ON CLASS ALGEBRAS

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Abstract. Let $F^aG$ be a twisted group algebra. A subalgebra of $F^aG$ generated by all class sums of partition $\mathcal{P}$ of $G$ is called a projective class algebra in $F^aG$ associated with partition $\mathcal{P}$. In this paper we study various partitions of $G$ determined by actions of certain operator groups on $G$ and construct projective class algebras depending on the actions. With regard to projective class algebras, we investigate structures of associated skew group algebras and fixed group algebras.

1. Introduction

Let $G$ be a finite group with identity $1$, $F$ be a field of characteristic $p \geq 0$ and $F^* = F \setminus \{0\}$ be the multiplicative group of $F$ with trivial $G$-action. For a 2-cocycle $\alpha$ in $Z^2(G, F^*)$, let $F^aG$ be the twisted group algebra of $G$ over $F$ with $F$-basis $\{u_g | g \in G\}$ such that $u_1 = 1 = 1_{F^aG}$ and $u_g u_x = \alpha(g, x) u_{gx}$ for all $g, x \in G$.

Let $\mathcal{P} = \{\mathcal{E}_g | g \in G\}$ be a partition of $G$ consisting of certain classes $\mathcal{E}_g$, and let $a^+_g = \sum_{x \in \mathcal{E}_g} u_x$ be the class sum in $F^aG$ containing $g$. An algebra $\Lambda$ over $F$ generated by all class sums is called the projective class algebra associated with partition $\mathcal{P}$. The projective class algebra is a subalgebra of the twisted group algebra $F^aG$, and we may write $\Lambda = \oplus_g F \mathcal{E}^+_g < F^aG$, where the sum is taken over all distinct classes $\mathcal{E}_g$ in $\mathcal{P}$.

Clearly $F^aG$ itself is a projective class algebra in $F^aG$. And the center algebra $Z(F^aG)$ is a projective class algebra in $F^aG$ associated with the partition $\mathcal{P}$ consisting of $\alpha$-regular classes of $G$. If $\alpha = 1$, the algebra generated by $\sum_{x \in \mathcal{E}_g} x$ in the group algebra $FG$ is a class algebra in $FG$ associated with partition $\mathcal{P} = \{\mathcal{E}_g | g \in G\}$. Moreover, for all

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\( \mathcal{E}_g \in \mathcal{P} \) if \( \mathcal{E}_g^{-1} = \{ x^{-1} | x \in \mathcal{E}_g \} = \mathcal{E}_{g^{-1}} \) belongs to \( \mathcal{P} \), \( \mathcal{E}_1 = \{ 1 \} \) and if \( \Lambda = \bigoplus g F_{0g}^\ast \) is a subalgebra of \( FG \) with unit element then \( \Lambda \) is said to be a Schur algebra in \( FG \), that is, a Schur algebra is a subalgebra of \( FG \) associated to a partition of \( G \). Schur algebras were introduced by Schur and Wielandt [10], and studied intensively by Tamaschke [8] and Roesler [7]. Class algebras were investigated by Wigner [9], and recently in [4], [3] and [1].

The purpose of this paper is to study projective class algebra \( \Lambda \) in \( F^\ast G \) associated with partition \( \mathcal{P} \) on \( G \). Let \( \Omega \) be an operator group acting on \( G \) by an action \( \tau \) and \( \mathcal{P} \) be the set of all \( \Omega \)-orbits of \( G \). Then \( \mathcal{P} \) forms a partition of \( G \), in this case, a projective class algebra in \( F^\ast G \) associated with \( \mathcal{P} \) will be called a projective class algebra by \( \tau \). In section 2, we study projective class algebras \( \Lambda \) in \( F^\ast G \) by various actions \( \tau : \Omega \to \text{Aut}(G) \) and investigate structures of the fixed subalgebra \( \Lambda^\tau \) by \( \tau \) and the associated skew group algebra \( \Lambda^+_\tau \). In section 3, by making use of some linear transformations \( \tau \) on \( G \) and on Galois group \( \text{Gal}(E/F) \), where \( E \) is an algebraic closure of \( F \), we will determine projective class algebras \( F^\ast G \# \text{Gal}(E/F) \) and \( F^\beta (G \times \text{Gal}(E/F)) \) for some 2-cocycle \( \beta \in Z^2(G, \text{Gal}(E/F), F^\ast) \).

1. Class algebra associated with operator group

In this paper we always assume that \( G \) is a finite group, \( F \) is a field and \( F^\ast G \) is a twisted group algebra of \( G \) over \( F \) with a 2-cocycle \( \alpha \in Z^2(G, F^\ast) \) such that \( u_g u_x = \alpha(g, x) u_{gx} \) for \( g, x \in G \). And we denote by \( \text{Aut}(G) \) the automorphism group of \( G \). When \( \Omega \) is any group acting on a group \( X \) under \( \tau : \Omega \to \text{Aut}(X) \), the semidirect product group \( X \rtimes \Omega \) afforded by \( \tau \) satisfies the multiplication rule \( (x_1, \omega_1)(x_2, \omega_2) = (x_1 \tau(\omega_1)x_2, \omega_1\omega_2) \) for \( x_i \in X, \omega_i \in \Omega \) \( (i = 1, 2) \). We denote by \( \text{St}_\tau(x) \) the stabilizer of \( x \in X \), i.e., \( \text{St}_\tau(x) = \{ \omega \in \Omega | \tau(\omega)(x) = x \} \).

Let \( \lambda : \Omega \to \text{Aut}(G) \) be an action of an operator group \( \Omega \) on \( G \), and let \( G^\gamma = \{ g \in G | \gamma(\omega)g = g \} \) for all \( \omega \in \Omega \) denote the fixed subgroup of \( G \) by \( \gamma \). For any \( \omega \in \Omega \), assume that each \( \gamma(\omega) \) extends to an algebra isomorphism \( \hat{\gamma}(\omega) \) of \( F^\ast G \) by \( \hat{\gamma}(\omega)(\sum g \in G r_g u_g) = \sum g \in G r_g u_{\gamma(\omega)g} \), where \( r_g \in F^\ast \). Then for the \( \hat{\gamma} : \Omega \to \text{Aut}(F^\ast G) \), we denote by \( (F^\ast G)^\gamma \) the fixed subalgebra \( \{ a \in F^\ast G | \gamma(\omega)a = a \} \) for all \( \omega \in \Omega \) of \( F^\ast G \) by \( \gamma \).

Theorem 1. Let \( \gamma : \Omega \to \text{Aut}(G) \) be an action of an operator group \( \Omega \), and assume \( \gamma(\omega) \) extends to an algebra isomorphism \( \hat{\gamma}(\omega) \) of \( F^\ast G \)
for any $\omega \in \Omega$. Then $\alpha(g,x) = \alpha(\tau(\omega)g, \tau(\omega)x)$ for any $g, x \in G$, and $(F^\alpha G)^\dagger = F^\alpha G^\dagger$, where the same symbol $\alpha$ is used for the restriction to $G^\dagger$. Thus the algebra $(F^\alpha G)^\dagger$ is a projective class algebra in $F^\alpha G$ by $\tau$.

**Proof.** Let $g, x \in G$ and $\omega \in \Omega$. Then $\tau(\omega)$ satisfies

$$\tau(\omega)u_g \tau(\omega)u_x = u_{\tau(\omega)g}u_{\tau(\omega)x} = \alpha\left(\tau(\omega)g, \tau(\omega)x\right) u_{\tau(\omega)g\tau(\omega)x}$$

while

$$\tau(\omega)(u_g u_x) = \alpha(g, x)u_{\tau(\omega)gx} = \alpha(g, x)u_{\tau(\omega)g\tau(\omega)x},$$

hence we have $\alpha(g, x) = \alpha\left(\tau(\omega)g, \tau(\omega)x\right)$.

Choose any $u_g \in (F^\alpha G)^\dagger$. Then $u_g = \tau(\omega)u_g = u_{\tau(\omega)g}$ for all $\omega \in \Omega$, and $1 = u_{\tau(\omega)g}u_{g^{-1}g^{-1}} = \alpha^{-1}(g, g^{-1})\alpha(\tau(\omega)g, g^{-1})u_{\tau(\omega)g^{-1}}$. Thus

$$g = \tau(\omega)g \quad \text{and} \quad \alpha(g, g^{-1}) = \alpha(\tau(\omega)g, g^{-1}).$$

But the former yields the latter, hence we have $g \in G^\dagger$ and $u_g \in F^\alpha (G^\dagger)$. Conversely, if $u_g \in F^\alpha (G^\dagger)$ then $g \in G^\dagger$ and $\tau(\omega)u_g = u_{\tau(\omega)g} = u_g$ for all $\omega \in \Omega$, hence $u_g \in (F^\alpha G)^\dagger$. This proves $(F^\alpha G)^\dagger = F^\alpha (G^\dagger)$.

Let $P = \{\mathcal{O}_\tau(g)\mid g \in G\}$ be the partition of $G$ consisting of orbits $\mathcal{O}_\tau(g) = \{\tau(\omega)g\mid \omega \in \Omega\}$ and let $o^+_g = \sum_{x \in \mathcal{O}_\tau(g)} u_x$ be the class sum. Consider the projective class algebra $\Lambda = \oplus_y F o^+_y$ associated with $P$, where the sum is taken over distinct classes in $G$. We now show that $\Lambda = (F^\alpha G)^\dagger$. Obviously a generator $o^+_y$ of $\Lambda$ is contained in $F^\alpha G$. And

$$\tau(\omega)(o^+_y) = \tau(\omega)\left(\sum_{x \in \mathcal{O}_\tau(g)} u_x\right) = \sum_{x \in \mathcal{O}_\tau(g)} u_{\tau(\omega)x} = \sum_{y \in \mathcal{O}_\tau(g)} u_y = o^+_y$$

for any $\omega \in \Omega$, where the third equality follows from that if we let $\tau(\omega)x = y$ then since $x \in \mathcal{O}_\tau(g)$ we can write $x = \tau(\omega')g$ for some $\omega' \in \Omega$ hence $y = \tau(\omega)x = \tau(\omega')g$ and $y \in \mathcal{O}_\tau(g)$. Thus $o^+_y \in (F^\alpha G)^\dagger$, and $\Lambda$ is contained in $(F^\alpha G)^\dagger$.

On the other hand if $a$ is any element in $(F^\alpha G)^\dagger$ then it forms $a = \sum_{k \in F, g \in G} k_i u_g$ and

$$a = \tau(\omega)\left(\sum_{k \in F} k_i u_{\tau(\omega)g}\right) = \sum_{k \in F, x \in \mathcal{O}_\tau(g)} k_i u_x = \sum_{k \in F} k_i o^+_g \in \Lambda$$

for all $\omega \in \Omega$. Therefore $(F^\alpha G)^\dagger = \Lambda$; this completes the proof. \(\square\)

If $\alpha = 1$ then each $\tau(\omega)$ always extends to an isomorphism of $FG$. The fixed subalgebra $(FG)^\dagger$ was studied in [3], and clearly $(FG)^\dagger = F(G^\dagger)$. 

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Let $\Lambda$ be any $F$-algebra and $\chi : \Omega \rightarrow \text{Aut}_F(\Lambda)$ be a homomorphism of groups. Let $\Lambda \star_{\chi} \Omega$ be the associated skew group algebra generated by $\{b_\omega \mid \omega \in \Omega\}$ with

\[(\lambda_1 b_\omega \cdot \lambda_2 b_\mu = \lambda_1 \chi(\omega) \lambda_2 b_{\omega \mu}) \quad \text{for} \quad \omega, \mu \in \Omega, \quad \lambda_i \in \Lambda, \quad i = 1, 2.\]

If $\Lambda = FG$ then $\chi : \Omega \rightarrow \text{Aut}_F(FG)$ can be regarded as the extended map from $\chi : \Omega \rightarrow G$, and it is easy to see $FG \star_{\chi} \Omega \cong F(G \times \chi \Omega)$, where $G \times \chi \Omega$ is the semidirect product group afforded by $\chi$, hence $FG \star_{\chi} \Omega$ is also a class algebra in $F(G \times \chi \Omega)$ \cite[(2.12)]{3}. Similarly when $\Lambda = F^aG$, we will prove that $F^aG \star_{\chi} \Omega$ is isomorphic to a certain twisted group algebra of $G \chi \Omega$ over $F$ by finding suitable 2-cocycle on $G \chi \Omega$.

**Theorem 2.** Let $\hat{\tau} : \Omega \rightarrow \text{Aut}_F(F^aG)$ be a homomorphism extended from the action $\tau : \Omega \rightarrow \text{Aut}(G)$. Let $\Lambda$ be a projective class algebra in $F^aG$ by $\tau$. Then there is a 2-cocycle $\beta$ on the semidirect product group $G \times_{\tau} \Omega$ such that the associated skew group algebra $\Lambda \star_{\tau} \Omega$ is a projective class algebra in $F^a(G \times_{\tau} \Omega)$.

**Proof.** For any $g, x \in G$ and $\omega, \mu \in \Omega$, the operation on $G \times_{\tau} \Omega$ is defined by $(g, \omega)(x, \mu) = (g \tau(\omega)x, \omega \mu)$, so $(g, \omega)^{-1} = (\tau(\omega)^{-1}g^{-1}, \omega^{-1})$ and $1_{G \times \Omega} = (1_G, 1_\Omega)$. Define a map

\[\beta : (G \times_{\tau} \Omega) \times (G \times_{\tau} \Omega) \rightarrow F^a \quad \text{by} \quad \beta(g, \omega, (x, \mu)) = \alpha(g, \tau(\omega)x)\]

Then due to Theorem 1, it can be seen that $\beta$ is a 2-cocycle over $G \times_{\tau} \Omega$:

\[
\begin{align*}
\beta((g, \omega), (x, \mu)(y, \nu)) &\cdot \beta((x, \mu), (y, \nu)) \\
&= \alpha(g, \tau(\omega)(x \tau(\mu)y)) \cdot \alpha(x, \tau(\mu)y) \\
&= \alpha(g, \tau(\omega)x \tau(\omega \mu)y) \alpha(x, \tau(\mu)y) \alpha(\tau(\omega)x, \tau(\omega \mu)y) \\
&\quad \cdot \alpha(\tau(\omega)x, \tau(\omega \mu)y)^{-1} \\
&= \alpha(g \tau(\omega)x, \tau(\omega \mu)y) \cdot \alpha(g, \tau(\omega)x) \cdot \alpha(x, \tau(\mu)y) \cdot \alpha(x, \tau(\mu)y)^{-1} \\
&= \beta((g, \omega)(x, \mu), (y, \nu)) \cdot \beta((g, \omega), (x, \mu)) \quad \text{for all} \quad y \in G, \nu \in \Omega.
\end{align*}
\]

If we define a relation $\sim$ on $G \times_{\tau} \Omega$ by $(g, \omega) \sim (x, \mu)$ whenever $x \in E_g$ and $\omega = \mu \in \Omega$, then $\sim$ is an equivalent relation on $G \times_{\tau} \Omega$. Let $C_{(g, \omega)}$ be the equivalence class containing $(g, \omega)$ and let $Q = \{C_{(g, \omega)} \mid (g, \omega) \in G \times_{\tau} \Omega\}$.
Consider the twisted group algebra \( F^\beta(G \times \Omega) \) of \( G \times \Omega \) having \( F \)-basis \( \{ z_{(g, \omega)} | g \in G, \omega \in \Omega \} \). Then it satisfies the multiplication rule that
\[
 z_{(g, \omega)} z_{(x, \mu)} = \beta \left( \left( g, \omega \right), \left( x, \mu \right) \right) z_{(g, \omega)}(x, \mu) = \alpha \left( g, \tau(\omega) x \right) z_{(g \tau(\omega), \omega \mu)}.
\]
Let \( s^+_{(g, \omega)} \) be the class sum (with respect to \( Q \)) in \( F^\beta(G \times \Omega) \). Then the algebra
\[
 \Gamma = \oplus_{(g, \omega) \in G \times \Omega} F s^+_{(g, \omega)} \quad \text{with} \quad s^+_{(g, \omega)} = \sum_{(x, \mu) \in C_{(g, \omega)}} z_{(x, \mu)}
\]
is a projective class algebra in \( F^\beta(G \times \Omega) \) associated with \( Q \). It suffices to show that the associated skew group ring \( \Lambda \ast_\tau \Omega \) is isomorphic to \( \Gamma \).

Since \( \Lambda \) is a projective class algebra in \( F^\alpha G \) by \( \tau \), we can write \( \Lambda = \oplus_g F o^+_g \), where \( \mathcal{P} = \{ E_g | g \in G \} \) is the partition of \( G \) afforded by \( \tau \) and \( o^+_g = \sum_{x \in E_g} u_x \) is the class sum in \( F^\alpha G \). Thus the skew group algebra \( \Lambda \ast_\tau \Omega = (\oplus_g F o^+_g) \ast_\tau \Omega \) is generated by \( \{ b_\omega | \omega \in \Omega \} \) with multiplication
\[
o^+_g b_\omega \cdot o^+_x b_\mu = o^+_g \tau(\omega) o^+_x \cdot b_{\omega \mu} \quad \text{for} \quad \omega, \mu \in \Omega \quad \text{and} \quad o^+_g, o^+_x \in \Lambda.
\]
Define a map
\[
 \theta : \Lambda \ast_\tau \Omega = (\oplus_g F o^+_g) \ast_\tau \Omega \rightarrow \Gamma = \oplus_{(g, \omega)} F s^+_{(g, \omega)}
\]
by \( \theta(o^+_g b_\omega) = s^+_{(g, \omega)} \). Clearly \( \theta \) is a bijection and satisfies
\[
 \sum_{(x, \mu) \in C_{(g, \omega)}} z_{(x, \mu)} = s^+_{(g, \omega)} = \theta(o^+_g b_\omega) = \theta(\sum_{x \in E_g} u_x b_\omega)
\]
for any \( g, y \in G \) and \( \omega, \mu \in \Omega \), because \( C_{(g, \omega)} = E_g \times \{ \omega \} \). Hence we have
\[
\theta \left( o^+_g b_\omega \right) \cdot \theta \left( o^+_y b_\mu \right) = s^+_{(g, \omega)} s^+_{(y, \mu)} = \sum_{(x, \nu) \in C_{(g, \omega)}} z_{(x, \nu)} \sum_{(k, \eta) \in C_{(y, \mu)}} z_{(k, \eta)}
\]
\[
= \sum_{(x, \nu) \in C_{(g, \omega)}} \sum_{(k, \eta) \in C_{(y, \mu)}} \alpha \left( x, \tau(\omega) k \right) z_{(x \tau(\omega) k, \omega \mu)}
\]
\[
= \theta \left( \sum_{x \in E_g} \sum_{k \in E_y} \alpha \left( x, \tau(\omega) k \right) u_{x \tau(\omega) k} b_{\omega \mu} \right) = \theta \left( \sum_{x \in E_g} u_x \sum_{k \in E_y} u_{\tau(\omega) k} \cdot b_{\omega \mu} \right)
\]
\[
= \theta \left( \sum_{x \in E_g} u_x \cdot \tau(\omega) \sum_{k \in E_y} u_k \cdot b_{\omega \mu} \right) = \theta \left( o^+_g \tau(\omega) o^+_y \cdot b_{\omega \mu} \right)
\]
\[
= \theta \left( o^+_g b_\omega \cdot o^+_y b_\mu \right),
\]
thus \( \Lambda \ast_\tau \Omega \cong \Gamma \) as is desired. \qed
In particular if $\Lambda = F^0 G$ then the map $\theta : F^0 G \ast_* \Omega \to F^3 (G \times_\tau \Omega)$ defined by $\theta(u_g b_\omega) = z_{(g, \omega)}$ is a bijective homomorphism because

$$\theta(u_g b_\omega \cdot u_x b_\mu) = \alpha(g, \tau(\omega)x)\theta(u_{g \tau(\omega)x} b_{\omega \mu})$$

$$= \alpha(g, \tau(\omega)x) \cdot z_{(g \tau(\omega)x, \omega \mu)} = \beta((g, \omega), (x, \mu)) \cdot z_{(g, \omega)(x, \mu)}$$

$$= z_{(g, \omega)} z_{(x, \mu)} = \theta(u_g b_\omega) \theta(u_x b_\mu).$$

Hence the next corollary follows immediately.

**COROLLARY 3.** Let the context be same as in Theorem 2 and $F^0 G$ be a twisted group algebra. Then there is a 2-cocycle $\beta$ on $G \times_\tau \Omega$ satisfying $F^0 G \ast_* \Omega \cong F^3 (G \times_\tau \Omega)$, hence $F^0 G \ast_* \Omega$ is a projective class algebra.

Obviously the dimension of $F^3 (G \times_\tau \Omega)$ is equal to $|\Omega|$ times the dimension of $F^0 G$, for $\dim F^3 (G \times_\tau \Omega) = |G \times_\tau \Omega| = |G||\Omega| = |\Omega| \dim F^0 G$. In next theorem we will observe dimensions of projective class algebras in $F^3 (G \times_\tau \Omega)$ and $F^0 G$.

**THEOREM 4.** Assume that a finite group $\Omega$ acts on $G$ by $\tau : \Omega \to \text{Aut}(G)$. Then there is a homomorphism $\theta : \Omega \to \text{Aut}(G \times_\tau \Omega)$ such that $\text{St}_\theta(g, \omega)$ equals $\text{St}_\tau(g)$ for all $(g, \omega) \in G \times_\tau \Omega$. And the dimension of projective class algebra by $\theta$ in a twisted group algebra of $G \times_\tau \Omega$ is equal to $|\Omega|$ times the dimension of projective class algebra by $\tau$ in a twisted group algebra of $G$.

**Proof.** Let $(g, \omega)$ be any element in $G \times_\tau \Omega$. If we define a map

$$\theta : \Omega \to \text{Aut}(G \times_\tau \Omega) \text{ by } \theta(\nu)(g, \omega) = (\tau(\nu)g, \omega)$$

for any $\nu \in \Omega$, then $\theta$ is a homomorphism because

$$\theta(\nu_1)\theta(\nu_2)(g, \omega) = (\tau(\nu_1)\tau(\nu_2)g, \omega) = (\tau(\nu_1 \nu_2)g, \omega) = \theta(\nu_1 \nu_2)(g, \omega)$$

for all $\nu_1 \in \Omega$. Thus it is easy to see $\text{St}_\theta(g, \omega) = \text{St}_\tau(g) < \Omega$, because $\nu \in \text{St}_\theta(g, \omega)$ if and only if $(g, \omega) = \theta(\nu)(g, \omega) = (\tau(\nu)g, \omega)$, or equivalently, $g = \tau(\nu)g$, if and only if $\nu \in \text{St}_\tau(g)$.

We now define a relation $\sim$ on $G \times_\tau \Omega$ by, for $(g, \omega)$ and $(x, \mu)$ in $G \times_\tau \Omega$, $(g, \omega) \sim (x, \mu)$ if there is $\nu \in \Omega$ such that $x = \tau(\nu)g$ and $\mu = \omega \in \Omega$. Clearly $\sim$ form equivalence classes $E$, and we denote by $s^\dagger_{(g, \omega)}$ the class sum of $E$ in $F^3 (G \times_\tau \Omega)$ for some 2-cocycle $\beta \in Z^2 (G \times_\tau \Omega, F^* \tau)$. Then $A = \oplus_{(g, \omega)} F s^\dagger_{(g, \omega)}$ (the sum is taken over distinct classes $E$ in $G \times_\tau \Omega$) is a projective class algebra by $\theta$ in $F^3 (G \times_\tau \Omega)$.
Since the equivalence class $E$ containing $(g, \omega)$ corresponds to the orbit $O_\theta(g, \omega)$ under $\theta$, for $O_\theta(g, \omega) = \{(\nu(g, \omega) | \nu \in \Omega) = \{((\tau(\nu)g, \omega) | \nu \in \Omega\} = \{(x, \mu)| (g, \omega) \sim (x, \mu)\}$, the dimension of $A$ is equal to the number of distinct orbits under $\theta$.

We recall a fact that if $Y$ is any group acting on a finite set $X$ by an operation $\tau : Y \to \text{Perm}(X)$ then the number of $Y$-orbits in $X$ equals $\sum_{x \in X} \frac{1}{|Y|} \cdot |\text{St}_\tau(x)|$ if $Y$ is finite.

Therefore the dimension of the projective class algebra $A$ by $\theta$ in $F^0(G \times, \Omega)$ is

$$\frac{1}{|\Omega|} \sum_{(g, \omega) \in G \times, \Omega} |\text{St}_\theta(g, \omega)| = \frac{1}{|\Omega|} \sum_{(g, \omega) \in G \times, \Omega} |\text{St}_\tau(g)| = |\Omega| \frac{1}{|\Omega|} \sum_{g \in G} |\text{St}_\tau(g)|,$$

which is indeed equal to $|\Omega|$ times the number of orbits under $\tau$, i.e., $|\Omega|$ times the dimension of projective class algebra by $\tau$. □

2. Class algebras associated with linear transformation

We let $F$ denote a field of any characteristic $p \geq 0$ and $E$ denote an algebraic closure of $F$ with Galois group $G = \text{Gal}(E/F)$. Let $\alpha \in Z^2(G, F^*)$. Then $E \otimes F^\alpha G = E^\alpha G$ is a twisted group algebra of $G$ over $E$ with the same basis $\{u_g | g \in G\}$ of $F^\alpha G$. In this section, we will discuss $E$-linear transformations on $G$ and $G$, and study projective class algebras of $E^\alpha G$ associated with the actions.

Let $n$ be a positive integer divisible by $\exp(G)$. For each $\sigma \in G = \text{Gal}(E/F)$, let $m(\sigma)$ be a positive integer satisfying

$$m(\sigma) \equiv 1 \pmod{n_p} \quad \text{and} \quad \varepsilon^\sigma_{n_p'} = \varepsilon^{m(\sigma)}_{n_p'}$$

where $\varepsilon_j$ ($j > 0$) denotes a primitive $j$-th root of unity in $E$, and $n_p$ and $n_p'$ are $p$- and $p'$-parts of $n$ such that $n = n_p n_p'$. If $p = 0$ or $p' = 0$ does not divide $|G|$ then $n_p = 1$ and $n_p' = n$. For each $g \in G$, choose $v(g) \in E$ such that

$$v(g)^n = t(g) \quad \text{where} \quad t(g) = \prod_{i=1}^{n-1} \alpha(g^i, g) \in F.$$
There are monomial transformations $K_{F^oG}$ and $S_{F^oG}$ of $E^oG$ (refer to [6]):

$$K_{F^oG} : G \to \text{Aut}(E^oG) \text{ by } u_g K_{F^oG}(x) = u_x^{-1} u_g u_x,$$

$$S_{F^oG} : G \to \text{Aut}(E^oG) \text{ by } u_g S_{F^oG}(\sigma) = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_g^{m(\sigma^{-1})}.$$

And also there are permutations $k_G$ and $s_G$ of $G$ such that:

$$k_G : G \to \text{Aut}(G) \text{ by } g k_G(x) = g^x;$$

$$s_G : G \to \text{Aut} \text{ by } g s_G(\sigma) = g^{m(\sigma^{-1})},$$

here we write $g^x = x^{-1} gx$. Furthermore, over the abstract direct product group $G \times G$ with multiplication $(\sigma, x) (\tau, y) = (\sigma \tau, xy)$, it is defined by $d_G = s_G \times k_G$ and $D_{F^oG} = S_{F^oG} \times K_{F^oG}$, hence

$$gd_G(\sigma, x) = (g^{m(\sigma^{-1})})^x,$$

$$u_g D_{F^oG}(\sigma, x) = v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} u_x^{-1} u_g^{m(\sigma^{-1})} u_x.$$

**Lemma 5.** The $K_{F^oG}$ is a (right) $G$-action on $E^oG$, while $k_G$ is a $G$-action on $G$. And $d_G$ [resp. $s_G$] is a $G \times G$ [resp. $G$]-action on $G$ if $G$ is abelian.

**Proof.** Let $g, x$ and $y$ be elements in $G$. Then the next two identities

$$u_g K_{F^oG}(x) \cdot u_y K_{F^oG}(x) = u_x^{-1} u_g u_x u_x^{-1} u_y u_x = \alpha(g, y) u_x^{-1} u_y u_x$$

$$= \alpha(g, y) u_{gy} K_{F^oG}(x),$$

$$u_g (K_{F^oG}(x) K_{F^oG}(y)) = u_x^{-1} u_g u_x K_{F^oG}(y) = u_y^{-1} u_g u_x u_y$$

$$= (u_x u_y)^{-1} u_g u_x u_y = \alpha^{-1}(x, y) u_{xy}^{-1} u_g \alpha(x, y) u_{xy} = u_g K_{F^oG}(xy)$$

imply that $K_{F^oG} : G \to \text{Aut}(E^oG)$ is a homomorphism. Also for any $\sigma, \tau \in G$, since $m(\sigma \tau) \equiv m(\sigma) m(\tau) \equiv m(\sigma \tau) \pmod{n}$, we have

$$g(d_G(\sigma, x) d_G(\tau, y)) = (g^{m(\tau^{-1})})^x d_G(\tau, y) = (g^{(\sigma^{-1})m(\tau^{-1})})^{xy} = gd_G(\sigma, xy)$$

and

$$gd_G(\sigma, x) \cdot y d_G(\sigma, x) = (g^{m(\sigma^{-1})})^{xy} d_G(\sigma, x) = ((gy)^{m(\sigma^{-1})})^x = gy d_G(\sigma, x),$$

hence $d_G$ is a homomorphism. The rest part of the proof is clear. \(\square\)

An element $g \in G$ is said to be $\alpha$-regular if $\alpha(g, x) = \alpha(x, g)$ for all $x$ in the centralizer $C_G(g)$ of $g$. Hence $g$ is $\alpha$-regular if and only if $u_g K_{F^oG}(x) = u_g$ for all $x \in G$ with $gx = xg$. A conjugacy class containing an $\alpha$-regular element is called an $\alpha$-regular class. A 2-cocycle $\alpha \in Z^2(G, F^*)$ is called normal if $\alpha$ satisfies $\alpha(g, x) = \alpha(x, x^{-1}gx)$ for any $\alpha$-regular $g \in G$ and any $x \in G$, or equivalently, $u_x^{-1} u_g u_x = u_x^{-1} gx$. 

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in terms of basis of algebra $F^\alpha G$. If, moreover $u_{x^{-1}} = u_{x^{-1}}$ for all $x$ then $\alpha$ is called standard. Thus $\alpha$ is standard if $\alpha(x, x^{-1}) = 1$ and $\alpha(x^{-1} g, x)\alpha(x^{-1}, g) = 1$ for any $\alpha$-regular element $g \in G$ and any $x \in G$ (refer to [5, p.97]).

We will study structures of the associated skew group algebras by actions $K_{F^\alpha G}$ and $s_G$ in next theorems. From now on, we denote the group generated by all $\alpha$-regular elements in $G$ by $G_0$.

**Theorem 6.** The algebra $E^\alpha G_0 *_{K_{F^\alpha G}} G$ is isomorphic to the twisted group algebra $E^\beta(G_0 \times_{K_0} G)$ for some $\beta \in Z^2(G \times_{kG} G, E^*)$. In particular if $\alpha$ is normal then $F^\alpha G_0 *_{K_{F^\alpha G}} G \cong F^\beta(G_0 \times_{kG} G)$ for $\beta \in Z^2(G \times_{kG} G, F^*)$.

**Proof.** The multiplication on $G \times_{kG} G$ satisfies

$$(g_1, x_1)(g_2, x_2) = (g_1, g_2 k_G(x_1), x_1 x_2) = (g_1, g_2^{x_1}, x_1 x_2)$$

for $g_i, x_i \in G$ ($i = 1, 2$). And from $\alpha \in Z^2(G, F^*)$, we have a 2-cocycle $\beta$ on $G \times_{kG} G$ satisfying $\beta((g_1, x_1), (g_2, x_2)) = \alpha(g_1, g_2 k_G(x_1)) = \alpha(g_1, g_2^{x_1})$ as in Theorem 2. Thus if we let $z_{(g, x)}$ be an $F$-basis of $F^\beta(G_0 \times_{kG} G)$, where $g \in G_0$, $x \in G$, then

$$z_{(g_1, x_1)} z_{(g_2, x_2)} = \beta((g_1, x_1), (g_2, x_2)) z_{(g_1 x_1)} z_{(g_2, x_2)}$$

$$= \alpha(g_1, g_2^{x_1}) z_{(g_1, g_2^{x_1}, x_1 x_2)}.$$  

On the other hand, let $\{b_x \mid x \in G\}$ be a basis of $F^\alpha G_0 *_{K_{F^\alpha G}} G$ afforded by the homomorphism $K_{F^\alpha G} : G \to \text{Aut}(E^\alpha G)$. Then it satisfies

$$u_{g_1} b_{x_1} \cdot u_{g_2} b_{x_2} = u_{g_1} \cdot u_{g_2} K_{F^\alpha G}(x_1) b_{x_1 x_2} = u_{g_1} u_{x_1} u_{g_2} b_{x_1 x_2}$$

(see (1)) where $\{u_g \mid g \in G_0\}$ is the $F$-basis of $F^\alpha G$. For any $g \in G_0$ and $x \in G$, we define a map

$$\bar{\theta} : F^\alpha G_0 *_{K_{F^\alpha G}} G \to F^\beta(G_0 \times_{kG} G) \quad \text{by} \quad \bar{\theta}(u_g b_{x}) = z_{(g, x)}.$$  

When $\alpha$ is normal then since $u_{x_1}^{-1} u_{g_j} u_{x_i} = u_{x_i}^{-1} g_j x_i$ for $g_j \in G_0$, we have

$$\bar{\theta}(u_{g_1} b_{x_1} \cdot u_{g_2} b_{x_2}) = \bar{\theta}(u_{g_1} u_{x_1}^{-1} u_{g_2} u_{x_1} b_{x_1} b_{x_2}) = \bar{\theta}(u_{g_1} u_{x_1}^{-1} g_j x_2 b_{x_1} b_{x_2})$$

$$= \alpha(g_1, g_2^{x_1}) z_{(g_1, g_2^{x_1}, x_1 x_2)} = z_{(g_1 x_1)} z_{(g_2 x_2)} = \bar{\theta}(u_{g_1} b_{x_1}) \bar{\theta}(u_{g_2} b_{x_2})$$

for any $u_{g_i} b_{x_i} \in F^\alpha G_0 *_{K_{F^\alpha G}} G$ ($i = 1, 2$). Hence $\bar{\theta}$ gives the isomorphism $F^\alpha G_0 *_{K_{F^\alpha G}} G \cong F^\beta(G_0 \times_{kG} G)$.

When $\alpha$ is not normal, there is a normal cocycle $\gamma \in Z^2(G, F^*)$ which is cohomologous to $\alpha$ (refer to [5, (2.6.2)]). Moreover since $E$ is
an algebraic closure, we may assume $\gamma \in Z^2(G, E^*)$ is standard (refer to [5, (2.6.4)]), and $\alpha(x, y) = f(x)f(y)f^{-1}(xy)\gamma(x, y)$ for a map $f : G \to E^*$ with $f(1) = 1$ for $x, y \in G$. Since every $\alpha$-regular element is $\gamma$-regular, we shall use the same notation $G_0$ for both sets of $\alpha$-regular elements and of $\gamma$-regular elements.

Define a 2-cocycle $\beta \in Z^2(G, E^*)$ with regard to $\gamma$ as before, that is, with $g_i \in G_0$, $x_i \in E^*$, $\beta\left( (g_1, x_1), (g_2, x_2) \right) = \gamma(g_1, g_2^{x_1})$ (see (3)), and using the same notations $\{ z_{(g, x)} \}$ for $E$-basis of $E^2(G \times k_{E^*}, G)$, it satisfies $z_{(g_1, x_1)}z_{(g_2, x_2)} = \gamma(g_1, g_2^{x_1}) z_{(g_1 g_2^{x_1}, x_1 x_2)}$. Thus the associated skew group algebra $E^* G_0 *_{K_{F^*}} G$ by $K_{F^*} : G \to \text{Aut}(E^* G)$ is isomorphic to

$E^* G_0 *_{K_{F^*}} G \cong E^2(G_0 \times k_{E^*}, G)$

(see (4)). We now suffice to show that $E^* G_0 *_{K_{F^*}} G \cong E^* G_0 *_{K_{F^*}} G$.

With $E$-basis $\{ u_x | x \in G \}$ and $\{ v_x | x \in G \}$ of $E^* G$ and $E^* G$ respectively, there is an $E$-algebra isomorphism $\theta : E^* G \to E^* G$ such that $\theta(u_x) = f(x) v_x$ for all $x \in G$.

Now let $\{ c_x | x \in G \}$ be a basis of $E^* G_0 *_{K_{F^*}} G$. Then $c_x$ satisfies

$v_{g_1} c_x \cdot v_{g_2} c_x = v_{g_1} c_1 v_{g_2} v_{g_2} c_x = v_{g_1} c_1 v_{g_2} c_x = \gamma(g_1, g_2^{x_1}) v_{g_2} c_x$,

because $\gamma$ is standard. We will define a map $\bar{\theta}$ induced from $\theta$ by

$\bar{\theta} : E^* G_0 *_{K_{F^*}} G \to E^* G_0 *_{K_{F^*}} G$, \quad $\bar{\theta}(u_g b_x) = \theta(u_g) c_x = f(g) v_g c_x$

for $g \in G_0$ and $x \in G$. We first note that, for $g_i \in G_0$ and any $x_i \in G$,

$u_{g_1} b_{x_1} u_{g_2} b_{x_2} = \alpha^{-1}(x_1, x_1) \alpha(x_1^{-1}, g_2) \alpha(x_1^{-1}, g_2, x_1) \alpha(g_1, g_2^{x_1}) u_{g_1 g_2^{x_1}} b_{x_1} b_{x_2}$

$= f^{-1}(x_1) f^{-1}(x_1^{-1}) f(1) \gamma^{-1}(x_1, x_1^{-1}) f(x_1^{-1}) f^{-1}(x_1^{-1}) g_2 f^{-1}(x_1^{-1}) g_2$

$\cdot \gamma(x_1^{-1}, g_2) f(x_1) f^{-1}(g_2^{x_1}) \gamma(x_1^{-1}, g_2, x_1)$

$\cdot f(g_1) f(g_2^{x_1}) f^{-1}(g_1 g_2^{x_1}) \gamma(g_1, g_2^{x_1}) u_{g_1 g_2^{x_1}} b_{x_1} b_{x_2}$

$= f(g_1) f(g_2^{x_1}) f^{-1}(g_1 g_2^{x_1}) \gamma(x_1, x_1^{-1}) \gamma(x_1^{-1}, g_2) \gamma(x_1^{-1}, g_2, x_1)$

$\cdot \gamma(g_1, g_2^{x_1}) u_{g_1 g_2^{x_1}} b_{x_1} b_{x_2}$

$= f(g_1) f(g_2) f^{-1}(g_1 g_2^{x_1}) \gamma(g_1, g_2^{x_1}) u_{g_1 g_2^{x_1}} b_{x_1} b_{x_2}$,

where the last equality is due to $\gamma$ standard. Then it follows that

$\bar{\theta}(u_{g_1} b_{x_1} u_{g_2} b_{x_2}) = f(g_1) f(g_2) f^{-1}(g_1 g_2^{x_1}) \gamma(g_1, g_2^{x_1}) \bar{\theta}(u_{g_1 g_2^{x_1}} b_{x_1} b_{x_2})$

$= f(g_1) f(g_2) \gamma(g_1, g_2^{x_1}) v_{g_1 g_2^{x_1}} b_{x_1} b_{x_2} = f(g_1) f(g_2) v_{g_1} b_{x_1} v_{g_2} b_{x_2}$

$= f(g_1) v_{g_1} b_{x_1} f(g_2) v_{g_2} b_{x_2} = \bar{\theta}(u_{g_1} b_{x_1}) \bar{\theta}(u_{g_2} b_{x_2})$. 

therefore we have \( E^0 \cdot G \ast K_{F \circ G} \cdot G \cong E^0 \cdot G \ast K_{F \circ G} \cdot G \cong E^0 \cdot (G \times sG, G) \). 

**Theorem 7.** If \( G \) is abelian then \( F^\alpha \cdot G \ast sG \cdot G \) is isomorphic to \( F^\beta (G \times xG, G) \) for some \( \beta \in Z^2(G \times sG, G, F^\ast) \).

*Proof.* For the algebra \( F^\alpha \cdot G \ast sG \cdot G \), we may consider that \( sG(\sigma) \) extends to an algebra isomorphism of \( F^\alpha \cdot G \) for each \( \sigma \in G \). Hence for all \( u_g \in F^\alpha \cdot G \) and \( \sigma \in G \), we have \( u_g sG(\sigma) = u_g^{m(\sigma)} \) thus

\[
 u_g sG(\sigma) \cdot u_x sG(\sigma) = (u_g u_x) sG(\sigma) = \alpha(g, x) u_{(gx)^m(\sigma)}
\]

\[
 = \alpha(g, x) \alpha^{-1}(g^{m(\sigma)}, x^{m(\sigma)}) u_g sG(\sigma) \cdot u_x sG(\sigma),
\]

which shows \( \alpha(g, x) = \alpha(g^{m(\sigma)}, x^{m(\sigma)}) = \alpha(g sG(\sigma), x sG(\sigma)) \) (see Theorem 2). Let \( \{ b_{\sigma} \mid \sigma \in G \} \) be a basis of \( F^\alpha \cdot G \ast sG \cdot G \) with multiplication

\[
 u_g b_{\sigma} u_x b_\tau = u_g u_x sG(\sigma) b_{\sigma} \tau = \alpha(g, x^{m(\sigma)}) u_{(gx)^m(\sigma)} b_{\sigma} \tau
\]

for any \( g, x \in G, \sigma, \tau \in G \).

Since multiplication on \( G \times sG \) satisfies \( (g, \sigma)(x, \tau) = (g \cdot x sG(\sigma), \sigma \tau) \)

\[
 = (g x^{m(\sigma)}, \sigma \tau),
\]

if we define a map

\[
 \beta : (G \times sG) \times (G \times sG) \to F^\ast \text{ by } \beta((g, \sigma), (x, \tau)) = \alpha(g, x^{m(\sigma)}),
\]

then it is routine to see that \( \beta \) is a 2-cocycle in \( Z^2(G \times sG, G, F^\ast) \). Let \( z_{(g, \sigma)} \) be the basis of \( F^\beta (G \times xG, G) \). Then it satisfies

\[
 z_{(g, \sigma)} z_{(x, \tau)} = \beta((g, \sigma), (x, \tau)) z_{(g, \sigma)(x, \tau)} = \alpha(g, x^{m(\sigma)}) z_{(g x^{m(\sigma)}, \sigma \tau)},
\]

thus the map \( \theta : F^\alpha \cdot G \ast sG \cdot G \to F^\beta (G \times sG, G) \) defined by \( \theta(u_g b_{\sigma}) = z_{(g, \sigma)} \) is an isomorphism because

\[
 \theta(u_g b_{\sigma} \cdot u_x b_\tau) = \alpha(g, x^{m(\sigma)}) \cdot z_{(g x^{m(\sigma)}, \sigma \tau)}
\]

\[
 = z_{(g, \sigma)} z_{(x, \tau)} = \theta(u_g b_{\sigma}) \chi(u_x b_\tau),
\]

this completes the proof. \( \square \)

Since any cocycle \( \alpha \) is cohomologous to a normal cocycle and cohomologous cocycles yield an isomorphism of twisted group algebras, we may assume \( \alpha \) is normal. We now will investigate structures of fixed twisted group algebras as projective class algebras.
THEOREM 8. (a) The fixed algebra $(F^\alpha G_0)^{K_{F^\alpha G}}$ equals $(F^\alpha G_0)^{k_G}$.

(b) The fixed algebra $(E^\alpha G)^{S_{F^\alpha G}}$ is generated by all elements $u_g \in E^\alpha G$ that satisfy $g^{m(\sigma)} = g$ for all $\sigma \in G = \text{Gal}(E/F)$.

Proof. Without loss of generality we may assume $\alpha \in Z^2(G, F^*)$ is normal. From the fixed algebra $(F^\alpha G_0)^{k_G}$, $k_G(x)$ is regarded as an extended algebra homomorphism of $F^\alpha G_0$ for each $x \in G$, hence $u_gk_G(x) = u_gk_G(x) = u_{g^x}$ for any $\alpha$-regular element $g$ in $G_0$. Moreover since $u_gK_{F^\alpha G}(x) = u_g^{-1}u_gu_x = u_gk_G(x)$, this shows that $K_{F^\alpha G}$ is the linearly extended mapping of $k_G$ to $\text{Aut}(F^\alpha G_0)$. Thus we have $(F^\alpha G_0)^{K_{F^\alpha G}} = (F^\alpha G_0)^{k_G}$.

Let $E_g$ be the $\alpha$-regular class containing $g \in G$. Since the class sums $o_g^+ = \sum_{y \in E_g} u_y$ constitute the center $Z(F^\alpha G)$ ([5, (2.6.3)]), the equality

$$o_g^+K_{F^\alpha G}u_x = u_x^{-1} \sum_{y \in E_g} u_yu_x = \sum_{y \in E_g} u_{x^{-1}y} = \sum_{z \in E_g} u_z = o_g^+$$

for any $u_x \in F^\alpha G$ shows that $o_g^+ \in (F^\alpha G_0)^{K_{F^\alpha G}}$ and $Z(F^\alpha G)$ is a subset of $(F^\alpha G_0)^{K_{F^\alpha G}}$. The other inclusion is clear, for $u_g \in (F^\alpha G_0)^{K_{F^\alpha G}}$ if and only if $g \in G_0$ and $u_g \in Z(F^\alpha G)$.

For the second statement, let $u_g$ be any $E$-basis element in $E^\alpha G$. Then $u_g$ is in $(E^\alpha G)^{S_{F^\alpha G}}$ if and only if $u_g = v(g)^{\sigma-1}v(g)^{-m(\sigma-1)}u_g^{m(\sigma-1)}$ for all $\sigma \in G$. This is equivalent to say that, $u_1$ equals

$$v(g)^{\sigma-1}v(g)^{-m(\sigma-1)}\prod_{i=1}^{m(\sigma-1)-1} \alpha(g^i, g)\alpha^{-1}(g, g^{-1})\alpha(g^{m(\sigma-1)}, g^{-1})u_g^{m(\sigma-1)-1},$$

which drives the following two identities that, for all $\sigma \in G$,

1. $1 = v(g)^{\sigma-1}v(g)^{-m(\sigma-1)}\prod_{i=1}^{m(\sigma-1)-1} \alpha(g^i, g)\alpha^{-1}(g, g^{-1})\alpha(g^{m(\sigma-1)}, g^{-1})$

2. $g = g^{m(\sigma-1)}$.

However we will show that (i) follows from (ii). In fact, since $v(g)^n = t(g) = \prod_{i=1}^{n-1} \alpha(g^i, g) \in F^*$ (see (2)), we have

$$\prod_{i=1}^{m(\sigma-1)n} \alpha(g^i, g) = \left( \prod_{i=1}^{n-1} \alpha(g^i, g) \right)^{m(\sigma-1)} = t(g)^{m(\sigma-1)}.$$
On the other hand, we also can write
\[ \prod_{i=1}^{m(\sigma^{-1})n} \alpha(g^i, g) = \left( \prod_{i=1}^{m(\sigma^{-1}) - 1} \alpha(g^i, g) \right)^n \cdot \prod_{i=1}^{n} \alpha(g^i, g) \]
\[ = \left( \prod_{i=1}^{m(\sigma^{-1}) - 1} \alpha(g^i, g) \right)^n \cdot \prod_{i=1}^{n} \alpha(g^i, g) \]
\[ = \left( \prod_{i=1}^{m(\sigma^{-1}) - 1} \alpha(g^i, g) \right)^n \cdot \prod_{i=1}^{n} \alpha(g^i, g) \cdot t(g). \]

Thus the above two identities give rise to
\[ \left( \prod_{i=1}^{m(\sigma^{-1}) - 1} \alpha(g^i, g) \right)^n \cdot \prod_{i=1}^{n} \alpha(g^i, g) \cdot t(g) = t(g)^{m(\sigma^{-1})}. \]

Since \( t(g) \in F^* \) and \( t(g)^{\sigma} = t(g) \) for all \( \sigma \in G, \ g = g^{m(\sigma^{-1})} \) in (ii) yields
\[ \left( v(g)^{\sigma^{-1}} v(g)^{-m(\sigma^{-1})} \cdot \prod_{i=1}^{m(\sigma^{-1}) - 1} \alpha(g^i, g) \alpha^{-1}(g, g^{-1}) \alpha(g^{m(\sigma^{-1})}, g^{-1}) \right)^n \]
\[ = t(g)^{\sigma^{-1}} t(g)^{-m(\sigma^{-1})} \cdot \left( \prod_{i=1}^{m(\sigma^{-1}) - 1} \alpha(g^i, g) \right)^n = 1, \]

and by taking \( n \)-th root of unity as 1, (i) follows immediately. Therefore \( (E^\alpha G)^{S_F \alpha G} \) is generated by \( u_g \in E^\alpha G \) with \( g^{m(\sigma)} = 1 \) for all \( \sigma \in G \). \( \Box \)

To observe an explicit example for Theorem 8, we recall that a finite group \( G \) is called an \( F \)-group if all irreducible \( E \)-characters of \( G \) have values in \( F \). Hence \( G \) is an abelian \( F \)-group if \( \text{Hom}(G, F^*) = \text{Hom}(G, E^*) \).

It was proved in [2] that \( G \) is an abelian \( F \)-group if and only if \( s_G(\sigma) \) fixes each \( g \) in \( G \) for all \( \sigma \in G \), i.e., \( g = g s_G(\sigma^{-1}) = g^{m(\sigma)} \). Thus, if \( G \) is an abelian \( F \)-group then the fixed algebra \( (E^\alpha G)^{S_F \alpha G} \) is generated by \( u_g \in E^\alpha G \) for all \( g \in G \). Hence the next corollary follows immediately.

**COROLLARY 9.** If \( G \) is an abelian \( F \)-group then \( (E^\alpha G)^{S_F \alpha G} = E^\alpha G \).

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