A CHARACTERIZATION OF $C^k \times (C^*)^\ell$
FROM THE VIEWPOINT OF
BIHOLOMORPHIC AUTOMORPHISM GROUPS

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ABSTRACT. We show that if a connected Stein manifold $M$ of dimension $n$ has the holomorphic automorphism group $\text{Aut}(M)$ isomorphic to $\text{Aut}(C^k \times (C^*)^{n-k})$ as topological groups, then $M$ itself is biholomorphically equivalent to $C^k \times (C^*)^{n-k}$. Besides, a new approach to the study of $U(n)$-actions on complex manifolds of dimension $n$ is given.

1. Introduction and results

This article is the outgrowth of the talk given by the first author at the Sixth International Conference on Several Complex Variables and Complex Geometry in Gyeong-Ju, Korea.

In the study of the holomorphic automorphism group $\text{Aut}(M)$ of a complex manifold $M$, it seems to be natural to direct our attention to not only the abstract group structure of $\text{Aut}(M)$ but also the topological group structure of $\text{Aut}(M)$ equipped with the compact-open topology. In fact, a well-known theorem of H. Cartan says that the topological group given as the holomorphic automorphism group of a bounded domain in $C^n$ has the structure of a Lie group, and this result enables us to make various kinds of detailed studies of bounded domains in $C^n$. On the other hand, in contrast to the case of bounded domains, the holomorphic automorphism group $\text{Aut}(C^k \times (C^*)^\ell)$ of the unbounded domain $C^k \times (C^*)^\ell$ is terribly big when $k + \ell \geq 2$, and can not have the

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structure of a Lie group. But, by looking at topological subgroups of \(\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^\ell)\) with Lie group structures, we can find a lead to apply the Lie group theory to the investigation of the problems related to the structure of \(\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^\ell)\).

In this article, we try to approach from this standpoint to the fundamental problem of what complex manifold has the holomorphic automorphism group isomorphic to \(\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^\ell)\) as topological groups. In fact, we can prove the following results. The details can be found in [11]:

**Main Theorem.** Let \(M\) be a connected Stein manifold of dimension \(n\). Assume that \(\text{Aut}(M)\) is isomorphic to \(\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^{n-k})\) as topological groups for some integer \(k\) with \(0 \leq k \leq n\). Then \(M\) is biholomorphically equivalent to \(\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}\).

As a consequence of the above theorem, we can obtain the fundamental result on the topological group structure of \(\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^\ell)\):

**Corollary.** If two pairs \((k, \ell)\) and \((k', \ell')\) of nonnegative integers do not coincide, then the topological groups \(\text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^\ell)\) and \(\text{Aut}(\mathbb{C}^{k'} \times (\mathbb{C}^*)^\ell')\) are not isomorphic.

It should be remarked that, as shown in Ahern-Rudin [1], the groups \(\text{Aut}(\mathbb{C}^n)\) and \(\text{Aut}(\mathbb{C}^m)\) are isomorphic as abstract groups precisely when \(n = m\). Also, as a consequence of the study of \(U(n)\)-actions on complex manifolds of dimension \(n\), Isaev-Kruzhilin [8] showed that exactly the same conclusion in the Main Theorem remains valid for the case of \(k = n\) without assuming the Steinness of \(M\).

Our method can be applied to the study of unitary group actions on complex manifolds. The following Theorems A and B give a different approach from Kaup [9] and Isaev-Kruzhilin [8] to the study of \(U(n)\)-actions on a complex manifold of dimension \(n\).

**Theorem A.** Let \(M\) be a connected Stein manifold of dimension \(n \geq 2\). Assume that \(U(n)\) acts effectively on \(M\) as a Lie transformation group through \(\rho\). Then \(M\) is biholomorphically equivalent to either \(B^n\) or \(\mathbb{C}^n\), where \(B^n\) denotes the unit ball in \(\mathbb{C}^n\).

**Theorem B.** Let \(M\) be a connected Stein manifold of dimension \(n \geq 2\). Assume that there are two injective continuous group homomorphisms \(\rho_1\) and \(\rho_2\) of \(U(n)\) into \(\text{Aut}(M)\). Then there exists an element \(\psi\) of \(\text{Aut}(M)\) such that \(\psi\rho_1(U(n))\psi^{-1} = \rho_2(U(n))\). More precisely, in this case one can choose an element \(\Psi\) of \(\text{Aut}(M)\) in such a way that

\[
\Psi\rho_1(u)\Psi^{-1} = \rho_2(u) \quad \text{or} \quad \Psi\rho_1(u)\Psi^{-1} = \rho_2(\bar{u}) \quad \text{for all} \quad u \in U(n),
\]
where $\bar{u}$ denotes the complex conjugate of a matrix $u$.

Our proof of the Main Theorem relies on the one hand on the theory of Reinhardt domains developed in Shimizu [16], [17] (cf. Krzuhilin [13]), on the other hand on the fundamental result on torus actions on complex manifolds due to Barrett-Bedford-Dadok [3].

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2. Basic concepts and notations

Let $M$ be a complex manifold. An automorphism of $M$ means a biholomorphic mapping of $M$ onto itself. We denote by $\text{Aut}(M)$ the topological group of all automorphisms of $M$ equipped with the compact-open topology. Let $G$ be a Lie group and consider a continuous group homomorphism $\rho : G \to \text{Aut}(M)$ of the Lie group $G$ into the topological group $\text{Aut}(M)$. Then the mapping

$$G \times M \ni (g, p) \longmapsto (\rho(g))(p) \in M$$

is continuous. It follows from Akhiezer [2] that this mapping is actually of class $C^\omega$, and therefore $G$ acts on $M$ as a Lie transformation group. In view of this, when a continuous group homomorphism $\rho : G \to \text{Aut}(M)$ of $G$ into $\text{Aut}(M)$ is given, we say that $G$ acts on $M$ as a Lie transformation group through $\rho$. Also, the action of $G$ on $M$ is called effective if $\rho$ is injective.

We denote by $U(k)$ the unitary group of degree $k$. Write $T^n = (U(1))^n$. The $n$-dimensional compact torus $T^n$ acts as a group of automorphisms on $\mathbb{C}^n$ by the standard rule $\alpha \cdot z = (\alpha_1 z_1, \ldots, \alpha_n z_n)$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in T^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. By definition, a Reinhardt domain $D$ in $\mathbb{C}^n$ is a domain in $\mathbb{C}^n$ which is stable under the action of $T^n$. Each element $\alpha$ of $T^n$ then induces an automorphism $\pi_{\alpha}$ of $D$ given by $\pi_{\alpha}(z) = \alpha \cdot z$, and the mapping $\rho_D$ sending $\alpha$ to $\pi_{\alpha}$ is an injective continuous group homomorphism of the torus $T^n$ into the topological group $\text{Aut}(D)$. The subgroup $\rho_D(T^n)$ of $\text{Aut}(D)$ is denoted by $T(D)$. 
3. Some lemmas and fundamental theorems

For later purpose, in this section we shall recall some lemmas and fundamental theorems. We refer the reader to [11] for the details.

Let $f$ be a holomorphic function on a Reinhardt domain $D$ in $\mathbb{C}^n$. Then $f$ can be expanded uniquely into a “Laurent series”

$$f(z) = \sum_{\nu \in \mathbb{Z}^n} a_\nu z^\nu,$$

which converges absolutely and uniformly on any compact set in $D$, where $z = (z_1, \ldots, z_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, and $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$.

The following lemma is a consequence of the uniqueness of the Laurent series expansion:

**Lemma 1.** Let $f$ be a holomorphic function on a Reinhardt domain $D$ in $\mathbb{C}^n$. If $f$ satisfies the condition that, for some $\nu_0 \in \mathbb{Z}^n$,

$$f(\alpha \cdot z) = \alpha^{\nu_0} f(z) \quad \text{for all } \alpha \in T^n \text{ and all } z \in D,$$

then $f$ has the form $f(z) = a_{\nu_0} z^{\nu_0}$.

We denote by $\Pi(\mathbb{C}^n)$ the group of all automorphisms of $\mathbb{C}^n$ of the form

$$\mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto (\alpha_1 z_1, \ldots, \alpha_n z_n) \in \mathbb{C}^n,$$

where $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^*)^n$. For a Reinhardt domain $D$ in $\mathbb{C}^n$, we denote by $\Pi(D)$ the subgroup of $\Pi(\mathbb{C}^n)$ consisting of all elements of $\Pi(\mathbb{C}^n)$ leaving $D$ invariant. Identifying $\Pi(\mathbb{C}^n)$ with the multiplicative group $(\mathbb{C}^*)^n$, we see that, when $\Pi(D)$ is regarded as a topological subgroup of $\text{Aut}(D)$, it is isomorphic to a closed Lie subgroup of $(\mathbb{C}^*)^n$. Using Lemma 1, we obtain the following characterization of $\Pi(D)$ as a subgroup of $\text{Aut}(D)$:

**Lemma 2.** Let $D$ be a Reinhardt domain in $\mathbb{C}^n$. Then $\Pi(D)$ is the centralizer $C_{\text{Aut}(D)}(T(D))$ of $T(D)$ in $\text{Aut}(D)$.

As stated in the introduction, our proof of the Main Theorem is based on the following fact by Shimizu [17], which is shown implicitly in the process of determining the automorphism groups of bounded Reinhardt domains in $\mathbb{C}^n$, and also on the fundamental result on torus actions on complex manifolds due to Barrett-Bedford-Dadok [3]:

Fundamental Theorem 1 ([17]). Let \( D \) be a bounded Reinhardt domain in \( \mathbb{C}^n \) and suppose that
\[
D \cap \{ z_i = 0 \} \neq \emptyset, \quad 1 \leq i \leq m,
\]
\[
D \cap \{ z_i = 0 \} = \emptyset, \quad m + 1 \leq i \leq n,
\]
or that \( D \subset \mathbb{C}^m \times (\mathbb{C}^*)^{n-m} \). If \( G \) is a connected compact subgroup of \( \text{Aut}(D) \) containing \( T(D) \), then there exists a transformation
\[
\varphi : \mathbb{C}^m \times (\mathbb{C}^*)^{n-m} \ni (z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) \in \mathbb{C}^m \times (\mathbb{C}^*)^{n-m},
\]
\[
\begin{cases}
  w_i = r_i z_{\sigma'(i)}(z'')^\nu_i'', & \text{if } 1 \leq i \leq m, \\
  w_i = r_i z_{\sigma''(i)}, & \text{if } m + 1 \leq i \leq n,
\end{cases}
\]
such that, for \( \tilde{D} = \varphi(D) \) and \( \tilde{G} = \varphi G \varphi^{-1} \subset \text{Aut}(\tilde{D}) \), one has
\[
\tilde{G} = U(k_1) \times \cdots \times U(k_s) \times U(k_{s+1}) \times \cdots \times U(k_t),
\]
\[
k_1 + \cdots + k_s + k_{s+1} + \cdots + k_t = n,
\]
\[
k_1 + \cdots + k_s = m,
\]
\[
k_{s+1} = \cdots = k_t = 1,
\]
where \( r_1, \ldots, r_n \) are positive constants, \( \sigma' \) and \( \sigma'' \) are permutations of \( \{1, \ldots, m\} \) and \( \{m + 1, \ldots, n\} \), respectively, \( z'' \) denotes the coordinates \( (z_{m+1}, \ldots, z_n) \), and \( \nu''_1, \ldots, \nu''_m \) are elements of \( \mathbb{Z}^{n-m} \).

From this, we obtain the following corollary which will play an important role in our proof of the Main Theorem:

Corollary. In the above theorem, if \( G \) is isomorphic to \( U(k) \times (U(1))^{n-k} \) as topological groups and if \( k \geq 2 \), then we have \( m \geq k \).

Fundamental Theorem 2 ([3]). Let \( M \) be a connected Stein manifold of dimension \( n \). Assume that \( T^n \) acts effectively on \( M \) as a Lie transformation group through \( \rho \). Then there exist a biholomorphic mapping \( F \) of \( M \) into \( \mathbb{C}^n \) and a continuous group automorphism \( \theta \) of the torus \( T^n \) such that
\[
F((\rho(\alpha))(p)) = \theta(\alpha) \cdot F(p) \quad \text{for all } \alpha \in T^n \text{ and all } p \in M.
\]
Consequently, \( D := F(M) \) is a Reinhardt domain in \( \mathbb{C}^n \), and one has \( F \rho(T^n)F^{-1} = T(D) \).

Lemma 3. In the above theorem, if \( M = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \), then we have \( D = F(M) = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \).
**Lemma 4.** Let $M$ be a connected Stein manifold of dimension $n$. If $N > n$, then there is no injective continuous group homomorphism of the torus $T^N$ into the topological group $\text{Aut}(M)$.

This lemma can be shown by using the fact that the group $T(D)$ is a maximal torus in $\text{Aut}(D)$, provided that $D$ is a bounded Reinhardt domain in $\mathbb{C}^n$ [16; Section 4, Proposition 1].

4. Proofs of the theorems and the corollary

For the sake of simplicity, we write $X_{k, \ell} = \mathbb{C}^k \times (\mathbb{C}^*)^\ell$ and $\Omega_k = X_{k, n-k}$ in this section.

**Proof of the Main Theorem.** Let us assume that there exists a topological group isomorphism $\Phi : \text{Aut}(\Omega_k) \rightarrow \text{Aut}(M)$. Since $\Omega_k$ is a Reinhardt domain in $\mathbb{C}^n$, we have the injective continuous group homomorphism $\rho_{\Omega_k} : T^n \rightarrow \text{Aut}(\Omega_k)$. Thus, we obtain an injective continuous group homomorphism $\Phi \circ \rho_{\Omega_k} : T^n \rightarrow \text{Aut}(M)$. Hence, by Fundamental Theorem 2 there exists a biholomorphic mapping $F$ of $M$ into $\mathbb{C}^n$ such that $D := F(M)$ is a Reinhardt domain in $\mathbb{C}^n$ and $F(\Phi \circ \rho_{\Omega_k})(T^n)F^{-1} = T(D)$. Therefore we may assume that $M$ is a Reinhardt domain $D$ in $\mathbb{C}^n$ and we have a topological group isomorphism $\Phi : \text{Aut}(\Omega_k) \rightarrow \text{Aut}(D)$ such that $\Phi(T(\Omega_k)) = T(D)$. Now we will proceed in steps.

1) $D$ has the form $D = \Omega_h$ after a suitable permutation of coordinates.
First we wish to show that $(\mathbb{C}^*)^n \subset D$. Since $\Phi : \text{Aut}(\Omega_k) \rightarrow \text{Aut}(D)$ is a topological group isomorphism and since $\Phi(T(\Omega_k)) = T(D)$, we see that $\Phi$ gives rise to a topological group isomorphism $\Phi : C_{\text{Aut}(\Omega_k)}(T(\Omega_k)) \rightarrow C_{\text{Aut}(D)}(T(D))$. Moreover, by Lemma 2 we have

$$C_{\text{Aut}(\Omega_k)}(T(\Omega_k)) = \Pi(\Omega_k) = \Pi(\mathbb{C}^n) \quad \text{and} \quad C_{\text{Aut}(D)}(T(D)) = \Pi(D).$$

Thus $\Pi(D)$ is a $2n$-dimensional Lie subgroup of the connected Lie group $\Pi(\mathbb{C}^n) = (\mathbb{C}^*)^n$, and therefore $\Pi(D) = \Pi(\mathbb{C}^n)$. By taking a point $z_0$ in $D \cap (\mathbb{C}^*)^n$, this shows that

$$(\mathbb{C}^*)^n = \Pi(\mathbb{C}^n) \cdot z_0 = \Pi(D) \cdot z_0 \subset D,$$

as required. Since $D$ is now a Stein subdomain of $\mathbb{C}^n$ containing $(\mathbb{C}^*)^n$, we see that $D$ has the form $D = \Omega_h$ after a suitable permutation of coordinates (cf. [15; p. 46, Theorem 1.5]), completing the proof of the assertion 1).
In the case of \( n = 1 \), it is easy to prove the Main Theorem. Therefore, in what follows, we assume that \( n \geq 2 \). Under this assumption, we next prove the following:

2) We have \( h \geq k \).

When \( k = 0 \), there is nothing to prove. To prove our assertion when \( k \neq 0 \), we divide the proof into the two cases of \( k = 1 \) and \( k \geq 2 \).

First consider the case of \( k \geq 2 \). Noting that \( \text{Aut}(\Omega_k) \) contains the subgroup \( U(k) \times (U(1))^{n-k} \), we set \( G = \Phi(U(k) \times (U(1))^{n-k}) \), which is a connected compact subgroup of \( \text{Aut}(D) \) containing \( T(D) \), because \( U(k) \times (U(1))^{n-k} \supset T(\Omega_k) \) and \( \Phi(T(\Omega_k)) = T(D) \). Take a bounded domain \( U \) in \( \mathbb{C}^n \) contained in \( D \) and put

\[
D_0 = \{ g(z) \in D \mid g \in G, \ z \in U \} = \bigcup_{g \in G} g(U) = \bigcup_{z \in U} G \cdot z.
\]

Then \( D_0 \) is a bounded Reinhardt domain in \( D \) and \( G \) can be regarded as a connected compact subgroup of the Lie group \( \text{Aut}(D_0) \) containing \( T(D_0) \). Since \( G \) is isomorphic to \( U(k) \times (U(1))^{n-k} \) and \( k \geq 2 \), we can apply the corollary to Fundamental Theorem 1 to \( D_0 \) and \( G \subset \text{Aut}(D_0) \). Therefore, after a suitable permutation of coordinates, we have for some \( m \geq k \),

\[
\emptyset \neq D_0 \cap \{ z_i = 0 \} \subset D \cap \{ z_i = 0 \}, \ 1 \leq i \leq m.
\]

This implies that \( \Omega_{m} \subset D \); and consequently, we have \( h \geq m \geq k \), as required.

Now consider the case of \( k = 1 \). The only thing which has to be proved now is that the topological groups \( \text{Aut}(\Omega_1) \) and \( \text{Aut}(\Omega_0) \) are not isomorphic. Suppose contrarily that we have an isomorphism \( \Phi : \text{Aut}(\Omega_1) \rightarrow \text{Aut}(\Omega_0) \). Then, by Fundamental Theorem 2 and Lemma 3, we may assume that we have a topological group isomorphism \( \Phi : \text{Aut}(\Omega_1) \rightarrow \text{Aut}(\Omega_0) \) such that \( \Phi(T(\Omega_1)) = T(\Omega_0) \). For \( s = 0, 1 \), let us set

\[
T'(\Omega_s) = \{ (1, \alpha_2, \ldots, \alpha_n) \in T(\Omega_s) \mid \alpha_2, \ldots, \alpha_n \in U(1) \}.
\]

Then \( \Phi(T'(\Omega_1)) \) is an \((n - 1)\)-dimensional subtorus of \( T'(\Omega_0) \); and hence, after a suitable change of coordinates by a transformation of the form

\[
\Omega_0 = (\mathbb{C}^*)^n \ni (z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) \in (\mathbb{C}^*)^n = \Omega_0,
\]

\[
w_i = z^{\nu_i}, \quad 1 \leq i \leq n,
\]

where \( \nu_1, \ldots, \nu_n \) are elements of \( \mathbb{Z}^n \), we have \( \Phi(T'(\Omega_1)) = T'(\Omega_0) \). This combined with the fact that \( \Phi : \text{Aut}(\Omega_1) \rightarrow \text{Aut}(\Omega_0) \) is a group isomorphism yields that \( \Phi \) maps the centralizer \( Z_1 \) of \( T'(\Omega_1) \) in \( \text{Aut}(\Omega_1) \) onto the centralizer \( Z_0 \) of \( T'(\Omega_0) \) in \( \text{Aut}(\Omega_0) \). Therefore, for the groups \( Z_0 \)
and $Z_1$, their commutator groups $[Z_0, Z_0]$ and $[Z_1, Z_1]$ must be isomorphic. To derive a contradiction, we here assert that $[Z_0, Z_0]$ is an abelian group, while $[Z_1, Z_1]$ is not an abelian group. We verify this only in the case of $n = 2$, because the verification in the case of $n > 2$ is almost identical. First of all, we can show that $Z_1$ and $Z_0$ are the groups of all elements

$$g_1 \in \text{Aut}(\Omega_1) = \text{Aut}(\mathbb{C} \times \mathbb{C}^*) \quad \text{and} \quad g_0 \in \text{Aut}(\Omega_0) = \text{Aut}((\mathbb{C}^*)^2)$$

having the forms

$$(*) \quad g_1(z) = (\alpha z_1 + \beta, f(z_1)z_2) \quad \text{and} \quad g_0(z) = (\alpha z_1, f(z_1)z_2)$$

respectively, where $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, and $f(z_1)$ is a nowhere vanishing holomorphic function that is defined on $\mathbb{C}$ for $g_1$ and on $\mathbb{C}^*$ for $g_0$. Take any two transformations $K_{\alpha,\beta,f}$ and $K_{\alpha',\beta',f'}$ of the form $(*)$ given by

$$K_{\alpha,\beta,f}(z) = (\alpha z_1 + \beta, f(z_1)z_2) \quad \text{and} \quad K_{\alpha',\beta',f'}(z) = (\alpha' z_1 + \beta', f'(z_1)z_2)$$

and write $[K_{\alpha,\beta,f}, K_{\alpha',\beta',f'}](z) = (K_1(z), K_2(z))$ in terms of the coordinates in $\mathbb{C}^2$, where $[\varphi, \psi] := \varphi^{-1} \circ \psi^{-1} \circ \varphi \circ \psi$ denotes the commutator of transformations $\varphi$ and $\psi$. Then, by direct calculations we have

$$K_1(z) = (\alpha \alpha' z_1 + \alpha \beta' - \beta \alpha' + \beta - \beta')/\alpha \alpha',$$

$$K_2(z) = \frac{f(\alpha' z_1 + \beta') f'(z_1)z_2}{f((\alpha \alpha' z_1 + \alpha \beta' - \beta \alpha' + \beta - \beta')/\alpha \alpha') f'(\alpha z_1)}.$$

In particular, considering the case of $(\beta, \beta') = (0, 0)$, we have

$$[K_{\alpha,0,f}, K_{\alpha',0,f'}](z) = (z_1, (f(\alpha' z_1) f'(z_1)z_2)/(f(z_1)f'(\alpha z_1))),$$

which implies that $[Z_0, Z_0]$ is abelian. On the other hand, consider three elements

$$P(z) = (\alpha z_1 + \beta, z_2), \quad Q(z) = (z_1, z_2 \exp z_1), \quad \text{and} \quad R(z) = (\gamma z_1, z_2 \exp z_1)$$

in $Z_1$. Then, using the computation result above, we obtain

$$[P, Q](z) = (z_1, z_2 \exp((1 - \alpha)z_1 - \beta)),$$

$$[P, R](z) = ((\alpha \gamma z_1 + \beta (1 - \gamma))/\alpha \gamma, z_2 \exp((1 - \alpha)z_1 - (\beta/\gamma))),$$

and therefore $[[P, Q], [P, R]]$ is not the identity mapping whenever $\beta(\alpha - 1)(\gamma - 1) \neq 0$. This implies that $[Z_1, Z_1]$ is not abelian, and our assertion that the topological groups $\text{Aut}(\Omega_1)$ and $\text{Aut}(\Omega_0)$ are not isomorphic is shown.

Summarizing our results obtained so far, we have shown that if $M$ is a connected Stein manifold of dimension $n$ and if the topological groups $\text{Aut}(M)$ and $\text{Aut}(\Omega_k)$ are isomorphic, then $M$ is biholomorphically equivalent to some $\Omega_h$ with $h \geq k$. 
Finally, we shall complete the proof by showing the following:
3) \( M \) is biholomorphically equivalent to \( \Omega_k \). Suppose that \( h \neq k \), and so \( h > k \) by 2). For the connected Stein manifold \( \Omega_k \) of dimension \( n \), we know that the topological groups \( \text{Aut}(\Omega_k) \) and \( \text{Aut}(\Omega_h) \) are isomorphic. Then, by letting \( M = \Omega_k \), an application of what we have shown just above yields that \( \Omega_k \) is biholomorphically equivalent to \( \Omega_p \) with \( p \geq h \). Since \( k < h \leq p \), this contradicts the fact that \( \Omega_s \) and \( \Omega_t \) are not homeomorphic when \( s \neq t \). We thus conclude that \( h = k \). \( \square \)

Proof of the Corollary to the Main Theorem. If \( k + \ell = k' + \ell' \), the topological groups \( \text{Aut}(X_{k,\ell}) \) and \( \text{Aut}(X_{k',\ell'}) \) are isomorphic precisely when \( (k, \ell) = (k', \ell') \) by our Main Theorem.

Now, suppose that \( k + \ell \neq k' + \ell' \), say, \( k + \ell < k' + \ell' \), and write \( n = k + \ell, \ n' = k' + \ell' \). If there exists a topological group isomorphism \( \Phi : \text{Aut}(X_{k',\ell'}) \rightarrow \text{Aut}(X_{k,\ell}) \), then we have an injective continuous group homomorphism \( \Phi \circ \rho_{X_{k',\ell'}} : T^{n'} \rightarrow \text{Aut}(X_{k,\ell}) \). Since \( X_{k,\ell} \) is a connected Stein manifold of dimension \( n < n' \), this contradicts the fact in Lemma 4. Therefore the topological groups \( \text{Aut}(X_{k,\ell}) \) and \( \text{Aut}(X_{k',\ell'}) \) are not isomorphic. \( \square \)

Proof of Theorem A. Choose a maximal torus \( T^n \) in \( U(n) \).

Then, by Fundamental Theorem 2 there exists a biholomorphic mapping \( F : M \rightarrow D \) of \( M \) onto a Reinhardt domain \( D \) in \( \mathbb{C}^n \) such that \( F \circ (T^n) F^{-1} = T(D) \). Set \( G = F \circ (U(n)) F^{-1} \) and take a bounded domain \( U \) in \( \mathbb{C}^n \) contained in \( D \). Then, \( D_0 := \{ g(z) \in D \mid g \in G, \ z \in U \} \) is a bounded Reinhardt domain in \( \mathbb{C}^n \) contained in \( D \) and \( G \) can be regarded as a connected compact subgroup of the Lie group \( \text{Aut}(D_0) \) containing \( T(D_0) \). Since \( G \) is isomorphic to \( U(n) \) and \( n \geq 2 \), we can apply Fundamental Theorem 1 and its corollary to \( D_0 \) and \( G \subset \text{Aut}(D_0) \). Therefore there exists a transformation
\[
\varphi : \mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) \in \mathbb{C}^n
\]
\[
w_i = r_i z_{\sigma(i)}, \quad 1 \leq i \leq n,
\]
such that, for \( \tilde{D}_0 = \varphi(D_0) \) and \( \tilde{G} = \varphi G \varphi^{-1} \subset \text{Aut}(\tilde{D}_0) \), we have \( \tilde{G} = U(n) \). Put \( \tilde{D} = \varphi(D) \). Then, since \( \tilde{D}_0 \) is a non-empty subdomain of \( \tilde{D} \), we see by the uniqueness theorem on holomorphic functions that \( U(n) = \tilde{G} \subset \text{Aut}(\tilde{D}) \), or \( g(\tilde{D}) = \tilde{D} \) for all \( g \in U(n) \). Being a Stein manifold, \( \tilde{D} \) is now to be of the form
\[
\tilde{D} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 < r \right\},
\]
where $0 < r \leq +\infty$. This shows that $\tilde{D}$, and hence $M$ is biholomorphically equivalent to either $B^n$ or $C^n$. $\square$

**Proof of Theorem B.** By Theorem A, we may assume that $M = B^n$ or $M = C^n$. When $M = B^n$, our assertion is a consequence of the conjugacy of maximal compact subgroups of the Lie group $\text{Aut}(B^n)$. Therefore, in what follows, we consider the case where $M = C^n$.

To prove our assertion, it suffices to prove that, for any injective continuous group homomorphism $\rho$ of $U(n)$ into $\text{Aut}(M)$, we have an element $\psi$ of $\text{Aut}(M)$ such that $\psi\rho(U(n))\psi^{-1} = U(n)$. Suppose that such $\rho$ is given. Choose a maximal torus $T^n$ in $U(n)$. Then, by Fundamental Theorem 2 and Lemma 3, there exists a biholomorphic mapping $F : M = C^n \rightarrow D = C^n$ such that $F\rho(T^n)F^{-1} = T(D)$. Set $G = F\rho(U(n))F^{-1}$. As in the proof of Theorem A, there exists a transformation

$$\varphi : C^n \ni (z_1, \ldots, z_n) \quad \mapsto \quad (w_1, \ldots, w_n) \in C^n$$

$$w_i = r_i z_{\sigma(i)}, \quad 1 \leq i \leq n,$$

such that, for $\tilde{D} = \varphi(D) = C^n$ and $\tilde{G} = \varphi G \varphi^{-1} \subset \text{Aut}(\tilde{D})$, we have $\tilde{G} = U(n)$. Therefore, putting $\psi = \varphi \circ F \in \text{Aut}(M)$, we have

$$\psi\rho(U(n))\psi^{-1} = \varphi(F\rho(U(n))F^{-1})\varphi^{-1} = \varphi G \varphi^{-1} = \tilde{G} = U(n),$$

as desired.

Finally, notice that every continuous, and hence analytic, group automorphism of $U(n)$ is an inner automorphism up to the complex conjugation in $U(n)$. Indeed, this follows from the following fact: Both the groups $\text{Aut}(U(1))$ and $\text{Aut}(SU(n))/\text{Int}(SU(n)) \ (n \geq 3)$ are the cyclic groups of order 2 generated by the complex conjugation $u \mapsto \bar{u}$ and $\text{Aut}(SU(2)) = \text{Int}(SU(2))$, where $\text{Aut}(L)$ (resp. $\text{Int}(L)$) denotes the group of all analytic automorphisms (resp. inner automorphisms) of a given Lie group $L$ (cf. [6]). Then one can find an element $u_o \in U(n)$ such that

$$\psi\rho_1(u)\psi^{-1} = \rho_2(u_o uu_o^{-1}) \quad \text{or} \quad \psi\rho_1(u)\psi^{-1} = \rho_2(u_o \bar{u} u_o^{-1})$$

for all $u \in U(n)$. Thus, the element $\Psi := \rho_2(u_o^{-1})\psi \in \text{Aut}(M)$ is a required one in Theorem B. $\square$
5. A remark

As mentioned in the introduction, Isaev [7] and Krantz [12] obtained the following theorem, which is a special case of \( k = n \) in our Main Theorem:

**Theorem I-K.** Let \( M \) be a connected Stein manifold of dimension \( n \). Assume that \( \text{Aut}(M) \) is isomorphic to \( \text{Aut}(\mathbb{C}^n) \) as topological groups. Then \( M \) is biholomorphically equivalent to \( \mathbb{C}^n \).

Let us recall the key point of their proof of this theorem. Firstly, by using Fundamental Theorem 2 and Lemma 2, they also prove that \( M \) must be biholomorphically equivalent to \( \mathbb{C}^h \times (\mathbb{C}^*)^{n-h} \) for some integer \( h \) with \( 0 \leq h \leq n \), as we did in the step 1) of the proof of the Main Theorem. Secondly, they verify that

1. the topological group \( \text{Aut}(\mathbb{C}^n) \) is connected; while
2. the topological group \( \text{Aut}(\mathbb{C}^h \times (\mathbb{C}^*)^{n-h}) \) is disconnected, provided that \( h \neq n \).

Consequently, since \( \text{Aut}(M) \) is now assumed to be isomorphic to \( \text{Aut}(\mathbb{C}^n) \) as topological groups, they conclude that \( M \) is in fact biholomorphically equivalent to \( \mathbb{C}^n \), completing the proof of Theorem I-K.

Here it should be remarked the following: In the case where \( 0 \leq h, k \leq n - 1 \) and \( h \neq k \), the assertion (2) above does not guarantee that \( \text{Aut}(\mathbb{C}^h \times (\mathbb{C}^*)^{n-h}) \) is isomorphic to \( \text{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}) \) as topological groups. So, it seems to be difficult to prove our Main Theorem with the same arguments as those in the proof of Theorem I-K.

In connection with this, we would like to ask the following two questions: For a given integer \( h \) with \( 0 \leq h \leq n \), we denote by \( C_h \) the cardinality of the set consisting of all connected components of the topological group \( \text{Aut}(\mathbb{C}^h \times (\mathbb{C}^*)^{n-h}) \). For instance, we have \( C_n = 1 \) by the assertion (1) above.

**Question 1.** Is it possible to determine the cardinality \( C_h \) by means of the integer \( h \)?

**Question 2.** Is it true that \( C_h = C_k \) if and only if \( \mathbb{C}^h \times (\mathbb{C}^*)^{n-h} \) is biholomorphically equivalent to \( \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \)?

Of course, for general domains in \( \mathbb{C}^n \), the answer to Question 2 is negative. In fact, there exists a family \( \{D_t\}_{t \in \mathbb{R}} \) of bounded strictly pseudoconvex domains in \( \mathbb{C}^n \) with smooth boundaries such that the automorphism group \( \text{Aut}(D_t) \) is the identity only for every \( t \in \mathbb{R} \) and
$D_s$ is not biholomorphically equivalent to $D_t$ if $s \neq t$ (cf. [4], [5]). Also, for generalized complex ellipsoids

$$E(k, \alpha) = \left\{ z \in \mathbb{C}^n \left| \sum_{i=1}^{k} |z_i|^2 + \left( \sum_{j=k+1}^{n} |z_j|^2 \right)^\alpha < 1 \right. \right\}$$

in $\mathbb{C}^n$, where $k \in \mathbb{Z}$ with $1 \leq k \leq n$ and $0 < \alpha \in \mathbb{R}$, we know that the Lie group $\text{Aut}(E(k, \alpha))$ is connected for every $(k, \alpha)$ and $E(k, \alpha)$ is not biholomorphically equivalent to $E(\ell, \beta)$ if $(k, \alpha) \neq (\ell, \beta)$ (cf. [10], [14]).

Anyway, it would be interesting to investigate these questions; however, these seem to be very difficult at this moment.

References


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