(\alpha-\beta)(\mu-1)x_2 & +\alpha[\lambda\mu(\alpha-1) + \lambda(1-\beta)]x_2 & +\alpha\lambda(\mu-\lambda\beta)(1-\beta) \\
+\alpha(\lambda-1)(\beta-1)x_3 & +\lambda^2(1-\beta)+(1-2\lambda)(\alpha-\beta)x_3 & +\beta(\lambda-\alpha)x_3 \\
-(\alpha-1)(\mu-1)x_1 & -(\lambda-1)(\alpha-\lambda)x_1 & \lambda(-\lambda x_1+2x_1)
\end{pmatrix}
$$

### 3.3. Schläfli double six

From Proposition 1, we know that $X_\phi$ and $Y_\phi$ are associated point sets and $\pi^{-1}(X_\phi)$ and $(\pi')^{-1}(Y_\phi)$ form a Schläfli double six on $Z_\phi$. From the matrix expression of $\phi_v$, we find that $Y_\phi$ consists of the following points:

$$
p'_1 = \begin{bmatrix} 1 & \beta - 1 \\ \alpha + \lambda\beta - \lambda - \beta & 0 \end{bmatrix},
$$

$$
p'_2 = \begin{bmatrix} \beta - 1 \\ \alpha + \lambda\beta - \lambda - \beta \end{bmatrix},
$$

$$
p'_3 = \begin{bmatrix} \lambda - \alpha \alpha\beta \end{bmatrix},
$$

$$
p'_4 = \begin{bmatrix} \lambda - \alpha \alpha\beta \end{bmatrix},
$$

$$
p'_5 = \begin{bmatrix} \lambda - \alpha \alpha\beta \end{bmatrix},
$$

$$
p'_6 = \begin{bmatrix} \lambda - \alpha \alpha\beta \end{bmatrix}.
$$

It is easy to verify that the $Y_\phi = \{p'_1, \ldots, p'_6\}$ is $PGL(3)$-equivalent (with the fixed ordering as above) to the standard matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & (\beta-1)(\lambda\beta-\alpha\mu) \\
0 & 0 & 1 & 1 & (\alpha-1)(\lambda\beta-\alpha\mu)
\end{pmatrix}
$$
Left-multiplying the above matrix by the inverse of the matrix formed by the last three columns and performing a \((\mathbb{C}^*)^6\)-action, we get:

\[
Y_\phi \sim_{\text{PGL}(3)} \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & \lambda & \mu & 0 & 1 & 0 \\
1 & \alpha & \beta & 0 & 0 & 1
\end{pmatrix} \sim_{S_6} \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & \lambda & \mu \\
0 & 0 & 1 & 1 & \alpha & \beta
\end{pmatrix}.
\]

Hence, up to projective equivalence, the association map (which is clearly an involution) \(X_\phi \mapsto Y_\phi\) is given by

\[
X(\lambda, \mu, \alpha, \beta) \mapsto X(\lambda, \alpha, \mu, \beta).
\]

### 3.4. Relation between configuration space of six points and trilinear forms

The configuration space \(\mathcal{P}\) of six points in general position is studied in [30]. It is proven that \(\mathcal{P}\) admits a compactification \(\overline{\mathcal{P}}\) via the addition of some configurations not in general position. Moreover, \(\overline{\mathcal{P}}\) admits an involution \(*\) (which is the association map applied to a point set) whose fixed points are precisely the configurations which lie on some (not necessarily smooth) conic. The quotient \(\overline{\mathcal{P}}/*\) is isomorphic to the linear space \(\mathbb{P}^4\). On the other hand, \(\mathcal{Q}\) admits an involution \(\tau\) which interchanges the vector spaces \(U\) and \(V\), inducing the involution \(X_\phi \leftrightarrow Y_\phi\). There is an obvious commutative diagram relating the quasijective varieties \(\mathcal{P}\) and \(\mathcal{Q}\):

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{S_6} & \mathcal{Q} \\
\mathbb{Z}/2 \downarrow & & \downarrow \mathbb{Z}/2 \\
\mathcal{P}/\langle * \rangle & \xrightarrow{S_6} & \mathcal{Q}/\langle \tau \rangle
\end{array}
\]

### 4. Multiple points and l-configurations on the plane

Since a smooth trilinear form \(\phi\) is completely determined by its determinantal variety \(X_\phi\), we expect the same of singular trilinear forms for which \(X_\phi\) is at least pure dimensional. In this and the next two sections, we deal with all the possible ways in which a zero dimensional length six subscheme \(X\) on the plane may degenerate from one in general position. \(X\) shall always refer to a zero dimensional length six subscheme of the projective plane with homogeneous coordinates \([x_1, x_2, x_3]\). We also call such a subscheme \(X\) in the plane a configuration.
4.1. Configurations in the plane

We include a short discussion on what happens if we have a configuration $X$ which does not give rise to a smooth trilinear form. When we specialize a scheme $X$ in general position to one with collections of three (or more) collinear points, one with multiple points or one lying on a conic, we can still do the following:

(i) compute the linear system $|W| := \langle f_1, \cdots, f_4 \rangle$ of homogeneous cubic polynomials vanishing at $X$,

(ii) compare the locus $B := \{ [u^*] \in \mathbb{P}(U^*) \mid f_k(u^*) = 0, k = 1, 2, 3, 4 \}$ of the linear system $|W|$ with $X$, and obtain the surface $Z$ defined by the rational mapping $\phi_{uw}(x_1, x_2, x_3) = [f_1, \cdots, f_4]$ (setting $z_k = f_k$ and using elimination theory, we can find the equation for $Z$ but we do not perform these calculations in this paper),

(iii) compute the syzygy matrix $\phi_u$ for the vector space $W$ by solving Equation (1) and obtain $X_\phi$ and the surface $Z_\phi$ from $\phi_w$.

If $X$ results in a smooth trilinear form, we have the situation in Theorem 2.

If $X$ results in a Duval trilinear form $\phi$ with $B = X = X_\phi$ and $Z = Z_\phi = \phi_{uw}(\mathbb{P}(U^*))$. It is then easy to see that $Z$ is the blowup of $\mathbb{P}(U^*)$ at $X$ followed by the contraction of the (rational) $(-2)$-curves on a smooth surface with an appropriate choice of coordinates for $\mathbb{P}(U^*)$. We can call these configurations Duval. However, we have various anomalies for non-Duval configurations of $X$:

(i) $X$ is such that $X \subset B$ (but not equal) and $B$ is not pure dimensional, i.e., $X$ is not even the base locus of any three dimensional linear system of cubics.

(ii) $B = X$, a cubic surface $Z$ is defined by the three dimensional linear system $|W|$ and the syzygy matrix $\phi_u$ is a $(3,3,4)$ trilinear form, but either $Z \neq Z_\phi$ or $X \neq X_\phi$ (or both).

(iii) $B = X$, again $Z$ and $\phi_u$ are obtainable with $Z = Z_\phi$ and $X = X_\phi$ but $Z$ has non-Duval singularities.

Hence, we need some definitions to distinguish between these non-Duval configurations:

**Definitions.**

(i) We say that a scheme $X$ is a base locus or $X$ is b.l. for short if the linear system of cubics vanishing at $X$ is three dimensional.

(ii) We say that a scheme $X$ is admissible if $X$ is b.l., $X = X_\phi$ and $Z = Z_\phi$ (hence inadmissible if $X$ is either non-b.l., $X \neq X_\phi$ or $Z \neq Z_\phi$).
(iii) We say that a scheme $X$ is Duval (or smooth) if it is admissible and $Z$ has only $A$ or $D$ singularities (respectively smooth).

An admissible scheme $X$ which is non-Duval is precisely one for which $Z_{\phi}$ has non-Duval singularities, i.e., $\phi$ is a reducible, conical or nodal trilinear form (See Section 7). Such a scheme $X$ has a point of multiplicity $\geq 3$ which is non-curvilinear (See Section 4.2).

Most inadmissible but b.l. schemes $X$ correspond to a unique trilinear form $\phi_{c.i.}$ called the "c.i." (complete intersection) trilinear form in Theorem 3 and Theorem 5. For $\phi_{c.i.}$, $X_{\phi} = \mathbb{P}(U^*)$, $Y_{\phi} = \mathbb{P}(V^*)$ and $Z_{\phi} = \mathbb{P}(W^*)$. On the other hand, there exists an inadmissible b.l. scheme $X$ which consists of a non-curvilinear multiplicity 6 point such that $X = X_{\phi}$ but $Z \neq Z_{\phi}$ since $Z$ is the twisted cubic curve in $\mathbb{P}(W^*)$ but $Z_{\phi} = \mathbb{P}(W^*)$. The corresponding trilinear form $\phi$ is not $\phi_{c.i.}$ (See Section 4.2.7).

All non-b.l. schemes $X$ contain a non-curvilinear multiple point of multiplicity $\geq 3$ or more than three points lying on a line. $B$ is always non pure dimensional (since the generators of the linear system $|W|$ necessarily share a common factor) and the relations among the $f_k$'s is not three dimensional and hence do not give us $(3, 3, 4)$ trilinear forms.

In Sections 5 and 6, we would derive all the Duval trilinear forms by considering a comprehensive list of configurations of six points on the plane and deciding which ones are admissible. In Section 7, we would derive all the other non-Duval trilinear forms. Before we do that, it is necessary for us to know the defining ideals of multiple points on the affine plane.

**4.2. On $l$-configurations and multiple points of $\mathbb{P}^2$**

To derive singular trilinear forms, we allow a smooth configuration to have degeneracies such as multiple points or collections of three collinear points. Hence, it is inevitable that we deal with points which are in the first and higher order infinitesimal neighborhoods of points in the affine plane Spec $\mathbb{C}[x, y]$. In the following paragraphs, we state the notations for describing multiple points in the plane.

**4.2.1. $l$-configurations.** If there exists a line which contains three or more points in $X$ (possibly with some multiple points), we say that $X$ has a $l$-configuration. If $X$ contains exactly one $l$-configuration consisting of three distinct points, the rational map $\varphi_{uv}$ defines a singular cubic surface with an unique $A_1$ singularity. More generally, we have more $A_1$ singularities on $Z$ if $X$ has more $l$-configurations (at most four).
If a line \( L \) contains four (or more) points in \( X \), then it necessarily lies in the base locus of the linear system vanishing at \( X \). Hence, \( X \) is not b.l.. It is easy to verify that the linear system does not generate a three dimensional syzygy. Hence, \textit{we shall always assume that an l-configuration (if any) contains exactly three points in \( X \)}.

\textbf{4.2.2. Double points.} It is well known that a double point \( P \) on the affine line \( \text{Spec} \, \mathbb{C}[x] \) with support at the origin is isomorphic to the scheme \( \text{Spec} \, \mathbb{C}[x]/(x^2) \). It can be embedded in the affine plane \( \mathbb{C}[x, y] \) such that its support is the origin \( p_1 \) and it lies on the \( x \)-axis, i.e., it is scheme-theoretically defined by the ideal \((y, x^2)\). Again, if \( X \) has only one double point but no \( l \)-configuration, then \( \phi_{uw} \) defines a singular surface with an unique \( A_1 \) singularity.

We say that \( P \) lies on a cubic curve \( C \) passing through \( p_1 \) if \( C \) is smooth at \( p_1 \) and the tangent line to \( C \) at \( p_1 \) is defined by \( y = 0 \), or if \( C \) is singular at \( p_1 \).

We say that the double point \( P = \text{Spec} \, \mathbb{C}[x, y]/(y-\alpha x, x^2) \) is an arrow \textit{lying on the line} \( y-\alpha x = 0 \). If \( P \) is a subscheme of a configuration \( X \) and \( X \) has another point \( q \neq p_1, p_2 \) lying on the line \( y-\alpha x = 0 \), we say that \( P \) is \textit{pointing at} \( q \). If \( P \) does not point at any point in \( X \), we say that \( P \) is \textit{generic}.

To represent a double point with support at \([0, 0, 1]\) and lying along the line \( x_2 - \alpha x_1 = 0 \), we use self-explanatory matrix expression
\[
\begin{pmatrix}
0^2 & -\alpha \\
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

\textbf{4.2.3. Multiplicity \( \geq 3 \) points in the plane.} It is well known (See [20]) that curvilinear multiplicity \( n \) points \( P \) supported at the origin in \( \text{Spec} \, \mathbb{C}[[x, y]] \) form an \((n-1)\) dimensional family and can be defined by the ideals \( \mathcal{I} = (y + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}, x^n) \) for \( a_i \in \mathbb{C} \). Of course, we can always assume \( a_1 = 0 \) by replacing \( y + a_1 x \) by \( y \). On the other hand, \( a_2 = 0 \) if and only if \( P \) contains a \textit{linear} (which [20] called \textit{aligned}) triple point at the origin, i.e., the triple point defined by the ideal \((y, x^3)\). In subsequent cases with \( n \geq 4 \), we consider only multiple points which lie on some cubic curve \( y + a_2 x^2 + a_3 x^3 \), i.e., with defining ideal \((y + a_2 x^2 + a_3 x^3, x^n)\), since we are interested in syzygy relations among a homogeneous cubic system.

\textbf{4.2.4. Triple points.} From the above section, curvilinear triple points \( P \) with support at the origin \( p_1 \) from a one dimensional family with defining ideals \((y - \beta x^2, x^3)\). If \( \beta \neq 0 \), we have a \textit{strictly curvilinear}
The classification of (3,3,4) trilinear forms

triple point (we call it a 3c), in which case \((y - \beta x^2, x^2) = (y - \beta x^2, xy)\) as well. If \(\beta = 0\), we have a linear triple point (we call it a 3l).

A cubic curve \(C\) through \(p_1\) contains a curvilinear triple point \(P\) scheme-theoretically if and only if \(C\) is singular at \(p\) or \(C\) is smooth and the local equation of \(C\) at \(p_1 = (0,0)\) is \(y - \beta x^2 = 0\), i.e., \(C\) is defined by \(y - \beta x^2 = 0\) in the local ring \(\mathbb{C}[x,y][x,y]\) at the origin. \(P\) is linear if and only if \(\beta = 0\), i.e., \(p_1\) is a point of inflection on \(C\). For convenience in computations, we always choose coordinates so that the arrow in a triple point (or higher order multiple point) lies on a coordinate line, for example \(x_2 = 0\).

To describe a strictly curvilinear triple point with support at \([0,0,1]\) (regarded as the origin in the affine plane \(x_3 = 1\)) and lying on the conic \(x_2 x_3 - \beta x_1^2 = 0\) (\(\beta \neq 0\)), we use the notation \((0^3, 0^1, -\beta^1, 0^0, 1^0, 0^0)\) (the second column indicates that the arrow in the triple point lies on \(x_2 = 0\)). To describe a linear triple point at \([0,0,1]\) lying along \(x_2 = 0\), we set \(\beta = 0\).

It is easy to check that if \(X\) is admissible and it contains a strictly curvilinear triple point \(P = 3c\) with a generic arrow, then \(Z_{\phi}\) has an \(A_2\) singularity. If \(P = 3l\) is linear, we have an additional \(A_1\) singularity.

There is only one isomorphism type \(\text{Spec} \ \mathbb{C}[x,y]/(x,y)^2\) of non-curvilinear triple point (we call it a 3s since the generators of its defining ideal are either singular or non-reduced curves). Since a family of 3c’s can be defined scheme-theoretically by \((y - \beta x^2, xy) = (y - \beta x^2, xy, y^2) = (x^2 - \frac{1}{3} y, xy, y^2)\) for \(\beta \in \mathbb{C}^*\), we get 3s as the limit of the family of 3c’s as \(\beta \to \infty\). It can also be thought of as the entire first order neighborhood of \(p_1\). Since the ideal defining \(P\) is generated by degree two polynomials, any curve containing \(P = 3s\) must be singular at \(p_1\). If \(X\) contains a 3s (but no 4-point or 5-point), then it is either non-b1. or it is admissible and \(\phi\) is reducible (either \(QP\) or \(CP\), see Section 7).

4.2.5. Points of multiplicity four. From Section 4.2.3, strictly curvilinear 4-points \(P = 4c\) in the plane are defined by ideals \(I = (y + a_2 x^2 + a_3 x^4)\) with \(a_2 \neq 0\) if \(P\) does not contain a 3l. We note that since \(a_2 \neq 0, x^2 y \in I, y^2 \in I\) and in fact \(I = (y + a_2 x^2 + a_3 x^4, x^4) = (y + a_2 x^2 + a_3 x^3, x^2 y) = (y + a_2 x^2 + a_3 x^4, y^2)\). Further, with power series expansion in \(\mathbb{C}[[x,y]]\), we also have \(I = (y + \alpha xy + \beta x^2, x^2 y)\) for some \(\alpha \in \mathbb{C}\) and \(\beta \in \mathbb{C}^*\).

On the other hand, if a 4-point contains a 3l but is itself non-linear (i.e., it does not lie on the line \(y = 0\)), its defining ideal is \((y + a_3 x^3, x^4) = (y + a_3 x^3, xy)\) for \(a_3 \neq 0\).
For convenience, we would call a 4-point which contains a 3c a 4c and one which contains a 3l a 4l (with the implicit understanding that the latter is not linear). This causes no confusion since we do not consider l-configurations with more than three points.

To describe a 4-point with support at \([0, 0, 1]\) and lying on the conic \(x_2x_3 + \alpha x_1^3 + \beta x_1^2 = 0\), we use the notation \(
\begin{pmatrix}
0^4 & 0 & \alpha & \beta \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 
\end{pmatrix}
\). The 4-point is a 4c if and only if \(\alpha \neq 0\); it is a 4l if and only if \(\alpha = 0\) but \(\beta \neq 0\).

In general, if \(X\) is admissible and has a curvilinear 4-point (but no 5-point), then \(Z_\phi\) has an \(A_3\), \(A_4\) (possibly with one more \(A_1\) singularity) or \(D_4\) singularity (See Section 6).

There are three isomorphism types of non-curvilinear 4-points on the plane:

(i) Spec \(\mathbb{C}[x, y]/(x^3, xy, y^2)\),

(ii) Spec \(\mathbb{C}[x, y]/(x^2, y^2) \cong \text{Spec} \mathbb{C}[x, y]/(xy, x^2 - y^2)\), or

(iii) Spec \(\mathbb{C}[x, y]/(x^2, y(x - y))\).

If \(X\) has a 4s of the first type (but no 5-point), then \(X\) is non-b.l. or it is admissible and \(\phi\) is reducible (of type \(QT\), see Section 7). If \(X\) has a 4s of the second or third type (but no 5-point), then \(X\) is non-b.l., inadmissible or it is admissible such that \(Z_\phi\) has a nodal line (i.e., \(\phi\) is \(NL\), see Section 7).

4.2.6. Points of multiplicity five. A strictly curvilinear 5-point in the plane is defined by some ideal \(\mathcal{I} = (y + a_2x^2 + a_3x^3 + a_4x^4, x^5)\) and we can replace \(x^5\) by either one of \(x^3y\) or \(xy^2\) if \(a_2 \neq 0\), i.e., if it contains a 3c. Since we are only dealing with cubic polynomials, the 5-points we are considering are of the type with \(a_4 = 0\).

If a 5-point contains a 3l, i.e., \(a_2 = 0\), then it is defined by \(\mathcal{I} = (y + a_3x^3, x^5) = (y + a_3x^3, x^2y)\), where \(a_3 \neq 0\).

Like the case of 4-points, we would call a 5-point which contains a 3c a 5c and one which contains a 3l a 5l.

To describe a 5c with support at \([0, 0, 1]\) and lying on the conic \(x_2x_3 + \alpha x_1^3 + \beta x_1^2 = 0\), we use the notation \(
\begin{pmatrix}
0^5 & 0 & \alpha & \beta \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 
\end{pmatrix}
\), where \(\alpha \neq 0\) if we have a 5c; and \(\alpha = 0\) but \(\beta \neq 0\) if we have a 5l.

In general, if \(X\) is admissible and has a curvilinear 5-point (but no 6-point), then \(Z\) has a \(D_5\) singularity.

There are considerably more variations of non-curvilinear 5-points \(P\) on the plane since the generators may be cubics or conics. Moreover, \(P\)
may not be a local complete intersection. Among them, the only ones giving us trilinear forms are (see Section 7):

(i) \( \text{Spec } \mathbb{C}[x,y]/(xy - x^3, x^2y, y^2) \) (a \( QT \) reducible trilinear form),
(ii) \( \text{Spec } \mathbb{C}[x,y]/(x^3, xy, y^3) \) (a \( CV \) reducible trilinear form),
(iii) \( \text{Spec } \mathbb{C}[x,y]/(x^3, x^2y, y^2) \) (a reducible \( CT \) trilinear form) and
(iv) \( \text{Spec } \mathbb{C}[x,y]/(y^3, y^2 - x^3, y^2 - \lambda xy) \) \((\lambda \neq 0)\) (a \( NL2 \) nodal trilinear form).

4.2.7. Points of multiplicity six. A strictly curvilinear 6-point in the plane is defined by some ideal \( I = (y + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5, x^6) \) with \( x^6 \) replaceable by either one of \( x^4y, x^2y^2 \) or \( y^3 \) if \( a_2 \neq 0 \), i.e., if it contains a 3c. We only consider points with \( a_4 = a_5 = 0 \).

If a 6-point contains a 3l, i.e., \( a_2 = 0 \), then it is defined by \( I = (y + a_3x^3, x^6) = (y + a_3x^3, x^3y) = (y + a_3x^3, y^2) \), where \( a_3 \neq 0 \).

Again, we would call a 6-point which contains a 3c a 6c and one which contains a 3l a 6l.

If \( X \) has a 6c, then \( X \) is always c.i. and \( Z \) has a unique \( A_5 \) singularity. If \( X \) has a 6l, then \( X \) is also c.i. and \( Z \) has a unique \( E_6 \) singularity. Hence, \( X_\phi \) is never isomorphic to 6c or 6l for any \((3, 3, 4)\) trilinear form \( \phi \). The latter statement on 6l lends support to the well known fact that an \( E_6 \) cubic surface cannot be determinantly represented.

Like the case for 5s, we have many non-isomorphic 6s. Among them, we have:

(i) \( \text{Spec } \mathbb{C}[x,y]/(x,y)^3 \) \((X \text{ is b.i. but inadmissible since it gives rise to a trilinear form with } X_\phi = X, Z_\phi = \mathbb{P}(W^*) \text{ but } Z \text{ is a twisted cubic in } \mathbb{P}(W^*) \),
(ii) \( \text{Spec } \mathbb{C}[x,y]/(y^2 - xy - x^3, x^2y, xy^2) \) (a \( CV \) reducible trilinear form),
(iii) \( \text{Spec } \mathbb{C}[x,y]/(y^2 - x^3, x^2y, xy^2) \) (a \( CT \) reducible trilinear form),
(iv) \( \text{Spec } \mathbb{C}[x,y]/(xy - y^3, x^3 - y^3, xy^2) \) (a \( CN \) conical trilinear form),
(v) \( \text{Spec } \mathbb{C}[x,y]/(y^2 - x^3, x^3 - x^2y, xy^2) \) (a \( CC \) conical trilinear form) and
(vi) \( \text{Spec } \mathbb{C}[x,y]/(y^3, xy^2, y(y - x^2), y^2 - xy - \lambda x^3) \) \((\lambda \neq 0)\) (a \( NL2 \) nodal trilinear form).

We would classify Duval trilinear forms in Sections 5 and 6 in Theorems 3 and 5 respectively, which all correspond to admissible configurations with at worst curvilinear multiple points. Non-Duval trilinear forms, which correspond to admissible configurations with non-curvilinear multiple points or non-pure dimensional configurations would be classified according to the geometry of \( Z_\phi \) in Section 7.
5. Duval trilinear forms with $A_{\leq 2}$ singularities

Since a Duval trilinear form $\phi$ is uniquely determined by its determinantal variety $X_\phi$, a zero dimensional length six subscheme of the projective plane, we can derive all Duval trilinear forms $\phi$ by exhausting all zero dimensional length six subschemes in the plane, possibly with only curvilinear multiple points, compute the linear systems $|W|$ passing through $X_\phi$ and then the syzygy matrices $\phi_u$’s using a computer algebra package (Maple). There are eighty three types of such schemes but only sixty types are admissible (thirty three with $A_{\leq 2}$ singularities in Theorem 2, twenty seven with $A_{\geq 3}$, $D_4$ or $D_5$ singularities in Theorem 4); the rest lie on some conic and lead to the unique c.i. trilinear form (ten types in Theorem 3 and thirteen types in theorem 5).

5.1. $A_1^{n_1} A_2^{n_2}$ trilinear forms

We label a Duval trilinear form $\phi$ by stating the singularity type of $Z_\phi$ and a suffix (a),(b)... etc, if there are more than one type of trilinear forms with isomorphic $X_\phi$ but non-isomorphic $X_\phi$.

A Duval trilinear form $\phi$ is characterized geometrically by $X_\phi$ which is a picture consisting of points, arrows, multiple points and l-configurations in the plane.

5.1.1. Notation. We use the following information to describe each type of trilinear form:

- an algebraic representation of $X_\phi$ by a $(3 \times 6)$ matrix (labeled simply as $X_\phi$) similar to the one for smooth trilinear forms in Theorem 2.
- the matrix $\phi_u$ for the trilinear forms $\phi$.

Theorem 3 (Duval Trilinear Forms with $A_{\leq 2}$ singularities). There are exactly thirty three types of non-$G$-equivalent Duval trilinear forms with $A_{\leq 2}$ singularities. Up to $G$-equivalence, the dimension of the family of the trilinear forms of type $A_1^{n_1} A_2^{n_2} (\ast)$ is $\max\{0, 4 - (n_1 + 2n_2)\}$, with the exception of the $A_2^2(a)$ and $A_2^2(b)$ trilinear forms which form one dimensional families.

\[
\begin{align*}
\text{1. } & [1, 0, 0] \quad [1, -\lambda, \mu] \quad [0, 1, 0] \\
& [1, \nu, -1] \quad [0, 1, -1] \\
& [0, 0, 1] \quad A_1(a) \quad X_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & \nu & -\lambda \\ 0 & 0 & 1 & -1 & -1 & \mu \end{pmatrix}
\end{align*}
\]
\[ \phi_u = \begin{pmatrix} \mu x_1 + \mu x_2 + (\lambda - 1)x_3 & 0 & 0 & \lambda x_1 + x_2 \\ (\lambda - \mu - 1 - \nu)x_1 & (\lambda - 1)(x_2 + \nu x_3) & 0 & - (\lambda + \nu)x_2 \\ (\lambda - \mu - 1 + \nu)x_1 & \nu(\lambda - \mu)x_1 - (1 - \nu)x_2 & x_1 & 0 \end{pmatrix}. \]

We have \((\lambda, \mu, \nu) \in (\mathbb{P}^1)^3\) with \(\lambda \neq 0, 1, \mu, -\nu, \mu - \nu + 1, \mu \nu, \infty, \mu \neq 0, -1, \infty\) and \(\nu \neq 0, 1, \infty\).

\[ 2. \quad [0, 1, 0] \quad A_1(b) \quad X_\phi = \begin{pmatrix} 0^2 & 1 & 0 & 1 & 1 & \lambda \\ 1 & 0 & 0 & -1 & 0 & -\mu \\ 0 & \alpha & 1 & 0 & -1 & 1 - \lambda \end{pmatrix}. \]

\[ \phi_u = \begin{pmatrix} d \\ (\lambda - 1)x_1 \\ \mu x_1 + x_2 + \nu x_3 \\ x_2 \\ 0 \\ x_1 \\ 0 \end{pmatrix}, \]

where \(d := \alpha(1 - \lambda)x_2 + [\lambda(\mu - 1) + \alpha \mu(1 - \lambda)]x_3, \quad e := [\lambda(\mu - 1) + \alpha \mu(1 - \lambda)]x_1 + [\lambda(\mu - 1) + \alpha(\lambda + \mu - 2\lambda \mu) + \alpha^2(\lambda - 1)(\mu - 1)]x_2 + (1 + \alpha)[\lambda(\mu - 1) + \alpha \mu(1 - \lambda)]x_3, \quad f := (\mu - 1)(\lambda - \lambda \alpha + \alpha)x_2 + [\lambda(\mu - 1) + \alpha \mu(1 - \lambda)]x_3. \]

We have \((\lambda, \mu, \alpha) \in (\mathbb{P}^1)^3\) with \(\lambda \neq 0, 1, \mu, \infty, \mu \neq 0, 1, \infty\) and \(\alpha \neq 0, 1, \infty, \frac{\lambda}{\lambda - 1}\). Also, \(\alpha \neq \frac{\lambda(\mu - 1)}{\mu(\lambda - 1)}\), i.e., \(\lambda \mu - \lambda + \alpha \mu - \lambda \alpha \mu \neq 0\) so that \(X_\phi\) does not lie on the conic \(\mu(\lambda - 1)(x_1^2 + x_2 x_3 + x_1 x_3) + (\lambda - 1)x_2 x_3 = 0\) through \([X_0]\).

\[ 3. \quad [1, 0, 0] \quad [1, -\lambda, \mu] \quad [0, 1, 0] \quad [1, 0, -1] \quad [0, 1, -1] \quad [0, 0, 1] \quad A_3^2(a) \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -\lambda \\ 0 & 0 & 1 & -1 & -1 & \mu \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} \mu x_1 + \mu x_2 + (\lambda - 1)x_3 & 0 & 0 & \lambda x_1 + x_2 \\ (\lambda - \mu - 1)x_3 & x_2 & 0 & -\lambda x_3 \\ (\lambda - \mu - 1)x_3 & 0 & x_1 & -x_3 \end{pmatrix}. \]

We have \((\lambda, \mu) \in (\mathbb{P}^1)^2\) with \(\lambda \neq 0, \mu, + 1, \infty\) and \(\mu \neq 0, -1, \infty\).

\[ 4. \quad [0, 1, 0] \quad [1, 0, 0] \quad A_3^2(b) \quad \begin{pmatrix} 0^2 & 1 & 0 & 1 & 1 & \lambda \\ 1 & 0 & 0 & -1 & 0 & -\mu \\ 0 & 0 & 1 & 0 & -1 & 1 - \lambda \end{pmatrix}. \]
\[ \phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_1 + x_2 + x_3 \\ (\lambda - 1)x_1 & \mu x_1 + x_2 + \mu x_3 & 0 & \lambda(\mu - 1)(x_1 + x_3) \\ 0 & x_2 & x_1 & 0 \end{pmatrix}. \]

We have \((\lambda, \mu) \in (\mathbb{P}^1)^2\) with \(\lambda \neq 0, 1, \mu, \infty\) and \(\mu \neq 0, 1, \infty\).

---

5. \([0, 1, 0]^2 [1, -1, 0]\)

\[ A_1^2(e) \quad X_\phi = \begin{pmatrix} 0^2 & \alpha & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} x_3 & 0 & x_2 & \alpha(x_1 + x_2) \\ \lambda x_1 & x_2 & \alpha(1 - \lambda)x_1 + \alpha x_3 \\ 0 & 0 & x_1 & x_3 \end{pmatrix}. \]

We have \((\lambda, \alpha) \in (\mathbb{P}^1)^2\) with \(\lambda \neq 0, -1, \infty\) and \(\alpha \neq 0, -1, \lambda, \infty\).

---

6. \([1, 0, 0]^2 [0, 0, 1]\)

\[ A_1^2(d) \quad X_\phi = \begin{pmatrix} 1^2 & 0 & 0^2 & 1 & 0 \\ 0 & \alpha_2 & 1 & 0 & 1 \\ 0 & 1 & 0 & \alpha_1 & 1 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} x_3 & 0 & (1 - \alpha_1)x_2 & x_2 \\ -\alpha_2 x_1 & x_2 & \alpha_2(1 - \alpha_1)x_2 & (\alpha_1\alpha_2 - \alpha_2 + 1)x_1 + \alpha_2(1 - \alpha_1)x_2 \\ 0 & x_3 & x_1 + x_3 & \alpha_2 x_1 + x_3 \end{pmatrix}. \]

We have \((\alpha_1, \alpha_2) \in (\mathbb{P}^1)^2\) with \(\alpha_1, \alpha_2 \neq 0, 1, \infty\). Also, \(\alpha_1\alpha_2 - \alpha_2 + 1 \neq 0\) so that \(X_\phi\) does not lie on the unique conic \(x_1 x_2 + \alpha_1 x_2 x_3 + (1 - \alpha_1)x_1 x_3 = 0\) which contains \(|X_\phi|\) and the arrow at \([0, 1, 0]\).

---

7. \([0, 1, 0]^2 [1, -1, 0] [1, 0, 0]\)

\[ A_2(a) \quad X_\phi = \begin{pmatrix} 1 & 0^2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}. \]
The classification of $[3,3,4]$ trilinear forms

\[
\phi_u = \begin{pmatrix} x_3 & 0 & -\alpha x_2 & x_1 + x_2 \\ 0 & x_2 & -x_1 & x_1 \\ 0 & \lambda x_3 & x_1 & x_3 \end{pmatrix}.
\]

We have $(\lambda, \alpha) \in (\mathbb{P}^1)^2$ with $\lambda \neq 0, -1, \infty$ and $\alpha \neq 0, 1, \infty$.

8. \quad \bullet \quad \bullet \quad \bullet

\begin{align*}
[A_2(b)] & \quad X_\phi = \begin{pmatrix} 0 & 0^3 & 0 & -\beta & 1 & \lambda \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \\
[0, 0, 0] & \quad \phi_u = \begin{pmatrix} x_2 & x_1 \\ \beta x_3 & (1 - \beta) x_3 \\ 0 & x_3 \end{pmatrix}.
\end{align*}

We have $(\lambda, \beta) \in (\mathbb{P}^1)^2$ defined by $\lambda \neq 0, 1, \infty$ and $\beta \neq 0, \infty$. Also, $\beta \neq 1/(1 - \lambda)$, i.e., $\lambda \beta - \beta + 1 \neq 0$, so that $X_\phi$ does not lie on the conic $x_1^2 - \lambda x_1 x_2 + (\lambda - 1) x_2 x_3 = 0$ which contains $|X_\phi|$ and the arrow at $[0, 0, 1]$.

9. \quad \bullet \quad \bullet \quad \bullet

\begin{align*}
[A_1(a)] & \quad X_\phi = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -\lambda \\ 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}, \\
[0, 0, 0] & \quad \phi_u = \begin{pmatrix} x_3 & 0 & 0 & \lambda x_1 + x_2 \\ 0 & x_2 & 0 & x_1 + x_3 \\ 0 & 0 & x_1 & x_2 + x_3 \end{pmatrix}.
\end{align*}

We have $\lambda \in \mathbb{P}^1$ with $\lambda \neq 0, 1, \infty$.

10. \quad \bullet \quad \bullet \quad \bullet

\begin{align*}
[A_1(b)] & \quad X_\phi = \begin{pmatrix} 0^2 & 1 & 0 & 1 & 1 & \lambda \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 - \lambda \end{pmatrix}, \\
[0, 0, 0] & \quad \phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_1 + x_2 + x_3 \\ (\lambda - 1)x_1 & x_2 & 0 & -\lambda(x_1 + x_3) \\ 0 & 0 & x_1 & x_3 \end{pmatrix}.
\end{align*}

We have $\lambda \in \mathbb{P}^1$ with $\lambda \neq 0, 1, \infty$. 
11. \[ \begin{array}{c}
\bullet & \bullet & \bullet \\
[0, 1, 0]^2 & [0, 0, 0]^2
\end{array} \]
\[ \begin{bmatrix} x_3 & x_2 & (1-\alpha)x_3 \\
x_1 & 0 & x_1 + x_2 \\
0 & x_1 & (1-\alpha)x_3 + x_1 + x_3
\end{bmatrix} \]
\[ \varphi_u = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \]
\[ X_\varphi = \begin{pmatrix}
1 & 0^2 & 1 & 0^2 & \alpha & 1 \\
0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1 & 0 & -1
\end{pmatrix} \]

We have \( \alpha \in \mathbb{P}^1 \) with \( \alpha \neq 0, 1, \infty \).

12. \[ \begin{array}{c}
\bullet \\
[0, 1, 0]^2
\end{array} \]
\[ \begin{bmatrix} x_3 & 0 & \alpha x_2 \\
x_1 & x_2 & 0 \\
0 & x_3 & x_1
\end{bmatrix} \]
\[ \varphi_u = \begin{pmatrix}
1^2 & 0 & 0^2 & 1 & 0^2 & 1 \\
0 & 1 & 1 & 0 & 0 & -1 \\
0 & 1 & 0 & \alpha & 1 & 0
\end{pmatrix} \]

We have \( \alpha \in \mathbb{P}^1 \) with \( \alpha \neq 0, \infty \). Also, \( \alpha \neq -1 \) so that \( X_\varphi \) does not lie on the conic \( x_1 x_2 - x_2 x_3 + x_3 x_1 = 0 \) which contains \([0, 1, 0]\) and the arrows \([0, 1, 0]^2\) and \([0, 0, 1]^2\).

13. \[ \begin{array}{c}
\bullet & \bullet & \bullet \\
[0, 0, 1]^2 & [1, -1, 0]^2 & [1, 0, 0]^2
\end{array} \]
\[ \begin{bmatrix} x_3 & \lambda(\lambda + 1)x_1 \\
\lambda x_1 - x_2 & \lambda + 1 \lambda x_1 + x_2 \\
0 & 0 & x_1 + x_3
\end{bmatrix} \]

We have \( \lambda \in \mathbb{P}^1 \) defined by \( \lambda \neq 0, -1, \infty \).

14. \[ \begin{array}{c}
\bullet & \bullet & \bullet \\
[0, 1, 0]^2 & [1, -1, 0]^2 & [1, 0, 0]^2
\end{array} \]
\[ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & -1
\end{bmatrix} \]

\[ X_\varphi = \begin{pmatrix}
1 & 0^2 & \alpha & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 1 & 0 & -1
\end{pmatrix} \]
\[ \phi_u = \begin{pmatrix} x_3 & 0 & x_2 & \alpha(x_1 + x_2) \\ 0 & x_2 & 0 & x_1 + x_3 \\ 0 & 0 & x_1 & x_3 \end{pmatrix} . \]

We have \( \alpha \in \mathbb{P}^1 \) with \( \alpha \neq 0, -1, \infty \).

---

15. \( [0, 0, 0]^2 [0, 0, 1] \)

\[ A_1 A_2(c) \]

\[ X_\phi = \begin{pmatrix} 1^2 & 0 & 0^2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \alpha & 1 & -1 \end{pmatrix} , \]

\[ \phi_u = \begin{pmatrix} x_3 & 0 & (1 - \alpha)x_2 & x_2 \\ 0 & x_2 & 0 & x_1 \\ 0 & x_3 & x_1 + x_3 & x_3 \end{pmatrix} . \]

We have \( \alpha \in \mathbb{P}^1 \) defined by \( \alpha \neq 0, 1, \infty \).

---

16. \( [0, 1, 0]^2 [1, 0, 0] [1, 1, 0] \)

\[ A_1 A_2(d) \]

\[ X_\phi = \begin{pmatrix} 1 & 0^2 & 1 & 0^2 & \alpha & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix} , \]

\[ \phi_u = \begin{pmatrix} x_3 & 0 & x_2 & x_1 + x_2 \\ 0 & x_2 & 0 & x_1 \\ 0 & -\alpha x_3 & x_1 & x_3 \end{pmatrix} . \]

We have \( \alpha \in \mathbb{P}^1 \) defined by \( \alpha \neq 0, 1, \infty \).

---

17. \( [1, 1, 1] [0, 1, 0] \)

\[ A_1 A_2(e) \]

\[ X_\phi = \begin{pmatrix} 0 & 0^3 & 0 & 0 & 1 & \lambda \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} , \]

\[ \phi_u = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ 0 & 0 & x_1 - x_2 & -x_1 + x_3 \\ 0 & x_3 & (1 - \lambda)x_2 & x_1 \end{pmatrix} . \]

We have \( \lambda \in \mathbb{P}^1 \) with \( \lambda \neq 0, 1, \infty \).
18. $A_1 A_2(f)$

$X_\phi = \begin{pmatrix} 0^3 & 0 & -1 & 0 & 1 & \lambda \\ 1 & 0 & 0 & 0 & 1 & \lambda \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$,

$\phi_u = \begin{pmatrix} x_2 & x_1 & \lambda x_1 \\ 0 & 0 & x_1 & x_3 \\ x_3 & x_3 & x_2 & x_1 \end{pmatrix}$.

We have $\lambda \in \mathbb{P}^1$ with $\lambda \neq 0, 1, \infty$.

19. $A_1 A_2(g)$

$X_\phi = \begin{pmatrix} 0^2 & 0 & 0^3 & 0 & -\beta & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$,

$\phi_u = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ -\beta x_3 & (1 - \beta)x_3 & x_1 & x_3 \\ 0 & x_3 & x_2 & x_1 \end{pmatrix}$.

We have $\beta \in \mathbb{P}^1$ defined by $\beta \neq 0, \infty$. Also, $\beta \neq 1$ so that $X_\phi$ does not lie on the conic $x_2x_3 - x_1^2 = 0$ which contains $[1,1,1]$ and the arrows at $[0,1,0]$ and $[0,0,1]$.

20. $A_1^4(a)$

$X_\phi = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix}$,

$\phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_1 + x_2 \\ 0 & x_2 & 0 & x_1 + x_3 \\ 0 & 0 & x_1 & x_2 + x_3 \end{pmatrix}$.

21. $A_1^4(b)$

$X_\phi = \begin{pmatrix} 1 & 0^2 & 1 & 0^2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix}$,

$\phi_u = \begin{pmatrix} x_3 & x_2 & 0 & 0 \\ x_1 & 0 & x_1 + x_2 & 0 \\ 0 & x_1 & 0 & x_1 + x_3 \end{pmatrix}$.
22. \( [0, 1, 0]^2 [1, -1, 0] [1, 0, 0] \)

\[ A_1^2 A_2(a) \]

\[ X_\phi = \begin{pmatrix} 1 & 0^2 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_1 + x_2 \\ 0 & x_2 & 0 & x_1 + x_3 \\ 0 & 0 & x_1 & x_3 \end{pmatrix}. \]

23. \( [1, 0, 0]^2 [0, 1, 0] \)

\[ A_1^2 A_2(b) \]

\[ X_\phi = \begin{pmatrix} 1^2 & 0 & 0^2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_2 \\ 0 & x_2 & 0 & x_1 \\ 0 & x_3 & x_1 + x_3 & x_3 \end{pmatrix}. \]

24. \( [1, 0, 0] [0, 1, 0]^2 [1, -1, 0] \)

\[ A_1^2 A_2(c) \]

\[ X_\phi = \begin{pmatrix} 1 & 0^2 & 1 & 0^2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} x_3 & 0 & x_2 & x_1 + x_2 \\ 0 & x_2 & 0 & x_1 \\ 0 & 0 & x_1 & x_3 \end{pmatrix}. \]

25. \( [0, 1, 0]^2 \)

\[ A_1^2 A_2(d) \]

\[ X_\phi = \begin{pmatrix} 1^2 & 0 & 0^2 & 1 & 0^2 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_2 \\ -x_1 & x_2 & 0 & x_1 \\ 0 & x_3 & x_1 & x_3 \end{pmatrix}. \]
26. \([0, 0, 1]^2\)

\[
A_1^2 A_2(e) \quad X_\phi = \begin{pmatrix} 0^3 & 0 & -1 & 0^2 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},
\]

\[
\phi_u = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ 0 & 0 & x_1 & x_3 \\ x_3 & x_3 & x_2 & x_1 \end{pmatrix}.
\]

\[\begin{array}{c|c}
27. & \\hline
\end{array}\]

\[
A_1^2 A_2(f) \quad X_\phi = \begin{pmatrix} 0^2 & 0 & 0^3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix},
\]

\[
\phi_u = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ 0 & x_3 & x_1 & x_3 \\ 0 & x_3 & x_2 & x_1 \end{pmatrix}.
\]

\[\begin{array}{c|c}
28. & \\hline
\end{array}\]

\[
A_2^2(a) \quad X_\phi = \begin{pmatrix} 1 & 0^2 & 1 & 0^2 & \alpha & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},
\]

\[
\phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_1 + x_2 \\ 0 & x_2 & 0 & x_1 \\ 0 & \alpha x_3 & x_1 & x_3 \end{pmatrix}.
\]

We have \(\alpha \in \mathbb{P}^1\) with \(\alpha \neq 0, -1, \infty\).

\[\begin{array}{c|c}
29. & \\hline
\end{array}\]

\[
A_2^2(b) \quad X_\phi = \begin{pmatrix} 0^3 & 0 & -1 & 0^3 & 0 & -\beta^3 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},
\]

\[
\phi_u = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ -\beta x_3 & 0 & x_1 & x_3 \\ 0 & x_3 & x_2 & x_1 \end{pmatrix}.
\]

We have \(\beta \in \mathbb{P}^1\) defined by \(\beta \neq 0, \infty\). Also, \(\beta \neq 1\) so that \(X_\phi\) does not lie on the conic \(x_2 x_3 - x_1^2 = 0\) which contains \([0, 1, 0]^3\) and the arrow at \([0, 0, 1]\).
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30. \[ A_1A_3^2(a) \]
\[ X_\phi = \begin{pmatrix} 1^2 & 0^2 & 1 & 0^2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \]
\[ \phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_1 + x_2 \\ 0 & x_2 & 0 & x_1 \\ 0 & 0 & x_1 & x_3 \end{pmatrix}. \]

31. \[ A_1A_3^2(b) \]
\[ X_\phi = \begin{pmatrix} 1^2 & 0 & 0^2 & 1 & 0^2 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \]
\[ \phi_u = \begin{pmatrix} x_2 & 0 & 0 & x_2 \\ 0 & x_2 & 0 & x_1 \\ 0 & 0 & x_1 & x_3 \end{pmatrix}. \]

32. \[ A_1A_3^2(c) \]
\[ X_\phi = \begin{pmatrix} 0^3 & 0 & -1 & 0^3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \]
\[ \phi_u = \begin{pmatrix} x_2 & x_1 & 0 & 0 \\ 0 & 0 & x_1 & x_3 \\ 0 & x_3 & x_2 & x_1 \end{pmatrix}. \]

33. \[ A_3^2(a) \]
\[ X_\phi = \begin{pmatrix} 1^2 & 0 & 0^2 & 1 & 0^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \]
\[ \phi_u = \begin{pmatrix} x_3 & 0 & 0 & x_2 \\ 0 & x_2 & 0 & x_1 \\ 0 & 0 & x_1 & x_3 \end{pmatrix}. \]
Remarks.

(i) It is well known that up to $PGL(4)$-equivalence, cubic surfaces with singularities of type $A_1^{n_1} A_2^{n_2}$ form a continuous family of dimension $4 - (n_1 + 2n_2)$. The only exception is the $A_2^3$ cubic surfaces, which form a one-dimensional family. However, the only closed (within the semi-stable locus) $PGL(4)$-orbits are the type $A_1^{n_1}$ orbits, $n_1 \leq 4$ and the unique $A_2^3$ orbit (See [1]). The dimensions of the families of trilinear forms of type $A_1^{n_1} A_2^{n_2}$ (up to $G$-equivalence) are similar to that of corresponding cubic surfaces. Also, trilinear forms of the types $A_2^2(a)$ and $A_2^2(b)$ form one-dimensional families up to $G$-equivalence. The relations among the closures of $G$-orbits would be investigated in [25].

(ii) The trilinear forms in the above theorem are automatically G.I.T. semi-stable due to the absence of unstable singularities in $Z_e$. They must all appear in the compactification of the moduli space of the smooth trilinear forms.

(iii) There are b.l. configurations $X$ for which the linear system through $X$ defines a cubic surface with only $A_{\leq 2}$ singularities which are notably missing from the above theorem. For example, if $X$ is a union of six distinct points lying on a smooth conic, then the linear system of cubics through $X$ define a cubic surface $Z$ with a unique $A_1$ singularity.

For these b.l. configurations, we have a complete list too (in Section 5.3).

5.2. Two explicit examples

We would derive the $A_2(b)$ and $A_2^2(b)$ trilinear forms in this section as they are sufficiently representative of other derivations. All other derivations of trilinear forms in Theorem 3 are easy since they involve only (at worst) double points.

In subsequent calculations, we let the homogeneous coordinates on $\mathbb{P}^2$ be $[x_1, x_2, x_3]$ and we denote the homogeneous ideal $(f_1, f_2, f_3, f_4) \subset \mathbb{C}[x_1, x_2, x_3]$ by $I$, local rings by $\mathcal{O}$ (with an appropriate subscript) and localizations of the ideal $I$ by $\mathcal{O}$ (with an appropriate subscript).

5.2.1. The $A_2(b)$ trilinear form. $X_{x}$ consists of a generic 3c with three other simple points. We assume without loss of generality that the generic 3c is supported at $[0, 0, 1]$ with its arrow lying on $x_2 = 0$ and two of the other points in $X_{x}$ are $[0, 1, 0]$ and $[1, 1, 1]$. Note that we cannot pick $[1, 0, 0]$ as an element of $X_{x}$ since it lies on $x_2 = 0$. The above configuration has an one dimensional stabilizer subgroup $H =$
\[
\begin{pmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  a - b & 0 & b
\end{pmatrix},
\]
where \(ab \neq 0\). The remaining point in \(X_\phi\) can be normalized by \(H\) as \([\lambda, 1, 0]\) with \(\lambda \neq 1\) (so that \([0, 0, 1], [1, 1, 1], [\lambda, 1, 0]\) are not collinear) and \(\lambda \neq 0\) (so that we do not have a double point \([0,1,0]^2\)). The stabilizer of the above configuration is then the trivial subgroup. Note that in the affine plane \(\text{Spec } \mathbb{C}[x,y]\) defined by \(x = x_1/x_3, y = x_2/x_3, x_3 \neq 0\), the localized ideal \(I_{[0,0,1]}\) is necessarily equal to \((y - \beta x^2, x^3)O_{[0,0,1]}\) for some \(\beta \neq 0\) (since the triple point is not linear). A simple calculation reveals that the unique conic \(C\) which contains the support of \(X_\phi\) and has tangent \(x_2 = 0\) at \([0,0,1]\) has equation \(x_1^2 - \lambda x_1 x_2 + (\lambda - 1) x_2 x_3 = 0\). The triple point at \([0,0,1]\) lies in \(C\) if and only if \(\beta = \frac{1}{1 - \lambda}\). Hence we need to impose the condition \(\beta \neq \frac{1}{1 - \lambda}\) in order that \(X_\phi\) does not lie on a conic.

![Diagram showing points and lines]

\([0,0,1]^3\) lies on \(x_2x_3 - \beta x_1^2 = 0\), where \(\beta \neq 0, \frac{1}{1 - \lambda}\).

The basis elements for \(W\) are thus:

\[
\begin{align*}
  f_1 &= x_1(x_1^2 - \lambda x_1 x_2 + (\lambda - 1)x_2 x_3), \\
  f_2 &= x_2(x_1^2 - \lambda x_1 x_2 + (\lambda - 1)x_2 x_3), \\
  f_3 &= x_3(x_2 x_3 - \beta x_1^2 + (\beta - 1)x_1 x_2), \\
  f_4 &= x_2 x_3(x_1 - x_2).
\end{align*}
\]

Denoting the image of \(f_i\) in \(I_{[0,0,1]}\) by \(\overline{f_i}\) with affine coordinates \(x := x_1/x_3\) and \(y := x_2/x_3\), we note that \(\beta f_1 + x f_2 = (\lambda \beta - \beta + 1)xy(1 - x)\), where \(\lambda \beta - \beta + 1 \neq 0\). Hence, \(xy \in I_{[0,0,1]}\) and we can see easily that \((\overline{f_1}, \overline{f_2}, \overline{f_3}, \overline{f_4}) = (y - \beta x^2, x^3)O_{[0,0,1]}\), a confirmation we need.

The syzygy matrix \(M_u\) can be found from Equation (1) with the \(f_k\)'s above.

5.2.2. The \(A_3^2(b)\) bilinear forms. We may assume that \(X\) is supported at \([0,1,0]\) and \([0,0,1]\) and that the arrows lie along \(x_3 = 0\) and \(x_2 = 0\).
respectively. It is easy to verify that the stabilizer subgroup in $PGL(3)$ of $|X|$ and arrows of $X$ is a two dimensional group \[
abla = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \]
where $abc \neq 0$. Since the triple point at $[0, 1, 0]$ is strictly curvilinear, it lies on a smooth conic \(x_2x_3 - \beta_1 x_1^2 = 0\) for some $\beta_1 \neq 0$. However, $\beta_1$ may be normalized to unity since we can scale $x_1$ and $x_2$ to achieve that. Similarly, we may assume that $[0, 0, 1]$ lies on \(x_3x_2 - \beta x_1^2 = 0\). However, in this case, $\beta$ cannot be set to unity. Hence, we get a one parameter family (up to $PGL(3)$ equivalence) of $A_2^1(b)$ subschemes in the plane given by $X_{\phi} = \{ [0, 1, 0]^3, [0, 0, 1]^3 \}$ where $[0, 1, 0]^3$ is defined by $(x - x_2^2, x_3^3)$ (where $x := x_1/x_2$, $z := x_3/x_2$) and $[0, 0, 1]^3$ is defined by $(y - \beta x_2^2, x_3^3)$ (where $x := x_1/x_3$, $y := x_2/x_3$) with $\beta \neq 1$. Such a subscheme admits an one dimensional stabilizer group \[
abla = \begin{pmatrix} ab & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & b^2 \end{pmatrix} \] in $PGL(3)$. Our calculations is compatible with the well known result that $A_2^1$ cubic surfaces form a one parameter family (up to $PGL(4)$ equivalence) and each such surface admits a one dimensional automorphism group. The special case $\beta = 1$ leads to an inadmissible configuration (See the following section). Applying the six linearly independent conditions on the vector space of cubic polynomials, we get the following basis elements for the vector space $W$:

\[
\begin{align*}
f_1 &= x_1(x_2x_3 - x_1^2), \\
f_2 &= x_2(x_2x_3 - x_1^2), \\
f_3 &= x_3(x_2x_3 - \beta x_1^2), \\
f_4 &= x_1x_2x_3.
\end{align*}
\]

The syzygy matrix $M_\alpha$ can be found from Equation (1) with the $f_k$'s above. 

5.3. Inadmissible configurations which are b.l.

We know that if $X$ is of length six and lies on a conic (reduced but not necessarily irreducible), then $X$ is the intersection of the conic with a cubic by Bezout’s Theorem. The Koszul complex of $X$ is a minimal free resolution:

\[
0 \rightarrow S(-5) \rightarrow S(-3) \oplus S(-2) \rightarrow S \rightarrow S(X) \rightarrow 0.
\]
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However, it is possible to have a non-minimal resolution starting with four linearly independent cubics. Although these four cubics still define a rational morphism \(\mathbb{P}(U^*) \to Z\), we have following result for the syzygy matrix \(\phi_u\):

**Theorem 4 (The Complete Intersection Trilinear Form \(\phi_{c,i}\)).** Suppose a b.i. scheme \(X\) (possibly containing infinitely near points) in \(\mathbb{P}^2_{[x_1,x_2,x_3]}\) is defined scheme-theoretically by the ideal \(I = I_1 + I_2 := (x_1q, x_2q, x_3q) + (f)\), where \(q\) and \(f\) define a conic and a cubic curve with no common component respectively, then \(X\) is inadmissible with \(\phi_u\) \(G\)-equivalent to the following trilinear form

\[
(\phi_{c.i.})_u = \begin{pmatrix}
0 & x_1 & x_2 \\
-x_1 & 0 & x_3 \\
-x_2 & -x_3 & 0
\end{pmatrix}.
\]

Furthermore, \(X_\phi = \mathbb{P}(U^*) \neq X, Y_\phi = \mathbb{P}(V^*) \) and \(Z_\phi = \mathbb{P}(W^*)\).

These configurations of \(X\) (for which \(Z\) has only \(A_{\leq 2}\) singularities), which we call "c.i." configurations (since \(X\) is the intersection of the saturation ideal \(I_1 = (q)\) of \(I_1\) with \(I_2 = (f)\)), are precisely one of the following:

1. \(A_1(C) : |X| = 1^6\)
   There is no \(l\)-configuration.

2. \(A_1^2(C) : |X| = 2 \cdot 1^4\)
   There is no \(l\)-configuration.

3. \(A_2(C) : |X| = 1^6\)
   There are exactly two \(l\)-configurations.

4. \(A_3(E) : |X| = 2 \cdot 1^4\)
   There are exactly two \(l\)-configurations.

5. \(A_3^3(F) : |X| = 2^3\)
   There are exactly two \(l\)-configurations.
\[ A_1^4(D) : |X| = 2^2 \cdot 1^2 \]
There are exactly two \(l\)-configurations.

\[ A_1^2 A_2^2(G) : |X| = 3^1 \cdot 1^3 \]
There are exactly two \(l\)-configurations.

\[ A_2^2(C) : |X| = (3c)^2 \]
There is no \(l\)-configuration.

\[ A_1 A_2^3(D) : |X| = 3^1 \cdot 2 \cdot 1 \]
There are exactly two \(l\)-configurations.

\[ A_2^3(B) : |X| = (3l)^2 \]
There are exactly two \(l\)-configurations.

**Proof.** Clearly, all \(U\)-linear relations among the generators of \(I_1\) are generated by the three relations \(x_3q x_2 - x_2q x_3 = 0, x_3q x_1 - x_1q x_3 = 0\) and \(x_2q x_1 - x_1q x_2 = 0\). On the other hand, if there is a \(U\)-linear relation \(\lambda_1 x_1q + \lambda_2 x_2q + \lambda_3 x_3q + \lambda_4 f = 0\) with \(\lambda_1 x_1q + \lambda_2 x_2q + \lambda_3 x_3q \neq 0\), then we can show easily that \(q\) and \(f\) have a common factor (which is not a unit), contradicting our hypothesis that \(X\) is zero dimensional. Hence, the syzygy matrix is

\[
\phi_u(u^*) = \begin{pmatrix}
0 & x_3 & -x_2 & 0 \\
x_3 & 0 & -x_1 & 0 \\
x_2 & -x_1 & 0 & 0
\end{pmatrix},
\]

which is \(G\)-equivalent to \(\phi_{c.i.}\).

In all cases except \(A_2^2(C)\), it is easy to see that \(X\) lies on a unique reducible conic \(q = 0\). The conic \(q = 0\) is irreducible for \(A_1(C)\) and \(A_1^2(C)\) configurations. The remaining \(A_2^2(C)\) configuration corresponds to the special case \(\beta = 1\) in the \(A_2^2(b)\) configuration, i.e., it lies on the irreducible conic \(q = x_2x_3 - x_1^2 = 0\). \(\square\)
Because of Theorem 4, some mildly degenerate configurations of $X$, i.e., those for which the linear system of cubics through $X$ define a surface with only $A_{2}^{2}$ singularities, are not realized in our compactification of the moduli of smooth trilinear forms.

6. Duval trilinear forms with $A_{23}$, $D_{4}$ or $D_{5}$ singularities

With exactly the same technique in Section 5, we can derive all Duval trilinear forms with $A_{23}$, $D_{4}$ and $D_{5}$ singularities. These singularities result in the instability of cubic surfaces but not necessarily that of trilinear forms. The instability of the trilinear forms in Theorem 5 will be dealt with in a different paper.

**Theorem 5** (Duval Trilinear Forms with $A_{23}$, $D_{4}$ and $D_{5}$ singularities). There are exactly twenty seven types of non-G-equivalent Duval trilinear forms with $A_{23}$, $D_{4}$ or $D_{5}$ singularities. Up to G-equivalence, the only types forming a one-parameter family are the $A_{3}$ trilinear forms.

\[ A_{3}(a) \quad (\alpha \neq 0, \infty) \]
\[ X_{\phi} = \begin{pmatrix} 1^{2} & 0 & 0 \alpha & 1 & 1 & 0 \end{pmatrix}, \]
\[ \phi_{u} = \begin{pmatrix} 0 & x_{3} & x_{2} & 0 \\ 0 & x_{3} & 0 & x_{1} \\ x_{3} & x_{1} & -x_{2} & -\alpha x_{1} & x_{2} \end{pmatrix}. \]

\[ A_{3}(b) \quad (\beta \neq 0) \]
\[ X_{\phi} = \begin{pmatrix} 1^{3} & 0 & 0 & \beta & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 \end{pmatrix}, \]
\[ \phi_{u} = \begin{pmatrix} 0 & x_{3} & 0 & x_{2} \\ 0 & x_{2} & x_{1} + x_{2} & x_{2} \\ x_{3} & -\beta x_{2} & -\beta x_{2} & x_{1} - \beta x_{2} \end{pmatrix}. \]

\[ A_{3}(c) \quad (\beta \neq 0) \]
\[ X_{\phi} = \begin{pmatrix} 1^{3} & 0 & 0 & 0 & \beta & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & \beta & 0 & 1 & 1 \end{pmatrix}, \]
\[
\phi_u = \begin{pmatrix}
0 & 0 & x_2 + x_3 & -x_2 \\
0 & x_3 & x_1 & 0 \\
x_2 - x_3 & x_1 + (2\beta - 1)x_3 & 0 & \beta x_1
\end{pmatrix}.
\]

4. \[ A_3(d) \quad (\lambda \neq 1, \infty) \]
\[
X_\phi = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
1^4 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix},
\]
\[
\phi_u = \begin{pmatrix}
-\lambda x_1 + x_3 & x_1 - x_2 & 0 & 0 \\
0 & 0 & x_1 & x_3 \\
x_1 + (1 - \lambda)x_2 & x_2 & x_2 & x_1
\end{pmatrix}.
\]

5. \[ A_1 A_3(a) \]
\[
X_\phi = \begin{pmatrix}
1^2 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix},
\]
\[
\phi_u = \begin{pmatrix}
x_3 & x_2 & 0 \\
x_3 & 0 & x_1 \\
x_3 & x_1 - x_2 & 0 & x_2
\end{pmatrix}.
\]

6. \[ A_1 A_3(b) \]
\[
X_\phi = \begin{pmatrix}
1^3 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix},
\]
\[
\phi_u = \begin{pmatrix}
0 & 0 & x_2 + x_3 & x_2 \\
0 & x_3 & x_1 & 0 \\
x_2 - x_3 & x_1 - x_3 & 0 & 0
\end{pmatrix}.
\]

7. \[ A_1 A_3(c) \]
\[
X_\phi = \begin{pmatrix}
1^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix},
\]
\[
\phi_u = \begin{pmatrix}
x_3 & x_2 & 0 \\
0 & x_2 & x_2 & x_1 \\
x_3 & x_2 & x_1 & 0
\end{pmatrix}.
\]
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8. $A_1A_3(d)$

$A_1A_3(d)$

$X_\phi = \begin{pmatrix}
1^3 & 0 & 0 & 0^2 & 1 & 0 \\
0 & 0 & -1 & 0 & -1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}$,

$\phi_u = \begin{pmatrix}
0 & x_3 & x_2 & x_3 \\
0 & x_2 & 0 & x_1 \\
x_3 & x_2 & x_1 + x_2 & x_3
\end{pmatrix}$.

9. $A_1A_3(e)$

$A_1A_3(e)$

$X_\phi = \begin{pmatrix}
1^3 & 0 & 0 & 0^2 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & -1
\end{pmatrix}$,

$\phi_u = \begin{pmatrix}
0 & 0 & x_2 - x_3 & -x_2 \\
0 & x_1 + x_3 & x_1 & 0 \\
x_3 & x_1 + x_2 & 0 & x_1
\end{pmatrix}$.

10. $A_1A_3(f)$

$A_1A_3(f)$

$X_\phi = \begin{pmatrix}
0^4 & 0 & -1 & 1 & 0^2 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$,

$\phi_u = \begin{pmatrix}
x_2 & x_1 & 0 & 0 \\
-x_3 & x_1 & x_3 & 0 \\
x_3 & x_2 & x_1 & x_2
\end{pmatrix}$.

11. $A_4(a)$

$A_4(a)$

$X_\phi = \begin{pmatrix}
1^3 & 0 & 0 & 0^2 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}$,

$\phi_u = \begin{pmatrix}
0 & x_3 & x_2 & 0 \\
0 & 0 & x_2 & x_1 \\
x_3 & x_2 & x_1 & x_2
\end{pmatrix}$.

12. $A_4(b)$

$A_4(b)$

$X_\phi = \begin{pmatrix}
1^3 & 0 & 0 & 0^2 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix}$,
\[ \phi_u = \begin{pmatrix} 0 & 0 & x_3 & x_2 \\ 0 & x_3 & x_1 & 0 \\ x_3 & x_1 + x_2 & x_2 & x_1 \end{pmatrix}. \]

13. \[ A_4(c) \]

\[ X_\phi = \begin{pmatrix} 1^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} 0 & x_3 & x_2 & 0 \\ 0 & x_2 & x_1 & x_3 \\ x_3 & 0 & -x_1 & x_2 \end{pmatrix}. \]

14. \[ A_4(d) \]

\[ X_\phi = \begin{pmatrix} 1^4 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} 0 & 0 & x_3 & x_2 \\ 0 & x_2 & x_1 & x_3 \\ x_3 & x_1 & 0 \end{pmatrix}. \]

15. \[ D_4(a) \]

\[ X_\phi = \begin{pmatrix} 1^4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} 0 & 0 & x_3 & x_2 \\ 0 & x_2 & 0 & x_1 \\ x_3 & x_1 & x_1 - x_2 & 0 \end{pmatrix}. \]

16. \[ D_4(b) \]

\[ X_\phi = \begin{pmatrix} 1^4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ \phi_u = \begin{pmatrix} 0 & 0 & x_3 & x_2 \\ 0 & x_2 & 0 & x_1 - x_2 \\ x_3 & x_1 & x_1 - x_2 & 0 \end{pmatrix}. \]
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17. \( A_1^2 A_3(a) \)

\[
A_1^2 A_3(a) \\
X\phi = \\
\begin{pmatrix}
1^2 & 0^2 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
\phi_u = \\
\begin{pmatrix}
0 & x_3 & x_2 & 0 \\
0 & x_3 & 0 & x_1 \\
x_3 & x_1 - x_2 & 0 & 0
\end{pmatrix}.
\]

18. \( A_1^2 A_3(b) \)

\[
A_1^2 A_3(b) \\
X\phi = \\
\begin{pmatrix}
1^3 & 0 & 0 & 0^2 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

\[
\phi_u = \\
\begin{pmatrix}
0 & x_3 & x_2 & 0 \\
0 & x_2 & 0 & x_1 \\
x_3 & x_2 & x_1 & 0
\end{pmatrix}.
\]

19. \( A_1^2 A_3(c) \)

\[
A_1^2 A_3(c) \\
X\phi = \\
\begin{pmatrix}
1^3 & 0 & 0 & 0^2 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix},
\]

\[
\phi_u = \\
\begin{pmatrix}
0 & 0 & x_2 - x_3 & x_2 \\
0 & x_2 + x_3 & x_1 & 0 \\
x_3 & x_1 & 0 & 0
\end{pmatrix}.
\]

20. \( A_1 A_4(a) \)

\[
A_1 A_4(a) \\
X\phi = \\
\begin{pmatrix}
1^3 & 0 & 0 & 0^2 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
\phi_u = \\
\begin{pmatrix}
0 & x_3 & x_2 & 0 \\
0 & 0 & x_2 & x_1 \\
x_3 & x_2 & x_1 & 0
\end{pmatrix}.
\]

21. \( A_1 A_4(b) \)

\[
A_1 A_4(b) \\
X\phi = \\
\begin{pmatrix}
1^3 & 0 & 0 & 0^2 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix},
\]
\[ \phi_u = \begin{pmatrix} 0 & 0 & x_3 & x_2 \\ 0 & x_3 & x_1 & 0 \\ x_3 & x_1 + x_2 & x_2 & 0 \end{pmatrix} . \]

22. \( A_1 A_4(c) \)

\[ X_\phi = \begin{pmatrix} 1^4 & 0 & 0 & 0 & 0^2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} , \]

\[ \phi_u = \begin{pmatrix} 0 & 0 & x_3 & x_2 \\ 0 & x_2 & x_1 & x_3 \\ x_3 & x_1 & 0 & 0 \end{pmatrix} . \]

23. \( A_1 A_4(d) \)

\[ X_\phi = \begin{pmatrix} 1^4 & 0 & 0 & 0 & 0^2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} , \]

\[ \phi_u = \begin{pmatrix} 0 & x_3 & x_2 & 0 \\ 0 & x_2 & 0 & x_1 \\ x_3 & 0 & x_1 & x_2 \end{pmatrix} . \]

24. \( A_5(a) \)

\[ X_\phi = \begin{pmatrix} 1^3 & 0 & 0 & 0^3 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} , \]

\[ \phi_u = \begin{pmatrix} 0 & 0 & -x_3 & x_2 \\ 0 & x_3 & x_1 & 0 \\ x_3 & x_1 & x_2 & x_1 \end{pmatrix} . \]

25. \( D_5(a) \)

\[ X_\phi = \begin{pmatrix} 1^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} , \]

\[ \phi_u = \begin{pmatrix} 0 & x_3 & 0 & x_2 \\ 0 & x_2 & x_3 & x_1 \\ x_3 & 0 & x_1 & 0 \end{pmatrix} . \]
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26. \[ D_5(b) \]
\[
X_\phi = \begin{pmatrix}
1^5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix},
\]
\[
\phi_u = \begin{pmatrix}
0 & x_3 & 0 & x_2 \\
0 & 0 & x_2 & x_1 \\
x_3 & x_2 & x_1 & 0
\end{pmatrix}.
\]

27. \[ A_1A_5(a) \]
\[
X_\phi = \begin{pmatrix}
1^3 & 0 & 0 & 0^3 & 0 & -1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]
\[
\phi_u = \begin{pmatrix}
0 & 0 & x_3 & x_2 \\
0 & x_3 & x_1 & 0 \\
x_3 & x_1 & x_2 & 0
\end{pmatrix}.
\]

6.1. Two explicit examples

We would present the derivations of \(A_3(d)\) and \(D_4\) trilinear forms.

6.1.1. The \(A_3(d)\) trilinear forms. This is the first instance, where we have a generic \(4c\). We assume that \(X_\phi\) contains \([0,0,1]^4\) and \([0,1,0]^4\), where \([0,0,1]^4\) lies on the conic \(x_2x_3 + ax_1x_2 + \beta x_1^2 = 0\). The stabilizer subgroup of the above configuration is the three dimensional group \(H' = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
d & 0 & c
\end{pmatrix}\). With \(H'\)-normalization, we may assume that \([0,0,1]^4\) lies on the parabola \(x_2x_3 - x_1^2 = 0\), i.e., \(a = 0\) and \(\beta = -1\). The stabilizer subgroup of \(\{[0,0,1]^4, [0,1,0]\}\), where \([0,0,1]^4\) is defined by \((y-x^2, y^2)\) is now \(H = \begin{pmatrix}
a & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & a^2
\end{pmatrix}\). We have one more point \((\mu, 1, \lambda)\) in \(X_\phi\) with \(\mu \neq 0\) since the support of \(X_\phi\) does not lie on any straight line, and \(\lambda \neq \mu^2\) since otherwise \(X_\phi\) lies on the conic \(x_2x_3 - x_1^2 = 0\) and we have a c.i. configuration. With \(H\)-normalization, we may assume \(\mu = 1\) and \(\lambda \neq 1\). Hence \(X_\phi = \{[0,0,1]^4, [0,1,0], [1,1,\lambda]\}\). The three dimensional
linear system $|W|$ is generated by:

\[
\begin{align*}
    f_1 &= (x_1 - x_2)(x_2 x_3 - x_1^2), \\
    f_2 &= (\lambda x_1 - x_3)(x_2 x_3 - x_1^2), \\
    f_3 &= x_1(x_2 x_3 - x_1^2) + (1 - \lambda)x_2^2, \\
    f_4 &= x_3(x_2 x_3 - x_1^2) + (1 - \lambda)x_2^2.
\end{align*}
\]

Note that $\mathcal{I}_{[0,0,1]} = (y - x^2, y^2)$, where $x = x_1/x_3$ and $y = x_2/x_3$ are the affine coordinates on $\mathbb{A}^2_{x,y}$. $\phi_u$ can be found from Equation (1).

In case when $\lambda = 1$ so that $X = \{ [0,0,1]^4, [0,1,0], [1,1,1] \}$ lies on the conic $x_2 x_3 - x_1^2 = 0$, the linear system through the c.i. configuration $X$ (labelled $A_3(F)$ in the next theorem) is generated by

\[
\begin{align*}
    f_1 &= x_1(x_2 x_3 - x_1^2), \\
    f_2 &= x_2(x_2 x_3 - x_1^2), \\
    f_3 &= x_3(x_2 x_3 - x_1^2), \\
    f_4 &= x_1 x_2(x_1 - x_2).
\end{align*}
\]

6.1.2. The $D_4$ trilinear forms. It is known that up to projective equivalence there are exactly two distinct $D_4$ cubic surfaces. We see that the same is true of $D_4$ trilinear forms. Firstly, we assume that $X_{\phi} = \{ [1,0,0]^{4c}, [0,0,1], [1,1,0] \}$, where $[1,0,0]^{4c}$ is defined by $(z + \alpha y^2 + \beta y^3, z^2, y^2 z)$, where $y = x_2/x_1$ and $z = x_3/x_1$.

Suppose $\beta \neq 0$, then we may assume by the normalization of the stabilizer subgroup of $X_{\phi}$ that $\alpha = 1$. In addition, the condition $[1,1,0] \in X_{\phi}$ forces $\beta = -1$. The linear system of homogeneous cubics passing through $\{ [1,0,0]^{3c}, [0,0,1], [1,1,0] \}$ is

\[
\begin{align*}
    f_1 &= x_1^2 x_3 + x_1 x_2^2 - x_2^3, \\
    f_2 &= x_1 x_3^2, \\
    f_3 &= x_2 x_3, \\
    f_4 &= x_2 x_3^2, \\
    f_5 &= x_1 x_2 x_3.
\end{align*}
\]

By inspection, the first four cubics are already in the linear system we are looking for since their localizations are in $\mathcal{I}_{[1,0,0]}$, $f_5$ cannot be in $I$ since otherwise $yz \in \mathcal{I}_{[1,0,0]}$ (where $y = x_2/x_1$ and $z = x_3/x_1$) and we get only a $3c$ at $[1,0,0]$. We obtain the $D_4(a)$ trilinear form from this system.

If $\beta = 0$, the above system also passes through $\{ [1,0,0]^{3c}, [0,0,1], [1,1,0] \}$. In this case, $\mathcal{I}_{[1,0,0]} = (z + y^2, z^2, y^2 z)$ necessarily. We note
that the localization of \( f_1 - f_5 = x_1^3x_3 + x_1x_2^3 - x_2^3 - x_1x_2x_3 = (x_1x_3 + x_2^3)(x_1 - x_2) \) is \( z + y^2 \). Hence, the system we need is \(|f_1 - f_5, f_2, f_3, f_4|\) and we get the \( D_4(b) \) trilinear form now.

### 6.2. Inadmissible configurations which are b.l.

Just like the case for trilinear forms with \( A_1 \) and \( A_2 \) singularities, there are also c.i. configurations of \( X \) which result in \( A_{\geq 3} \), \( D_4 \), \( D_5 \) or \( E_6 \) singularities on \( Z \). They are listed in the following:

**Theorem 6 (Inadmissible Configurations which are b.l.).** The inadmissible configurations \( X \) that give rise to \( A_{\geq 3} \), \( D_4 \), \( D_5 \) or \( E_6 \) singularities in \( Z \) are exactly one of the following thirteen types:

1. \( A_3(E) : |X| = 2 \cdot 1^4 \)
   There are exactly two \( l \)-configurations.

2. \( A_3(F) : |X| = 4c \cdot 1^2 \)
   There is no \( l \)-configuration.

3. \( A_1 A_3(G) : |X| = 2^2 \cdot 1^2 \)
   There are exactly two \( l \)-configurations.

4. \( A_1 A_3(H) : |X| = 4c \cdot 2 \)
   There is no \( l \)-configuration.

5. \( A_4(E) : |X| = 3c \cdot 1^3 \)
   There are exactly two \( l \)-configurations.

6. \( A_4(F) : |X| = 5c \cdot 1 \)
   There is no \( l \)-configuration.

7. \( D_4(C) : |X| = 2^3 \)
   There is exactly one \( l \)-configuration.
\[ A_1 A_4(E) : |X| = 3c \cdot 2 \cdot 1 \]
There are exactly two l-configurations.

\[ A_1 A_4(F) : |X| = 4l \cdot 1^2 \]
There are exactly two l-configurations.

\[ A_5(B) : |X| = 6c \]
There is no l-configuration.

\[ D_5(C) : |X| = 4c \cdot 2 \]
There is exactly one l-configuration.

\[ A_1 A_5(B) : |X| = 4l \cdot 2 \]
There are exactly two l-configurations.

\[ E_6(A) : |X| = 6l \]
There is exactly one l-configuration.

Proof. Omitted.

7. Non-Duval trilinear forms

We have classified all Duval trilinear forms (which correspond to admissible configurations on the plane) as well as all inadmissible configurations with curvilinear multiple points (although they are not determinantal varieties of trilinear forms). To classify non-Duval trilinear forms, we do so solely from the geometry of the singularities of \( Z_\phi \). This is because only most of these trilinear forms correspond to admissible configurations \( X \) (necessarily zero dimensional by definition) with non-curvilinear points of multiplicities 3, 4, 5 and 6. Some have non-pure
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dimensional $X_{\phi}$. A one dimensional scheme in the plane does not determine uniquely a (3,3,4) trilinear form as the vector space of syzygies is always larger than three.

If a trilinear form $\phi$ is not Duval, then $Z_{\phi} = \{ [w^*] \mid h(w^*) = 0 \}$, where $h := \det(\phi_w)$, is exactly one (and only one) of the following:

(i) $Z_{\phi} = \mathbb{P}(W^*)$,
(ii) $Z_{\phi}$ is reducible, there are various possibilities here,
(iii) $Z_{\phi}$ is a cone over an irreducible cubic curve on the plane and it has either a triple point at the vertex (i.e., an $\tilde{E}_6$ singularity in case the cubic curve is smooth), or it has a nodal line (joining the node or the cusp of the singular curve to the vertex),
(iv) $Z_{\phi}$ is not a cone and it has a nodal line, there are two such surfaces up to projectivity.

The classification of non-Duval trilinear forms is organized along the line of the above mutually exclusive cases. Since we are dealing with (3,3,4) trilinear forms, we can omit the degenerate trilinear forms from our classification, i.e., those for which $\phi_u$, $\phi_v$, or $\phi_w$ is not injective, because they can be regarded as lower dimensional trilinear forms. Moreover, they are automatically G.I.T. unstable and thus are not required for moduli construction. We adopt the notations of [3] and [4] for non-Duval trilinear forms. Also, since we classify non-Duval trilinear forms according to the geometry of $Z_{\phi}$, we state the $\phi_w$ matrices in the classification theorems. They are also more convenient for G.I.T. instability test.

### 7.1. Spatial trilinear forms

We define a spatial trilinear form $\phi$ as one for which $Z_{\phi} = \mathbb{P}(W^*)$, equivalently $h = 0$.

**Theorem 7.** Suppose $\phi$ is spatial, then exactly one of the following holds:

1. $\phi$ is degenerate and $\phi_w$ is $G$-equivalent to a matrix with a row or column consisting entirely of zeros.
2. $\phi$ is degenerate and $\phi_w$ is $G$-equivalent to the c.i. trilinear form

   $$(\phi_{c.i.})_w = \begin{pmatrix} 0 & z_4 & z_3 \\ -z_4 & 0 & z_2 \\ -z_3 & -z_2 & 0 \end{pmatrix}.$$ 

3. $\phi_w$ is $G$-equivalent to $\phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ 0 & 0 & z_1 \\ z_3 & z_2 & 0 \end{pmatrix}$.
4. \( \phi_w \) is \( G \)-equivalent to \( \phi_w = \begin{pmatrix} 0 & 0 & z_2 \\ 0 & p_{23}(134) & 0 \\ z_4 & z_3 & z_1 \end{pmatrix} \) or its transpose, where \( p_{23}(134) \) denotes the linear polynomial \( a_{231}z_1 + a_{233}z_3 + a_{234}z_4 \). It is non-vanishing in the above matrix.

Proof. The proof is a simple exercise in the computation of determinants and is omitted. \( \square \)

7.2. Reducible trilinear forms

If a cubic surface \( Z \) is reducible, then it is exactly one of the following:

1. \( QP : Z \) is a union of a non-singular quadric and a plane whose intersection is a smooth conic,
2. \( CP : Z \) is a union of a quadric cone and a plane whose intersection is a smooth conic,
3. \( QT : Z \) is a union of a non-singular quadric and a plane whose intersection is a union of two transverse lines,
4. \( CV : Z \) is a union of a quadric cone and a plane whose intersection is a union of two transverse lines through the cone’s vertex,
5. \( CT : Z \) is a union of a quadric cone and a plane whose intersection is a generator of the cone (a double line),
6. \( PPP : Z \) is a union of three distinct planes not passing through any common line,
7. \( \overline{PPP} : Z \) is a union of three distinct planes intersecting along a line,
8. \( P^2P : Z \) is a union of a double plane and another distinct plane,
9. \( P^3 : Z \) is a triple plane.

In our classification of reducible trilinear forms, we do not mention the geometry of \( Y_\phi \) as they are always one-dimensional. As such, their geometry does not in any way characterize \( \phi \). For trilinear forms of types \( PPP, \overline{PPP}, P^2P \) and \( P^3 \), both \( X_\phi \) and \( Y_\phi \) are one dimensional so there is no mention of both these determinantal varieties.

We have the following classification for reducible trilinear forms:

Theorem 8 (Non-degenerate Reducible Trilinear Forms).

\( QP \): If a trilinear form \( \phi \) is of type \( QP \), then \( \phi_w \) is necessarily \( G \)-equivalent to one of the following:

1. \( QP(a) : \phi_w = \begin{pmatrix} 0 & z_4 & 0 \\ z_3 & z_2 & z_2 + z_4 \\ z_2 & -z_2 & z_1 \end{pmatrix} \), or its transpose.
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\[|X_\phi| = 3s \cdot 1^3, \text{ where } |X_\phi| \text{ is in general position and } 3s \text{ is defined by the local ideal } (x^2, xy, y^2)(\text{See Section 4.2}).\]

2. \(QP(b) : \phi_w = \begin{pmatrix} 0 & z_4 & 0 \\ z_3 & 0 & z_2 + z_4 \\ z_2 & z_2 & z_1 \end{pmatrix}, \text{ or its transpose.}\)

\[|X_\phi| = 3s \cdot 2 \cdot 1, \text{ where } |X_\phi| \text{ is not collinear and the arrow is generic.}\]

3. \(QP(c) : \phi_w = \begin{pmatrix} 0 & z_4 & 0 \\ z_3 & 0 & z_2 + z_4 \\ z_2 & z_3 & z_1 \end{pmatrix}, \text{ or its transpose.}\)

\[|X_\phi| = 3s \cdot 3c, \text{ where the } 3c \text{ contains a generic arrow.}\]

4. \(QP(\text{abc}) : \phi_w = \begin{pmatrix} 0 & z_4 & 0 \\ z_3 & 0 & z_2 + z_4 \\ z_2 & 0 & z_1 \end{pmatrix}, \text{ or its transpose,}\)

\(X_\phi \text{ is a union of a line and a } 3s \text{ not on the line.}\)

5. \(QP(\text{d}) : \phi_w = \begin{pmatrix} 0 & z_4 & z_3 \\ z_4 & 0 & z_2 \\ z_2 & z_1 & z_2 + a_{334}^2 z_4 \end{pmatrix}, \text{ or its transpose,}\)

where \(a_{334} \neq -\frac{1}{4}\) (so that \(Z_\phi\) is not \(CP\)). \(X_\phi\) is isomorphic to the union of a line and two other distinct points, both not lying on the line.

\(\text{CP} : \) If a trilinear form \(\phi\) is of type \(CP\), then \(\phi_w\) is necessarily \(G\)-equivalent to one of the following:

1. \(CP(a) : \phi_w = \begin{pmatrix} 0 & z_4 & 0 \\ z_3 & z_2 & z_2 \\ z_2 & -z_2 & z_1 \end{pmatrix}, \text{ or its transpose.}\)

\[|X_\phi| = 3s \cdot 1^3, \text{ where the three simple points are collinear (but not the } 3s).\]

2. \(CP(b) : \phi_w = \begin{pmatrix} 0 & z_4 & 0 \\ z_3 & z_2 & z_2 \\ z_2 & 0 & z_1 \end{pmatrix}, \text{ or its transpose.}\)

\[|X_\phi| = 3s \cdot 2 \cdot 1, \text{ where } |X_\phi| \text{ is not collinear and the arrow is pointing at the simple point.}\]

3. \(CP(c) : \phi_w = \begin{pmatrix} 0 & z_4 & 0 \\ z_3 & 0 & z_2 \\ z_2 & z_3 & z_1 \end{pmatrix}, \text{ or its transpose.}\)
\[ |X_\phi| = 3s \cdot 3l, \text{ where the } 3l \text{ contains a generic arrow.} \]

4. \( CP(abc) : \phi_w = \begin{pmatrix} 0 & z_4 & 0 \\ z_3 & 0 & z_2 \\ z_2 & 0 & z_1 \end{pmatrix} \), or its transpose.

\( X_\phi \) is the union of a line and a 3s not lying on the line.

5. \( CP(d) : \phi_w = \begin{pmatrix} 0 & z_4 & z_3 \\ z_4 & 0 & z_2 \\ z_2 & z_1 & z_2 - \frac{1}{4} z_4 \end{pmatrix} \), or its transpose.

\( X_\phi \) is isomorphic to the union of a line and an arrow whose support is not on the line.

**QT** : If a trilinear form \( \phi \) is of type QT, then \( \phi_w \) is necessarily \( G \)-equivalent to one of the following:

1. \( QT(a) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_2 & z_1 - z_2 \\ z_3 & z_1 & 0 \end{pmatrix} \), or its transpose.

\[ |X_\phi| = 4s \cdot 1^2, \text{ where } |X_\phi| \text{ is not collinear and } 4s \text{ is defined by the local ideal } (x^3, xy, y^2). \]

2. \( QT(b) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_2 & z_1 \\ z_3 & z_1 & 0 \end{pmatrix} \), or its transpose.

\[ |X_\phi| = 4s \cdot 2, \text{ where } 4s \text{ is defined by the local ideal } (x^3, xy, y^2) \text{ and the arrow is generic.} \]

3. \( QT(c) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_2 & z_2 + z_3 \\ z_3 & z_1 & 0 \end{pmatrix} \), or its transpose.

\[ |X_\phi| = 5s \cdot 1, \text{ where } 5s \text{ is defined by the local ideal } (xy - x^3, x^2y, y^2). \]

4. \( QT(cs) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_2 & z_3 \\ z_3 & z_1 & 0 \end{pmatrix} \), or its transpose.

\( X_\phi \) is the union of a conic with an embedded point.

5. \( QT(acs) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_2 & z_2 \\ z_3 & z_1 & 0 \end{pmatrix} \), or its transpose.

\( X_\phi \) is the union of a line with an embedded point and another isolated point.
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6. $QT(acs) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_2 & 0 \\ z_3 & z_1 & z_2 \end{pmatrix}$, or its transpose.

$X_\phi$ is the union of a line with two distinct embedded points.

7. $QT(abcs) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_2 & 0 \\ z_3 & z_1 & 0 \end{pmatrix}$, or its transpose.

$X_\phi$ is the union of two lines with an embedded point lying on only one of the lines.

8. $QT(d) : \phi_w = \begin{pmatrix} 0 & z_4 & z_3 \\ z_4 & 0 & p_{23}(123) \\ z_3 & z_2 & z_1 \end{pmatrix}$, or its transpose,

where $a_{232} \neq -1$ (so that $Z_\phi$ is not CV). $X_\phi$ is isomorphic to the union of a line with an embedded point and a simple point not lying on the line.

CV : If a trilinear form $\phi$ is of type CV, then $\phi_w$ is necessarily $G$-equivalent to one of the following:

1. $CV(a) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_3 - z_4 & z_1 \\ z_3 & z_2 & z_1 \end{pmatrix}$, or its transpose.

$|X_\phi| = 5s \cdot 1$, where $5s$ is defined by the local ideal $(x^3, xy, y^3)$.

2. $CV(b) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_3 - z_4 & z_1 \\ z_4 & z_2 & z_3 \end{pmatrix}$, or its transpose.

$|X_\phi| = 6s$, where $6s$ is defined by the local ideal $(y^2 - xy - x^3, x^2y, xy^2)$.

3. $CV(c) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_3 & z_3 - z_4 & z_1 \\ z_4 & z_2 & 0 \end{pmatrix}$, or its transpose.

$X_\phi$ is isomorphic to a line with an arrow whose support is on the line (but the arrow does not lie along the line).

4. $CV(abcs) : \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_3 - z_4 & 0 \\ z_2 & z_2 & 0 \end{pmatrix}$, or its transpose.
$X_\phi$ is isomorphic to a union of three distinct and non-concurrent lines.

5. $CV(d): \phi_w = \begin{pmatrix} 0 & z_4 & z_3 \\ z_4 & 0 & z_1 - z_2 + z_3 \\ z_3 & z_2 & z_1 \end{pmatrix}$.

$X_\phi$ is isomorphic to a line with an arrow whose support is on the line (but the arrow does not lie along the line).

**CT**: If a trilinear form $\phi$ is of type $CT$, then $\phi_w$ is necessarily $G$-equivalent to one of the following:

1. $CT(a): \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_3 & z_1 \\ z_3 & z_2 & 0 \end{pmatrix}$, or its transpose.

$|X_\phi| = 5s \cdot 1$, where $5s$ is defined by the local ideal $(x^3, x^2y, y^2)$.

2. $CT(b): \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_3 & z_2 \\ z_3 & z_2 & z_1 \end{pmatrix}$, or its transpose.

$|X_\phi| = 6s$, where $6s$ is defined by the local ideal $(y^2 - x^3, x^2y, xy^2)$.

3. $CT(c): \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_3 & 0 \\ z_3 & z_2 & z_1 \end{pmatrix}$, or its transpose.

$X_\phi$ is isomorphic to a line with an embedded arrow (i.e., the entire arrow lies on the line scheme-theoretically).

4. $CT(abcs): \phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & z_3 & 0 \\ z_3 & z_2 & 0 \end{pmatrix}$, or its transpose.

$X_\phi$ is isomorphic to the union of a double line and a line.

5. $CT(d): \phi_w = \begin{pmatrix} 0 & 0 & -z_3 \\ z_4 & 0 & z_2 - z_3 \\ z_3 & z_2 & z_1 \end{pmatrix}$.

$X_\phi$ is isomorphic to a line with an embedded arrow.

**PPP**: If a trilinear form $\phi$ is of type $PPP$, then $\phi_w$ is necessarily $G$-equivalent to one of the following:

1. $PPP(a): \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_2 & p_{23}(124) \\ z_4 & p_{32}(123) & z_1 \end{pmatrix}$. 

$X_\phi$ is isomorphic to a line with an embedded arrow.
2. \(PPP(b): \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_4 & z_1 \\ z_3 & p_{32}(13) & p_{33}(12) \end{pmatrix}\) or its transpose.

3. \(PPP(c): \phi_w = \begin{pmatrix} 0 & z_4 & z_3 \\ z_4 & 0 & z_2 \\ z_3 & z_1 & \lambda(z_1 + z_2) \end{pmatrix}\), where \(\lambda\) is a (possibly vanishing) scalar.

4. \(PPP(d): \phi_w = \begin{pmatrix} 0 & z_4 & z_3 \\ z_4 & 0 & z_2 \\ 0 & z_1 & 0 \end{pmatrix}\) or its transpose.

\(\overline{PPP}\): If a trilinear form \(\phi\) is of type \(\overline{PPP}\), then \(\phi_w\) is necessarily \(G\)-equivalent to one of the following:

1. \(\overline{PPP}(a): \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_4 & z_1 \\ z_3 + z_4 & z_2 & p_{33}(124) \end{pmatrix}\).

2. \(\overline{PPP}(b): \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_4 & z_2 \\ z_3 + z_4 & p_{32}(12) & z_1 \end{pmatrix}\) or its transpose.

\(\overline{P^2P}\): If a trilinear form \(\phi\) is of type \(\overline{P^2P}\), then \(\phi_w\) is necessarily \(G\)-equivalent to one of the following:

1. \(\overline{P^2P}(a): \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_4 & z_1 \\ z_3 & p_{33}(124) & z_2 \end{pmatrix}\).

2. \(\overline{P^2P}(b): \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_4 & p_{32}(124) \\ z_3 & z_2 & z_1 \end{pmatrix}\) or its transpose.

3. \(\overline{P^2P}(c): \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_4 & z_2 \\ z_4 & p_{32}(123) & z_1 \end{pmatrix}\) or its transpose.
4. $P^2P(d) : \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_4 & z_1 \\ z_4 & z_2 & p_{33}(12) \end{pmatrix}$ or its transpose.

5. $P^2P(e) : \phi_w = \begin{pmatrix} 0 & 0 & z_3 \\ 0 & z_4 & p_{23}(12) \\ z_4 & z_2 & z_1 \end{pmatrix}$

6. $P^2P(f) : \phi_w = \begin{pmatrix} 0 & z_4 & -z_3 \\ z_4 & 0 & z_2 \\ z_3 & z_2 & z_1 \end{pmatrix}$.

$P^3$ : If a trilinear form $\phi$ is of type $P^3$, then it is necessarily $G$-equivalent to the following:

$$\phi_w = \begin{pmatrix} 0 & 0 & z_4 \\ 0 & z_4 & z_2 \\ z_4 & z_3 & z_1 \end{pmatrix}.$$

Outline of proof. The theorem is the result of a tremendous amount of computations and book-keeping. We would just provide an outline of the strategy. The three main steps are as follows:

**First Classification.** The first step is to assume that $z_4$ is a factor of $h = \det(\phi_w)$. Using the properties of determinants, we find that there are (up to row and column operations and transpositions) 10 types of such matrices:

(a) four types which are generically $QP$, namely

$$QP1 = \begin{pmatrix} 0 & z_4 & p_{13} \\ z_4 & 0 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \quad QP2 = \begin{pmatrix} 0 & 0 & z_4 \\ p_{21} & z_4 + p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix},$$

$$QP3 = \begin{pmatrix} 0 & z_4 & p_{13} \\ z_4 & 0 & p_{23} \\ p_{31} & p_{32} & z_4 + p_{33} \end{pmatrix}, \quad QP4 = \begin{pmatrix} 0 & 0 & z_4 \\ p_{21} & z_4 + p_{22} & p_{23} \\ z_4 + p_{31} & p_{32} & p_{33} \end{pmatrix},$$

(b) one type which is generically $CP$, $CP = \begin{pmatrix} 0 & 0 & z_4 \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$,

(c) one type which is generically $QT$, $QT = \begin{pmatrix} 0 & 0 & z_4 \\ z_4 & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix},$ and
(d) four types which are generically \( PPP \), namely

\[
\begin{align*}
PPP1 &= \begin{pmatrix} z_4 & 0 & p_{13} \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, &
PPP2 &= \begin{pmatrix} z_4 & 0 & p_{13} \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & z_4 + p_{33} \end{pmatrix}, \\
PPP3 &= \begin{pmatrix} z_4 & 0 & p_{13} \\ 0 & 0 & p_{23} \\ p_{31} & z_4 + p_{32} & p_{33} \end{pmatrix} &
PPP4 &= \begin{pmatrix} z_4 & 0 & p_{13} \\ 0 & 0 & z_4 + p_{23} \\ p_{31} & z_4 + p_{32} & p_{33} \end{pmatrix},
\end{align*}
\]

where \( p_{jk} \)'s are linear polynomials in only \( z_1, z_2 \) and \( z_3 \).

We then further assume that \( \phi \) is non-degenerate to reduce the above generic forms to simpler forms by setting some of the \( p_{jk} \)'s as \( z_1, z_2 \) or \( z_3 \) and performing suitable \( G \)-actions.

**Geometry of \( X_\phi \).** The second step is to investigate the geometry of \( X_\phi \)'s (all the \( Y_\phi \)'s are one dimensional). When \( X_\phi \) is zero dimensional (hence admissible with necessarily non-curvilinear multiple points, see Section 4.2), \( \phi \) is uniquely determined by the isomorphism type of the multiple points and subsequent syzygy computations. This happens for the generic \( QP2, QP4, CP \) and \( QT \) trilinear forms in cases (a), (b) and (c) above. When \( X_\phi \) is one dimensional (as in the generic \( QP1, QP3 \) cases and all the \( PPP \) cases), there are no multiple points involved and a one dimensional \( X_\phi \) does not characterize a trilinear form via the syzygy method. We reduce these trilinear forms by suitable \( G \)-actions and organize them. These trilinear forms are always designated the last case of each category (from \( QP \) to \( CT \) trilinear forms).

**Degenerations.** The last (and the most messy) step is to allow the generic trilinear forms to degenerate. There are basically two types of degenerations: one in which \( Z_\phi \) remains unchanged but \( X_\phi \) specializes, possibly to a one dimensional scheme (as in \( QP(a) \rightarrow QP(b) \rightarrow QP(c) \rightarrow QP(abc) \)), and another in which \( Z_\phi \) itself degenerates (as in \( QP(a) \rightarrow CP(a) \) (or \( QT(a) \) \rightarrow \( CV(a) \) \rightarrow \( CT(a) \)). All other degenerations are combinations of these two types. Of course, there is considerable overlap among the different degeneration pictures. For example, the generic \( CP \) and \( QT \) trilinear forms in cases (b) and (c) are specializations of \( QP2 \) trilinear forms and the generic \( QP2 \) and \( QP4 \) trilinear forms are actually \( G \)-equivalent.

Finally, we write down all the reducible trilinear forms in the theorem after performing suitable \( G \)-actions and some organization. \( \square \)

### 7.3. Conical trilinear forms

We say that \( \phi \) is **conical** if \( Z_\phi \) is a cone over an irreducible cubic curve on the plane, equivalently, \( h \) is an irreducible cubic polynomial in only
\( z_2, z_3 \) and \( z_4 \) if the vertex of the cone is \([1, 0, 0, 0]\). [3] labels these conical cubic surfaces as \( \hat{E}_6 \), \( CN \) and \( CC \) (when the irreducible cubic curve is respectively smooth, nodal or cuspidal). We have the following theorem for conical trilinear forms:

**Theorem 9.** Suppose \( \phi \) is conical, then exactly one of the following holds:

1. **Type \( \hat{E}_6 \)**: \( \phi_w : W^* \to U \otimes V \) has rank 3 (hence \( \phi \) is degenerate), \( Z_\phi \) is a cone over a smooth plane cubic curve and it has an \( \hat{E}_6 \) singularity at the vertex. \( \phi \) is actually a smooth \((3, 3, 3)\) trilinear form.

2. **Degenerate Types \( CN \) or \( CC \)**: \( \phi_w : W^* \to U \otimes V \) has rank 3 (hence \( \phi \) is degenerate), \( Z_\phi \) is a cone over an irreducible nodal or cuspidal plane cubic curve and it has a nodal line joining the node or the cusp to the vertex.

3. **Type \( CN \)**: \( \phi_w \) is \( G \)-equivalent to the following:

\[
\phi_w = \begin{pmatrix} 0 & z_3 & z_4 \\ z_3 & z_4 & -z_1 \\ z_4 & z_1 + z_2 & z_3 \end{pmatrix}.
\]

\( Z_\phi \) is a cone (with vertex at \([1, 0, 0, 0]\)) over an irreducible nodal cubic curve (with singularity at \([2, z_3, z_4] = [1, 0, 0]\)). \( X_\phi \) and \( Y_\phi \) are both isomorphic to a multiplicity 6 point defined by the local ideal \((xy - y^3, x^3 - y^3, xy^2)\).

4. **Type \( CC \)**: \( \phi_w \) is \( G \)-equivalent to the following:

\[
\phi_w = \begin{pmatrix} 0 & z_3 & z_3 + z_4 \\ z_3 & -z_4 & -z_1 - z_2 \\ z_4 & z_1 + z_2 & z_1 \end{pmatrix}.
\]

\( Z_\phi \) is a cone (with vertex at \([1, 0, 0, 0]\)) over an irreducible cuspidal cubic curve (with singularity at \([2, z_3, z_4] = [1, 0, 0]\)). \( X_\phi \) and \( Y_\phi \) are both isomorphic to a multiplicity 6 point defined by the local ideal \((y^2 - x^3, x^3 - x^2y, xy^2)\).

In particular, a Type \( \hat{E}_6 \) trilinear form is necessarily degenerate.

**Proof.** Suppose a conical trilinear form \( \phi \) is such that \( Z_\phi \) is a cone (with vertex at \([1, 0, 0, 0]\)) over an irreducible cubic curve \( C \), equivalently \( h \) is a cubic polynomial in only \( z_2, z_3 \) and \( z_4 \), then \( \phi_w \) has either rank 3 (cases (1) and (2)) or is \( G \)-equivalent to the following generic form (after
a simple computation of determinants):
\[
\phi_w = \begin{pmatrix}
0 & z_3 & z_4 \\
z_3 & p_{22}(24) & -z_1 \\
z_4 & z_1 + p_{32}(24) & p_{33}(234)
\end{pmatrix}.
\]

It is easy to check that \( C \) necessarily has a singularity at \([z_2, z_3, z_4] = [1, 0, 0]\). The curve singularity is a node (or cusp) if and only if \( \Delta = a_{22}^2 + 4a_{32}a_{23} \neq 0 \) (respectively \( \Delta = 0 \)) with not all of \( a_{22}, a_{32}, a_{33} \) vanishing. In either cases, \( X_\phi \) and \( Y_\phi \) are both isomorphic to a non-curvilinear multiplicity 6 point. If \( \Delta \neq 0 \), we can obtain the normalized \( \phi_w \) in case (3) after computing a suitable set of generators (stated in the statement of the theorem) for the defining ideal of \( X_\phi \) and the syzygy matrix as in the case of Duval trilinear forms. Similarly, we can compute the normalized form for \( CC \) trilinear form in case (4) if \( \Delta = 0 \).

If \( a_{22} = a_{32} = a_{33} = 0 \), \( h \) is a polynomial in only \( z_3 \) and \( z_4 \), \( \phi \) is generically \( PPP \) but not conical.

7.4. Nodal trilinear forms

We say that \( \phi \) is nodal if it is irreducible, it is not conical and it has a nodal line. There are exactly two nodal cubic surfaces up to projectivity, namely, \( NL1 \) defined by \( h = z_1z_3^2 + z_2z_4^2 = 0 \) and \( NL2 \) defined by \( h = z_1z_3^2 + z_2z_3z_4 + z_4^2 = 0 \) (which is a specialization of \( NL1 \), see [3]). The two nodal cubic surfaces are both singular along the line \([z_1, z_2, 0, 0]\) but are distinguished (and hence characterized) by their Hessians (\( NL1 \) has Hessian \( z_3^2z_4^2 \) but \( NL2 \) has Hessian \( z_3^2 \)) since the latter is the determinant of the second fundamental form and hence a projective invariant.

We would say that \( \phi \) is \( NL1 \) (or \( NL2 \)) if \( Z_\phi \) is isomorphic to \( NL1 \) (respectively \( NL2 \)). We also use suffixes \((a), (b)\) etc. to label non-\( G \)-equivalent nodal trilinear forms which have isomorphic \( Z_\phi \). We have the following theorem for \( NL1 \) trilinear forms:

**Theorem 10 (NL1 Trilinear forms).** There are exactly six \( G \)-equivalence classes of \( NL1 \) trilinear forms:

1. \( NL1(a) \): \( \phi \) is \( G \)-equivalent to \( \phi_w = \begin{pmatrix}
0 & z_3 & z_2 \\
z_3 & z_4 & 0 \\
z_4 & 0 & z_1
\end{pmatrix} \).

\( X_\phi \) is a union of a smooth conic and an isolated point and \( Y_\phi \) is a union of a line with an isolated point.
2. \(NL1(b)\) : \(\phi\) is \(G\)-equivalent to \(\phi_w = \begin{pmatrix} 0 & z_3 & z_4 \\ z_3 & z_4 & 0 \\ z_2 & 0 & z_1 \end{pmatrix}\).

Note that \(NL1(a)\) and \(NL1(b)\) are equivalent up to the transposition of \(\phi_w\)'s.

3. \(NL1(c)\) : \(\phi\) is \(G\)-equivalent to \(\phi_w = \begin{pmatrix} 0 & z_3 & z_4 \\ z_3 & z_2 & 0 \\ z_4 & 0 & z_1 \end{pmatrix}\).

\(|X_\phi| = |Y_\phi| = 4s \cdot 1^2\) (neither collinear), where \(4s\) is scheme-theoretically defined by \((x^2, y^2)\).

4. \(NL1(d)\) : \(\phi\) is \(G\)-equivalent to \(\phi_w = \begin{pmatrix} 0 & z_3 & z_4 \\ z_3 & z_2 & z_2 \\ z_4 & -z_2 & z_1 \end{pmatrix}\).

\(|X_\phi| = |Y_\phi| = 4s \cdot 1^2\) (neither collinear), where \(4s\) is scheme-theoretically defined by \((x^2, y(x-y))\).

5. \(NL1(e)\) : \(\phi\) is \(G\)-equivalent to \(\phi_w = \begin{pmatrix} 0 & z_3 + z_4 & -8z_4 \\ z_3 - z_4 & 0 & z_1 + z_2 \\ 8z_4 & z_1 + z_2 & -16z_1 \end{pmatrix}\).

\(|X_\phi| = |Y_\phi| = 4s \cdot 2\), where \(4s\) is scheme-theoretically defined by \((xy, x^2 - y^2)\).

6. \(NL1(f)\) : \(\phi\) is \(G\)-equivalent to \(\phi_w = \begin{pmatrix} 0 & 4z_3 + 2z_4 & z_4 \\ 4z_3 - 2z_4 & 0 & z_1 + 2z_2 \\ -z_4 & z_1 + 4z_2 & z_1 \end{pmatrix}\).

\(|X_\phi| = |Y_\phi| = 4s \cdot 2\), where \(4s\) is scheme-theoretically defined by \((x^2, y(x-y))\).

\(Z_\phi\) is defined by \(h = z_1z_2^2 + z_2z_4^2 = 0\) in all the above cases.

Sketch of proof. We begin by imposing the restriction \(h = z_1z_2^2 + z_2z_4^2\) for a \(NL1\) trilinear form and assuming without loss of generality that \(\phi_w\) has the form

\[\phi_w = \begin{pmatrix} p_{11}(24) & z_3 + p_{12}(124) & p_{13}(24) \\ z_3 + p_{21}(124) & p_{22}(24) & p_{23}(24) \\ p_{31}(24) & p_{32}(24) & z_1 + p_{33}(24) \end{pmatrix}\.

Regarding \(h\) as a quadratic polynomial with indeterminate \(z_3\) over \(\mathbb{C}[z_1, z_2, z_4]\), we find that \(p_{33} = 0\),

coefficient of \(z_3 = z_1(p_{12} + p_{21}) - (p_{13}p_{32} + p_{24}p_{31}) = 0\)
and more importantly,

"constant" coefficient = \( |A| := \begin{vmatrix} p_{11}(124) & p_{12}(124) & p_{13}(24) \\ p_{21}(124) & p_{22}(124) & p_{23}(24) \\ p_{31}(24) & p_{32}(24) & z_1 \end{vmatrix} = -z_2 z_4^2 \).

With a computer algebra package (Maple in our case), we can obtain a comprehensive list of all NL1 trilinear forms satisfying the above conditions. After applying G-operations and syzygy computations in each case, we have the results in the theorem.

We have a similar result for NL2 trilinear forms:

**Theorem 11 (NL2 Trilinear forms).** There are exactly four G-equivalence classes of NL2 trilinear forms:

1. **NL2(a)**: \( \phi \) is G-equivalent to \( \phi_w = \begin{pmatrix} 0 & z_3 & z_4 \\ z_3 & z_4 & -z_2 \\ z_4 & 0 & z_1 \end{pmatrix} \).
   
   \( X_\phi \) is a smooth conic with an embedded point and \( Y_\phi \) is a line with an arrow whose support lies on the line.

2. **NL2(b)**: \( \phi \) is G-equivalent to \( \phi_w = \begin{pmatrix} 0 & z_3 & z_4 \\ z_3 & z_4 & 0 \\ z_4 & -z_2 & z_1 \end{pmatrix} \).
   
   Note that NL2(a) and NL2(b) are equivalent up to the transposition of \( \phi_w \)’s.

3. **NL2(c)**: \( \phi \) is G-equivalent to \( \phi_w = \begin{pmatrix} 0 & z_3 & z_4 \\ z_3 & z_4 & -z_2 - (1 + \mu)z_2 \\ z_4 & \lambda z_1 + \mu z_2 & z_1 \end{pmatrix} \),
   where \( \lambda \neq 0 \), \( \mu \in \mathbb{C} \).
   \( |X_\phi| = |Y_\phi| = 5s \cdot 1 \), where 5s is defined by the local ideal \( (y^3, y^2 + \lambda x^3, (1 + \mu)y^2 + \lambda xy) \) and \( (y^3, y^2 - \lambda x^3, \mu y^2 + \lambda xy) \) respectively. As \( \lambda \to 0 \), NL2(c) specializes to NL2(d).

4. **NL2(d)**: \( \phi \) is G-equivalent to \( \phi_w = \begin{pmatrix} z_4 & z_3 & -(1 + \lambda)z_2 \\ z_3 & 0 & z_4 \\ \lambda z_2 & z_4 & z_1 \end{pmatrix} \),
   \( \lambda \neq 0, -1 \).
   \( |X_\phi| = |Y_\phi| = 6s \), where 6s is defined by the local ideal \( (y^3, xy^2, y(y-x^2), y^2 - xy - \delta x^3) \) (with \( \delta = -1 - \lambda \) for \( X_\phi \) but \( \delta = \lambda \) for \( Y_\phi \)).
   As \( \lambda \to 0 \), NL2(d) specializes to NL2(a). As \( \lambda \to -1 \), NL2(d) specializes to NL2(b).
$Z_\Phi$ is defined by $h = z_1 z_3^2 + z_2 z_3 z_4 + z_4^3 = 0$ in all the above cases.

The proof is similar to the case of NL1 trilinear forms with the replacement $z_1(p_{12} + p_{21}) - p_{13}p_{32} - p_{23}p_{31} = z_2 z_4$ and $|A| = -z_3^3$. ∎

References

The classification of (3,3,4) trilinear forms


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