A CONSTRUCTION OF MAXIMAL COMMUTATIVE SUBALGEBRA OF MATRIX ALGEBRAS

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Abstract. Let \((B, m_B, k)\) be a maximal commutative \(k\)-subalgebra of \(M_n(k)\). Then, for some element \(z \in \text{Soc}(B)\), a \(k\)-algebra \(R = B[Z, Y]/I\), where \(I = (m_B X, m_B Y, X^2 - z, Y^2 - z, XY)\) will create an interesting maximal commutative \(k\)-subalgebra of a matrix algebra which is neither a \(C_1\)-construction nor a \(C_2\)-construction. This construction will also be useful to embed a maximal commutative \(k\)-subalgebra of matrix algebra to a maximal commutative \(k\)-subalgebra of a larger size matrix algebra.

1. Introduction

Let \((B, m_B, k)\) be a maximal commutative \(k\)-subalgebra of \(M_n(k)\). In this paper, we are interested in the following problem:

“How can we construct a maximal commutative \(k\)-subalgebra \((R, m, k)\) of \(M_n(k)\) for some \(n\) with \(m < n\)?”

In [2], Brown introduced a construction to produce a maximal commutative \(k\)-subalgebra \((R, m, k)\) of \(M_{m+1}(k)\) from \((B, m_B, k)\). In this paper, we will present a construction to produce a maximal commutative \(k\)-subalgebra \((R, m, k)\) of \(M_{m+2}(k)\) from \((B, m_B, k)\). In fact, the \(k\)-subalgebra \(R\) has dimension two more than the dimension of \(B\). We will call this construction a \(C_2^2\)-construction.

Moreover, we will show the \(C_1\)-construction (\(C_2\)-construction) does not imply the \(C_2^2\)-construction in the next section and we can conclude that \(C_2^2\)-construction is another construction to produce maximal commutative \(k\)-subalgebras of matrix algebra.

Recall the \(C_1\)-construction and \(C_2\)-construction given by Brown and Call in [1] and [2].

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**Definition 1.1.** Let \((B, m_B, k)\) be a finite dimensional commutative local \(k\)-algebra with identity and \(N\) a finitely generated faithful \(B\)-module. Then \(R = B \oplus N^f\) is a commutative \(k\)-algebra and \(M = B^f \oplus N\) is a faithful \(R\)-module via the following multiplications:

\[
\alpha(b, n_1, \ldots, n_\ell) = (\alpha b, \alpha n_1, \ldots, \alpha n_\ell)
\]

\[
(b, n_1, \ldots, n_\ell)(b', n'_1, \ldots, n'_\ell) = (bb', n_1 b' + n'_1 b, \ldots, n_\ell b' + n'_\ell b)
\]

\[
(b_1, \ldots, b_\ell, n)(b, n_1, \ldots, n_\ell) = (b_1 b, \ldots, b_\ell b, nb + \sum_{i=1}^\ell n_i b_i).
\]

Moreover, it is known that \(R \cong \text{Hom}_R(M, M)\) via the regular representation. Thus, \(R\) is isomorphic to a maximal commutative subalgebra of \(M_n(k)\), where \(n = \text{dim}_k(M)\). The \(k\)-algebra \(R\) of this form is called a \(C_1\)-construction.

The next theorem presents an equivalent condition to be a \(C_1\)-construction and the proof can be found in [1].

**Theorem 1.2.** Let \((R, m, k)\) be a commutative local \(k\)-algebra. Then, \(R\) is a \(C_1\)-construction if and only if there is an ideal \(I\) satisfying the following conditions:

1. \(\text{Ann}_R(I) = I\)
2. \(0 \to I \to R \to R/I \to 0\) splits as \(k\)-algebras.

Throughout this paper, the socle of an algebra \(R\) will be denoted by \(\text{Soc}(R)\) and the following theorem can be found in [2].

**Theorem 1.3.** Let \((B, m_B, k)\) be a finite dimensional commutative local \(k\)-algebra with identity and \(N\) a finitely generated faithful \(B\)-module. Suppose \(B \cong \text{Hom}_B(N, N)\) via the regular representation. Then there exists an element \(z \in \text{Soc}(B)\) with \(\text{dim}_k( Nz ) = 1\).

The following definition can be found in [3] and is a kind of generalization of the definition of \(C_2\)-construction in [2].

**Definition 1.4.** Let \((B, m_B, k)\) be a finite dimensional commutative local \(k\)-algebra with identity. If \(R \cong B[X]/(m_B X^p - z)\) for some \(z \in \text{Soc}(B) - \{0\}\) and a positive integer \(p > 1\), then the \(k\)-algebra \(R\) of this form is called a \(C_2\)-construction.

Here is an equivalent condition to be a \(C_2\)-construction and can be found in [3].

**Theorem 1.5.** Let \((R, m, k)\) be a commutative local \(k\)-algebra. Then, \(R\) is a \(C_2\)-construction if and only if \(R\) contains a commutative
$k$-subalgebra $(B, m_B, k)$ and an element $x \in m$ satisfying the following conditions:

1. $0 \neq x^p \in \text{Soc}(B)$ for some positive integer $p > 1$,
2. $m_B x = (0)$,
3. $\text{dim}_k(R) = \text{dim}_k(B) + p - 1$.

2. $C_2^2$-construction

In this section, we will introduce a method to produce a maximal commutative $k$-subalgebra $(R, m, k)$ of $M_{m+2}(k)$ from a maximal commutative $k$-subalgebra $(B, m_B, k)$ of $M_m(k)$.

**Theorem 2.1.** Let $(B, m_B, k)$ be a finite dimensional commutative local $k$-algebra with identity and $N$ a finitely generated faithful $B$-module. Suppose $B \cong \text{Hom}_B(N, N)$ via the regular representation. Let $R = B[X, Y]/(m_BX, m_BY, X^2 + Y^2 = z, XY = 0)$ and let $z \in \text{Soc}(B) - \{0\}$ with $\text{dim}_k(Nz) = 1$. If we let $M = N \oplus Nz \oplus Nz$, then the $k$-algebra $R$ is isomorphic to $\text{Hom}_R(M, M)$ via the regular representation. In other words, $R$ is isomorphic to a maximal commutative $k$-subalgebra of $M_N(k)$, where $n = \text{dim}_k(M)$.

**Proof.** Obviously, $M = N \oplus Nz \oplus Nz$ is a $B[X, Y]$-module via the following operations:

$(n, n_1 z, n_2 z)b = (nb, n_1 zb, n_2 zb)$

$(n, n_1 z, n_2 z)x = (n_1 z, nz, n_2 z^2) = (n_1 z, nz, 0)$

$(n, n_1 z, n_2 z)y = (n_2 z, n_1 z^2, nz) = (n_2 z, 0, nz)$

for all $n, n_1, n_2 \in N$ and $b \in B$. If we let $x$ and $y$ be the images of $X$ and $Y$ in $R$, then $M$ is an $R$-module via the following operations:

$(n, n_1 z, n_2 z)x = (n_1 z, nz, n_2 z^2) = (n_1 z, nz, 0)$

$(n, n_1 z, n_2 z)y = (n_2 z, n_1 z^2, nz) = (n_2 z, 0, nz)$.

Now let

$(n, 0, 0)\bar{b} + \alpha x + \beta y = (0, 0, 0)$

for $n \in N$ and $\alpha, \beta \in k$. Then

$(nb, n\alpha z, n\beta z) = (0, 0, 0)$

and hence by the faithfulness of $N$, we obtain

$b = 0, \quad \alpha = \beta = 0,$

which implies $M$ is a finitely generated faithful $R$-module.
Let \( f \in \text{Hom}_R(M, M) \) and define \( g : N \to M \) and \( h : M \to N \) by
\[
g(n) = (n, 0, 0), \quad h(n, n_1z, n_2z) = n
\]
for \( n, n_1, n_2 \in N \). Then, obviously \( g \) and \( h \) are \( B \)-module homomorphisms and hence the composition map \( \pi = hfg \) is a \( B \)-module homomorphism.

Since \( B \cong \text{Hom}_B(N, N) \) via the regular representation, \( \pi = \mu_a \) for some \( a \in B \). Thus,
\[
h(f(n, 0, 0)) = \pi(n) = \mu_a(n) = na.
\]
This implies that there are two functions
\[
\phi_1 : N \to Nz, \quad \phi_2 : N \to Nz
\]
such that
\[
f(n, 0, 0) = (na, \phi_1(n), \phi_2(n)).
\]
Then, it is easy to show that \( \phi_1 \) and \( \phi_2 \) are \( B \)-module homomorphisms.

Since \( \text{dim}_k(Nz) = 1 \), there exists an element \( n' \in N \) such that \( \{n'z\} \) is a basis of \( k \)-vector space \( Nz \). Thus, there exist \( \gamma_1, \gamma_2 \in k \) such that
\[
\phi_1(n') = \gamma_1n'z, \quad \phi_2(n') = \gamma_2n'z.
\]
Then, \( a + \gamma_1x + \gamma_2y \in R \) and we want to show
\[
f = \mu_a + \gamma_1x + \gamma_2y.
\]
To prove this, it is enough to show the following two identities:

1. \( f(0, n_1z, n_2z) = (\gamma_1n_1z + \gamma_2n_2z, n_1az, n_2az) \),
2. \( \phi_1(n) = \gamma_1n_z, \quad \phi_2(n) = \gamma_2n_z \).

It suffices to show the identity (1) for \( n_1 = n', n_2 = n' \). In fact,
\[
f(0, n_1z, n_2z) = f((n_1, 0, 0)x + (n_2, 0, 0)y)
= f((n_1, 0, 0))x + f((n_2, 0, 0))y
= (n_1a, \phi_1(n_1), \phi_2(n_1))x + (n_2a, \phi_1(n_2), \phi_2(n_2))y
= (\phi_1(n_1), n_1az, \phi_2(n_1)z) + (\phi_2(n_2), \phi_1(n_2)z, n_2az)
= (\gamma_1n_1z, n_1az, 0) + (\gamma_2n_2z, 0, n_2az)
= (\gamma_1n_1z + \gamma_2n_2z, n_1az, n_2az).
\]
Thus, identity (1) is satisfied.

For identity (2), note that
\[
(\gamma_1n_z, naz, 0) = f(0, n, 0) = f((n, 0, 0)x) = f(n, 0, 0)x
= (na, \phi_1(n), \phi_2(n))x = (\phi_1(n), naz, \phi_2(n)z)
\]
and from these identities, we obtain
\[
\phi_1(n) = \gamma_1n_z.
\]
Similarly, we have the following identities:
\[ (\gamma_2nz, 0, naz) = f(0, 0, nz) = f((n, 0, 0)y) = f(n, 0, 0)y = (na, \phi_1(n), \phi_2(n))y = (\phi_2(n), \phi_1(n)z, naz). \]

Thus we have
\[ \phi_2(n) = \gamma_2nz. \]

The identity (2) is thus satisfied and finally we obtain
\[ f(n, n_1z, n_2z) = (na + n_1a_1z + n_2a_2z, n_1a_1z + n_1a_1z + n_2a_2z) = (na, n_1az, n_2az) + (n_1a_1z, n_1a_1z, 0) + (n_2a_2z, 0, n_2a_2z) = (na, n_1az, n_2az) + (n_1a_1z, n_1a_1z, 0) + (n_2a_2z, n_2a_2z, n_2a_2z) = (n, n_1z, n_2z)(a + \gamma_1x + \gamma_2y) = \mu_{a+\gamma_1z+\gamma_2y}(n, n_1z, n_2z). \]

Therefore, we have the following result:
\[ f = \mu_{a+\gamma_1z+\gamma_2y}. \]

Since \( M \) is a faithful \( R \)-module, \( R \) is isomorphic to \( \text{Hom}_R(M, M) \) via the regular representation and hence \( R \) is isomorphic to a maximal commutative \( k \)-subalgebra of \( M_n(k) \), where \( n = \dim_k(M) \).

We will call the \( k \)-algebra \( R \) of the form in Theorem 2.1 a \( C_2^2 \)-construction.

If \((B_1, m_{B_1}, k)\) is a commutative \( k \)-algebra which is isomorphic to \( k \)-algebra \((B, m_B, k)\). Then, \( B \)-module \( N \) is a \( B_1 \)-module via \( nb_1 = n\phi(b) \), where \( \phi : B \to B_1 \) is an isomorphism from \( B \) to \( B_1 \) and \( \phi(b) = b_1 \). Thus, the following corollary can be proved.

**Corollary 2.2.** Let \((B, m_B, k)\) be a finite dimensional commutative local \( k \)-algebra with identity and \( N \) a finitely generated faithful \( B \)-module. Suppose \( B \) is isomorphic to \( \text{Hom}_B(N, N) \) via the regular representation. Let \((B_1, m_{B_1}, k)\) be a commutative \( k \)-algebra which is isomorphic to \( B \). If we let \( R = B_1[X, Y]/(m_{B_1}X, m_{B_1}Y, X^2 - z, Y^2 - z, XY), z \in \text{Soc}(B_1) - \{0\} \) with \( \dim_k(Nz) = 1 \), and \( M = N \oplus Nz \oplus Nz \), then the \( k \)-algebra \( R \) is isomorphic to \( \text{Hom}_R(M, M) \) via the regular representation. Thus, \( R \) is isomorphic to a maximal commutative \( k \)-subalgebra of \( M_n(k) \), where \( n = \dim_k(M) \).

**Remark 2.3.** With the \( C_2^2 \)-construction, a maximal commutative \( k \)-subalgebra \( B \) of \( M_n(k) \) with \( \dim_k(B) = t \) can be embedded in a maximal commutative \( k \)-subalgebra \( R \) of \( M_{m+2}(k) \) with \( \dim_k(R) = t+2 \).
Moreover, if \( t < m \), then we can construct a maximal commutative \( k \)-subalgebra of matrix algebra whose dimension is not greater than the size of the matrix by applying \( C_2^2 \)-construction.

For example, the \( k \)-algebra \( R \) in [4] by Courer is a maximal commutative \( k \)-subalgebra of \( M_{14}(k) \) whose dimension is 13. Now, by applying \( C_2^2 \)-construction successively, we can construct maximal commutative \( k \)-subalgebras \( R_1, R_2, \ldots, R_t \) of \( M_{16}(k), M_{18}(k), \ldots, M_{14+2t}(k) \) for \( t \in \mathbb{N} \), respectively. Here, the dimension of \( R_t \) is obviously \( 13 + 2t \) for each \( t \in \mathbb{N} \).

Here is an example of a \( C_2^2 \)-construction. We will let \( E_{ij} \) be the \((i, j)\)-th matrix unit.

**Example 2.4.** Let \( B \) be a local \( k \)-subalgebra of \( M_4(k) \) having maximal ideal consisting of the following matrices:

\[
\tau = \begin{pmatrix}
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & a & 0 & 0 \\
c & b & a & 0
\end{pmatrix}
\]

for \( a, b, c \in k \). Then, \( B \) is a local maximal commutative \( k \)-subalgebra of \( M_4(k) \).

Note that we can embed \( B \) into \( M_6(k) \) via the following \( k \)-algebra homomorphism:

\[
f \left[ \begin{pmatrix}
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & a & 0 & 0 \\
c & b & a & 0
\end{pmatrix} + \alpha I_4 \right] = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 & 0 \\
b & a & 0 & 0 & 0 & 0 \\
c & b & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} + \alpha I_6.
\]

If \( B_1 = f(B) \), then the map \( f : B \to B_1 \) is a \( k \)-algebra isomorphism.

Now let

\[
X = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad Y = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Then, the following identities hold:

\[
m_{B_1}X = (0) = m_{B_1}Y, \quad X^2 = E_{41} = Y^2, \quad XY = 0
\]
and hence
\[ (m_B, X, m_B, Y, X^2 - E_{41}, Y^2 - E_{41}, XY) = (0). \]
Thus, the \( k \)-algebra \( R \) is given as follows:
\[ R = B_1[X, Y] = B_1[X, Y]/(m_B, X, m_B, Y, X^2 - E_{41}, Y^2 - E_{41}, XY). \]
In fact,
\[ R = \{ r + \alpha I_6 \mid \alpha \in k \}, \]
where \( r \) is of the form
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\alpha & 0 & 0 & 0 & 0 \\
b & a & 0 & 0 & 0 \\
c & b & a & 0 & d \\
d & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 0
\end{pmatrix}
\]
for some \( a, b, c, d, e \in k \). Thus, the \( k \)-algebra \( R \) is a \( C_2^2 \)-construction. Therefore, we construct a maximal commutative \( k \)-subalgebra \( R \) which is a \( C_2^2 \)-construction and of dimension 6 from a maximal commutative \( k \)-subalgebra \( B \) of dimension 4.

REMARK 2.5. In the above Example, a maximal commutative \( k \)-subalgebra \( B \) of \( M_4(k) \) is embedded to a maximal commutative \( k \)-subalgebra \( R \) of \( M_6(k) \).

From the definition of \( C_2^2 \)-construction, the following property can be obtained.

Theorem 2.6. Let \((R, m, k)\) be a finite dimensional local commutative \( k \)-algebra. Then, \( R \) is a \( C_2^2 \)-construction if and only if there exist a commutative \( k \)-subalgebra \((B, m_B, k)\) and elements \( x, y \in m \) satisfying the following properties:

1. \( x^2 = y^2 \in \text{Soc}(B) - \{0\} \).
2. \( xy = 0 \).
3. \( m_B x = (0) = m_B y \).
4. \( \text{dim}_k(R) = \text{dim}_k(B) + 2 \).

Proof. Suppose \( R \) is a \( C_2^2 \)-construction. Then, by the definition of \( C_2^2 \)-construction, there exist a finite dimensional local commutative \( k \)-algebra \((B, m_B, k)\) and a finitely generated faithful \( B \)-module \( N \) such that
\[ R = B[X, Y]/(m_B X, m_B Y, X^2 - z, Y^2 - z, XY) \]
for some $z \in Soc(B) - \{0\}$ with $\dim_k(Nz) = 1$. Let $x$ and $y$ be the image of $X$ and $Y$, respectively. Then, the conditions (1), (2), (3) and (4) can be shown by straightforward calculations. Conversely, suppose there exist a $k$-subalgebra $B$ and elements $x$ and $y$ in $m$ such that the given conditions are satisfied. Let $x^2 = y^2 = z \in Soc(B)$ and define a map

$$\psi : B[X,Y]/(m_B X, m_B Y, X^2 - z, Y^2 - z, XY) \rightarrow R$$

by

$$\psi(b + I) = b, \quad \psi(X + I) = x, \quad \psi(Y + I) = y,$$

where $b \in B$ and $I = (m_B X, m_B Y, X^2 - z, Y^2 - z, XY)$. Then, obviously $\psi$ is a $k$-algebra homomorphism. Suppose $b + cX + dY + I \in ker\psi$. Then, we have

$$\psi(b + cX + dY + I) = b + cx + dy = 0.$$ 

Here, we may assume $c, d \in k$ since $m_B x = m_B y = (0)$. If $b \neq 0$, then $b \notin m_B$. For, if $b \in m_B$, then

$$cz = bx + cx^2 + dxy = 0, \quad dz = by + cxy + dy^2 = 0$$

and $c = 0 = d$. This implies $b = 0$ which is impossible. Thus, $b \notin m_B$ and the element $b + cx + dy$ is a unit which is impossible. Thus, $b = 0$ and $cx + dy = 0$. If $c \neq 0$, then $c^{-1}$ exists and hence $x + (c^{-1}d)y = 0$. By multiplying by $x$ each side, we get

$$0 = x^2 + (c^{-1}d)xy = z.$$ 

This is also impossible and we should have $c = 0$. Finally we have $dy = 0$ and we can show $d = 0$ from the identity $dz = dy^2 = 0$. Thus,

$$b = c = d = 0$$

and this implies $b + cX + dY + I = I$ and $\psi$ is a monomorphism. Since $\dim_k(im(\psi)) = \dim_k(B[x,y])$ and $\dim_k(R) = \dim_k(B) + 2$, the map $\psi$ should be an isomorphism and we can conclude the $k$-algebra $R$ is a $C_2^3$-construction.

Now, in the rest of this paper, we want to prove $C_1$-construction does not imply $C_2^3$-construction for each $i = 1, 2$.

**Corollary 2.7.** $C_1$-construction does not imply $C_2^3$-construction.
Proof. Let \((R, m, k)\) be a Schur algebra of size 4. That is, the element 
\[ r \in R \]
is of the following form:
\[
\begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & c & d \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} + \alpha I_4
\]
for some \(a, b, c, d \in k\). It is known in [1] that the Schur algebra of size 4 is a \(C_1\)-construction. Note that the index of \(m\) is 2 and hence by Theorem 2.6, the \(k\)-algebra \(R\) can’t be a \(C_2^3\)-construction. For, if \(R\) is a \(C_2^3\)-construction, then there exist elements \(x\) and \(y\) in \(m\) whose squares are not zero. But, this is impossible since the index of \(m\) is 2. \(\square\)

Corollary 2.8. \(C_2\)-construction does not imply \(C_2^3\)-construction.

Proof. Let \(R = m \oplus kI_4\) be a maximal commutative \(k\)-subalgebra of \(M_4(k)\) such that each element \(r \in m\) is of the following form:
\[
r = \begin{pmatrix}
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
b & a & 0 & 0 \\
c & b & a & 0 \\
\end{pmatrix}
\]
for some \(a, b, c \in k\). If we let \(B = k[E_{41}]\) and \(s = E_{21} + E_{32} + E_{43}\), then
\begin{enumerate}
\item \(s^3 \in Soc(B) - \{0\}\)
\item \(mBs = \{0\}\)
\item \(dim_k(R) = dim_k(B) + 2\).
\end{enumerate}
This implies \(R\) is a \(C_2\)-construction.

Now, suppose \(R\) is a \(C_2^3\)-construction. Then \(R\) contains a \(k\)-subalgebra \(B\) and an element \(x \in m\) satisfying the following conditions:
\[x^2 \in Soc(B) - \{0\}, \ m_Bx = (0)\]
If we let
\[u = E_{21} + E_{32} + E_{43}, \ v = E_{31} + E_{42}, \ w = E_{41},\]
then \(x = au + bv + cw\) for some \(a, b, c \in k\). Note that
\[x^2 = a^2v + abw \in Soc(B) \subseteq m_B.\]
Since \(m_Bx = (0)\), we obtain \(a = 0\) from the identities
\[a^3E_{41} = x^3 = 0.\]
Therefore, \(x = bv + cw\) and \(x^2 = 0\) which is impossible since \(x^2 \in Soc(B) - \{0\}\). Now, we can conclude that the\(k\)-algebra \(R\) is not a \(C_2^3\)-construction. \(\square\)
Remark 2.9. Example 2.4 and the proof of Corollary 2.8 show that
a $C_2^2$-construction can be constructed from a maximal commutative $k$-
subalgebra that is not a $C_2^2$-construction.

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