ALGEBRAIC NUMBERS, TRANSCENDENTAL NUMBERS AND ELLIPTIC CURVES DERIVED FROM INFINITE PRODUCTS

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ABSTRACT. Let $k$ be an imaginary quadratic field, $\mathbb{h}$ the complex upper half plane, and let $\tau \in \mathbb{h} \cap k$, $p = e^{\pi i \tau}$. In this article, using the infinite product formulas for $g_2$ and $g_3$, we prove that values of certain infinite products are transcendental whenever $\tau$ are imaginary quadratic. And we derive analogous results of Berndt-Chan-Zhang ([4]). Also we find the values of $\prod_{n=1}^{\infty} \frac{1-p^{2n-1}}{(1+p^{2n-1})^8}$ and $\rho \prod_{n=1}^{\infty} (1 + p^{2n})^{12}$ when we know $j(\tau)$. And we construct an elliptic curve $E: y^2 = x^3 + 3x^2 + (3 - \frac{1}{3^{2/3}})x + 1$ with $j = j(\tau) \neq 0$ and $P = (16^2p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{24}, 0) \in E$.

§1. Introduction

The elliptic modular function $j(\tau)$ is transcendental for algebraic number $\tau$ in the upper half plane which is not an imaginary quadratic irrationality. On the other hand, it is known from the theory of complex multiplication that $j(\tau)$ is algebraic for any imaginary quadratic $\tau$ ([16], [21]).

Ramanujan [19] introduced the following functions:

$$P(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) z^n,$$

$$Q(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) z^n,$$
\[ R(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)z^n, \]

with \( \sigma_k(n) = \sum_{d|n} d^k \) and \( z \in \mathbb{C} \).

In 1967 Mahler conjectured that \( J(z) = \frac{1728Q(z)^3}{Q(z)^3 - R(z)^2} \) is transcendental for any algebraic value of \( z \), \( 0 < |z| < 1 \). It was proved by Barré-Sirieix, Diaz, Gramain, Philibert ([1]). The algebraic independence of the numbers \( J(z) \), \( J'(z) \), \( J''(z) \) was shown in [18]. The result is a consequence of the main result in [18] on numbers \( z, P(z), Q(z), R(z) \).

Meanwhile in [13], [14] and [15], we dealt with certain algebraic integers as values of elliptic functions constructed from Weierstrass \( \wp \)-function by using infinite products.

In this article, by means of infinite product formulas of the Weierstrass \( \wp \)-function we prove that \( \eta(\tau)^4 \frac{Q(p^2)}{R(p^2)} \) and \( \frac{J'(\tau)}{\eta(\tau)^4} \) are algebraic numbers (Theorem 10) with \( \eta(\tau) \) the Dedekind eta-function, and we consider the zeros of the Weierstrass \( \wp \)-function and infinite products (Theorem 2). In [13] we derived analogous results of Berndt-Chan-Zhang ([4]), which would be a generalization in the case \( m \) even. Here we derive similar results (Theorem 9) of [13]. Furthermore, we prove that certain values of infinite product are algebraic integers (Corollary 5, Theorem 8).

Also, we justify that \( [\mathbb{Q}(\prod_{n=1}^{\infty}(\frac{1}{1+p^{2n-1}})^8) : \mathbb{Q}(j(\tau))] \leq 6 \) (resp. \( [\mathbb{Q}(p^2 \prod_{n=1}^{\infty}(1+p^{2n})^{24}) : \mathbb{Q}(j(\tau))] \leq 3 \)), and we get the polynomials

\[ M(x) = x^6 - 3x^5 + \left( 6 - \frac{j}{256} \right) x^4 + \left( 7 + \frac{j}{128} \right) x^3 + \left( 6 - \frac{j}{256} \right) x^2 - 3x + 1 \]

and

\[ N(z) = z^3 + 3z^2 + (3 - \frac{j}{256})z + 1 \]

satisfying \( M(\prod_{n=1}^{\infty}(\frac{1}{1+p^{2n-1}})^8) = 0 \) and \( N(16^2p^2 \prod_{n=1}^{\infty}(1+p^{2n})^{24}) = 0 \) when \( \tau \in \mathfrak{h} \cap k \) and \( j = j(\tau) \neq 0 \). From this we construct an elliptic curve

\[ E : y^2 = x^3 + 3x^2 + (3 - \frac{j}{256})x + 1 \]

with a point \( P = (16^2p^2 \prod_{n=1}^{\infty}(1+p^{2n})^{24}, 0) \in E \) (Theorem 11).
Finally, we consider the equations of Fricke functions (Theorem 12). Throughout the article we adopt the following notations:

- \( k \) an imaginary quadratic field
- \( \mathbb{H} \) the complex upper half plane
- \( \tau \in \mathbb{H} \cap k \)
- \( \Lambda_{\tau} = \mathbb{Z} + \tau \mathbb{Z} \)
- \( p = e^{\pi i \tau} \)
- \( \mathbb{Q} \) the set of all algebraic numbers in \( \mathbb{C} \)
- \( \varphi(z) := \varphi(z, \Lambda_{\tau}) = \frac{1}{z} + \sum_{\omega \in \Lambda_{\tau} \setminus \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \), a Weierstrass \( \varphi \)-function (relative to \( \Lambda_{\tau} \))
- \( G_k(\Lambda_{\tau}) := G_k(\tau) = \sum_{\omega \in \Lambda_{\tau} \setminus \{0\}} \frac{1}{\omega} \), the Eisenstein series with weight \( k \)
- \( g_2(\tau) = 60G_4(\tau) \)
- \( g_3(\tau) = 140G_6(\tau) \)
- \( \Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = (2\pi)^{12}\eta(\tau)^{24} \)
- \( j(\tau) = \frac{1728g_2(\tau)^3}{\Delta(\tau)} \)
- \( E \) an elliptic curve
- \( \rho = \prod_{n=1}^{\infty} \left( \frac{1-p^{-2n-1}}{1+p^{-2n-1}} \right)^8 \)
- \( \kappa = 16^2p^2 \prod_{n=1}^{\infty} \left( 1+p^{2n} \right)^{24} \).

§2. Values of infinite products

**Proposition 1.** ([8, p.86], [13], [17, p.140]) Let \( \tau \in k \cap \mathbb{H} \).

(a) \( \varphi \left( \frac{\tau}{2} \right) = -\frac{\pi^2}{3} \prod_{n=1}^{\infty} \left( 1-p^{-2n} \right)^4 \left( \prod_{n=1}^{\infty} \left( 1+p^{-2n-1} \right)^8 +16p \prod_{n=1}^{\infty} \left( 1+p^{-2n} \right)^8 \right) \).

(b) \( \varphi \left( \frac{\tau+1}{2} \right) = -\frac{\pi^2}{3} \prod_{n=1}^{\infty} \left( 1-p^{-2n} \right)^4 \left[ \prod_{n=1}^{\infty} \left( 1+p^{-2n-1} \right)^8 -32p \prod_{n=1}^{\infty} \left( 1+p^{-2n} \right)^8 \right] \).

(c) \( \varphi \left( \frac{1}{2} \right) = \frac{\pi^2}{3} \prod_{n=1}^{\infty} \left( 1-p^{-2n} \right)^4 \left( 2 \prod_{n=1}^{\infty} \left( 1+p^{-2n-1} \right)^8 -16p \prod_{n=1}^{\infty} \left( 1+p^{-2n} \right)^8 \right) \).

(d) \( g_2(\tau) = \frac{4\pi^4}{3} \prod_{n=1}^{\infty} \left( 1-p^{-2n} \right)^8 \left[ \prod_{n=1}^{\infty} \left( 1+p^{-2n-1} \right)^{16} -16p \prod_{n=1}^{\infty} \left( 1+p^{-2n} \right)^8 +256p^2 \prod_{n=1}^{\infty} \left( 1+p^{-2n} \right)^{16} \right] \).
(e) \( g_2(\tau) = \frac{8\pi^6}{27} \prod_{n=1}^{\infty} (1 - p^{2n})^{12} \left( \prod_{n=1}^{\infty} (1 + p^{2n-1})^{24} - 24p \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} (1 + p^{2n})^8 - 384p^2 \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 (1 + p^{2n})^{16} + 4096p^3 \prod_{n=1}^{\infty} (1 + p^{2n})^{24} \right). \)

(f) \( j(\tau) = \frac{1}{p^2} \left[ \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} - 16p \prod_{n=1}^{\infty} (1 + p^{n})^8 + 256p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{16} \right]^3. \)

Jacobi ([23, p.470]) showed that

\[ \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 - \prod_{n=1}^{\infty} (1 - p^{2n-1})^8 = 16p \prod_{n=1}^{\infty} (1 + p^{2n})^8. \]

By the Jacobi relation, we derive from Proposition 1 that

\[
\varrho \left( \frac{\tau}{2} \right) \]

(1) \[= \frac{\pi^2}{3} \prod_{n=1}^{\infty} (1 - p^{2n})^4 \left( -2 \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 + \prod_{n=1}^{\infty} (1 - p^{2n-1})^8 \right), \]

(2) \[= \frac{\pi^2}{3} \prod_{n=1}^{\infty} (1 - p^{2n})^4 \left( \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 - 2 \prod_{n=1}^{\infty} (1 - p^{2n-1})^8 \right), \]

(3) \[= \frac{\pi^2}{3} \prod_{n=1}^{\infty} (1 - p^{2n})^4 \left( \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 + \prod_{n=1}^{\infty} (1 - p^{2n-1})^8 \right), \]

(4) \[= \frac{4\pi^4}{3} \prod_{n=1}^{\infty} (1 - p^{2n})^8 \left[ \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16} - \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 (1 - p^{2n-1})^8 \right], \]
\[ g_3(\tau) = \frac{4\pi^6}{27} \prod_{n=1}^{\infty} (1-p^{2n})^{12} \left( -2 \prod_{n=1}^{\infty} (1+p^{2n-1})^{24} \right. \\
+ 3 \prod_{n=1}^{\infty} (1+p^{2n-1})^{16} (1-p^{2n-1})^8 \\
+ 3 \prod_{n=1}^{\infty} (1+p^{2n-1})^8 (1-p^{2n-1})^{16} - 2 \prod_{n=1}^{\infty} (1-p^{2n-1})^{24} \right), \]

\[ j(\tau) = \frac{1}{p^2} \left[ \prod_{n=1}^{\infty} (1+p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1-p^{2n-1})^{16} \right. \\
- \prod_{n=1}^{\infty} (1+p^{2n-1})^8 (1-p^{2n-1})^8 \right]^3, \]

\[ \varphi''(\frac{\tau}{2}) = 6\varphi(\frac{\tau}{2})^2 - \frac{1}{2} g_2(\tau) \quad ([2, p.332]) \]
\[ = 2\pi^2 \prod_{n=1}^{\infty} (1-p^{2n})^8 (1+p^{2n-1})^8 \]

\[ \times \left( \prod_{n=1}^{\infty} (1+p^{2n-1})^8 - \prod_{n=1}^{\infty} (1-p^{2n-1})^8 \right), \]

\[ \varphi''(\frac{\tau + 1}{2}) = 2\pi^2 \prod_{n=1}^{\infty} (1-p^{2n})^8 (1-p^{2n-1})^8 \]
\[ \times \left( \prod_{n=1}^{\infty} (1-p^{2n-1})^8 - \prod_{n=1}^{\infty} (1+p^{2n-1})^8 \right), \]

\[ \varphi''(\frac{1}{2}) = 2\pi^2 \prod_{n=1}^{\infty} (1-p^{2n})^8 (1-p^{2n-1})^8 (1+p^{2n-1})^8. \]

By using different expressions of these infinite products, Borcherds studied the denominator functions of generalized Kac-Moody algebra ([5], [6]).

Since \( \Delta(\tau) \) is nonzero and \( \prod_{n=1}^{\infty} (1+p^{2n})(1-p^{2n-1})(1+p^{2n-1}) = 1 \), we see that \( \prod_{n=1}^{\infty} (1-p^{2n}), \prod_{n=1}^{\infty} (1+p^{2n}), \prod_{n=1}^{\infty} (1+p^{2n-1}), \prod_{n=1}^{\infty} (1-p^{2n-1}) \) are nonzero. So, by (1)-(6), we end up with the following:

**Theorem 2.** Let \( \tau \in \mathfrak{h} \cap \mathfrak{h} \).

(a) \( \varphi\left(\frac{\tau}{2}\right) = 0 \iff \rho = 2 \iff \varphi\left(\frac{1}{2}\right) = -\varphi\left(\frac{\tau + 1}{2}\right) \)
\[ = \pi^2 \prod_{n=1}^{\infty} (1-p^{2n})^4 (1+p^{2n-1})^8. \]
(b) \( \varphi \left( \frac{\tau + 1}{2} \right) = 0 \iff \rho = \frac{1}{2} \iff \varphi \left( \frac{1}{2} \right) = -\varphi \left( \frac{\tau}{2} \right) \)
\[
= \pi^2 \prod_{n=1}^{\infty} (1 - p^{2n})^4 (1 - p^{2n-1})^8.
\]
(c) \( \varphi \left( \frac{1}{2} \right) = 0 \iff \rho = -1 \iff \varphi \left( \frac{\tau}{2} \right) = -\varphi \left( \frac{\tau + 1}{2} \right) \)
\[
= \pi^2 \prod_{n=1}^{\infty} (1 - p^{2n})^4 (1 - p^{2n-1})^8.
\]
(d) \( g_2(\tau) = j(\tau) = 0 \iff \rho^3 = -1 \text{ with } \rho \neq -1. \)
(e) \( g_3(\tau) = 0 \iff \rho = \frac{1}{2}, -1 \text{ or } 2. \)
(f) \( \varphi'' \left( \frac{1}{2} \right), \varphi'' \left( \frac{\tau}{2} \right) \) and \( \varphi'' \left( \frac{\tau + 1}{2} \right) \) are all nonzero and distinct.

Here we refer to [21, p.63] for (f).

Note that Eichler and Zagier [11] found the values of \( z \) which are zeros of \( \varphi(z, \tau) \) (\( z \in \mathbb{C} \)), i.e., the zeros of \( \varphi(z, \tau)(\tau \in \mathfrak{h}, z \in \mathbb{C}) \) are given by
\[
z = m + \frac{1}{2} + n\tau \pm \left( \frac{\log(5 + 2\sqrt{6})}{2\pi i} + 144\pi i \sqrt{6} \int_{\tau}^{i\infty} \frac{\Delta(t)}{E_6(t)^{3/2}} dt \right)
\]
(\( m, n \in \mathbb{Z} \)), where \( E_6(t) \) and \( \Delta(t) \) (\( t \in \mathfrak{h} \)) denote the normalized Eisenstein series of weight 6 and unique normalized cusp form of weight 12 on \( SL_2(\mathbb{Z}) \), respectively, and the integral is to be taken over the line \( t = \tau + i\mathbb{R}_+ \) in \( \mathfrak{h} \).

§3. Algebraic and transcendental numbers

Let \( \alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) with \( b \) mod \( d \) and \( |\alpha| \) be the determinant of \( \alpha \), and let
\[
(7) \quad \phi_\alpha(\tau) := |\alpha|^{12} \frac{\Delta (\alpha(\tau))}{\Delta (\tau)} = |\alpha|^{12} d^{-12} \frac{\Delta (\alpha \tau)}{\Delta (\tau)}.
\]

Then we recall the following fact.

**Proposition 3**. ([16]) For any \( \tau \in k \cap \mathfrak{h} \), the value \( \phi_\alpha(\tau) \) is an algebraic integer, which divides \( |\alpha|^{12} \).

First, we consider
\[
\frac{\Delta(\tau)}{\Delta \left( \frac{\tau}{2} \right)} = \frac{(2\pi)^{12} p^2 \prod_{n=1}^{\infty} (1 - p^{2n})^{24}}{(2\pi)^{12} p \prod_{n=1}^{\infty} (1 - p^n)^{24}} = p \prod_{n=1}^{\infty} (1 + p^n)^{24}
\]
and
\[
\Delta(\frac{\tau}{2}) = \frac{(2\pi)^{12} p \prod_{n=1}^{\infty} (1 - p^n)^{24}}{(2\pi)^{12} p^2 \prod_{n=1}^{\infty} (1 - p^{2n})^{24}} = p^{-1} \frac{1}{\prod_{n=1}^{\infty} (1 + p^n)^{24}}.
\]

Put \( \alpha_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \).

By (7) we get
\[
\phi_{\alpha_1}(\frac{\tau}{2}) = 2^{12} \frac{\Delta(\tau)}{\Delta(\frac{\tau}{2})} = 2^{12} \frac{\eta(\tau)^{24}}{\eta(\frac{\tau}{2})^{24}},
\]

from which and Proposition 3 we see that \( \sqrt{2} p^{1/4} \prod_{n=1}^{\infty} (1 + p^n) \) is an algebraic integer.

Using Proposition 3, we get that both
\[
\sqrt{a} \frac{\eta(\frac{a\tau + b}{d})}{\eta(\tau)} \quad \text{and} \quad \sqrt{a} \frac{\eta(\tau)}{\eta(\frac{a\tau + b}{d})}
\]
are algebraic integers.

Similarly, we get the following properties.

**Proposition 4.** ([13]) Let \( \tau \in k \cap \mathfrak{h} \). Then the following assertions hold:

(a) \( \sqrt{2} p^{1/4} \prod_{n=1}^{\infty} (1 + p^n), p^{-1/4} \prod_{n=1}^{\infty} (1 - p^n), p^{-1/2} \prod_{n=1}^{\infty} (1 + p^{2n-1}), \)
\( \sqrt{2} \prod_{n=1}^{\infty} (1 + p^n)(1 - p^{2n-1}), p^{-1/2} \prod_{n=1}^{\infty} (1 + p^{2n-1}) \) and
\( \sqrt{2} \prod_{n=1}^{\infty} (1 + p^n)(1 + p^{2n-1}) \) are algebraic integers.

(b) \( \frac{3 \varphi(\frac{\tau}{2})}{\pi^2 \eta(\tau)^4}, \frac{\varphi(\frac{\tau + 1}{2})}{\pi^2 \eta(\tau)^4}, \frac{\varphi(\frac{1}{2})}{\pi^2 \eta(\tau)^4}, \frac{g_2(\tau)}{\pi^2 \eta(\tau)^4}, \frac{27 g_3(\tau)}{\pi^2 \eta(\tau)^4} \) and \( \varphi(\frac{\tau - 1}{2}) - \varphi(\frac{\tau + 1}{2}) - \varphi(\frac{1}{2}) \) are algebraic integers.

In general, set \( \alpha_t = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \) and \( \alpha_t = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \) with \( a \in \mathbb{Z}^+ \). Then we get again by (7) that
\[
\phi_{\alpha_1}(\frac{\tau}{2})^{1/24} = \sqrt{ap}^{1/4}(a-1)^{\infty} \prod_{n=1}^{\infty} \left( \sum_{m=0}^{a-1} p^{mn} \right),
\]

(10) \( \phi_{\alpha_1}(\frac{\tau}{2})^{1/24} = p^{1/4}(a-1)^{\infty} \frac{1}{\prod_{n=1}^{\infty} \left( \sum_{m=0}^{a-1} p^{mn} \right)} \).
are algebraic integers.

In 1949, Gelfond and Schneider independently solved the Hilbert 7-th problem concerning the transcendence of \( 2^{\sqrt{2}} \). They actually proved the following strong transcendence criterion. For \( \alpha, \beta \in \mathbb{Q} \) with \( \alpha \neq 0,1 \) and \( \beta \notin \mathbb{Q} \), \( \alpha^\beta \) is transcendental ([20], [22]). Thus, for \( \tau \in \mathfrak{h} \), the Gelfond-Schneider theorem yields that \( e^{i\alpha \tau} = (-1)^{-i\alpha} \) is transcendental whenever \( i\alpha \) is algebraic of degree at least 2 over \( \mathbb{Q} \). This leads us to the fact that

\[
(11) \quad p = e^{\pi i \tau} \text{ is a transcendental number.}
\]

**Corollary 5.** Let \( \tau \in \mathfrak{h} \).

(a) If \( a \in \mathbb{Q} - \{-\frac{1}{24}\} \) then \( p^a \prod_{n=1}^{\infty}(1 + p^n) \) and \( p^a \prod_{n=1}^{\infty}(1 + p^{2n-1}) \) are transcendental numbers.

(b) If \( a \in \mathbb{Q} - \{0\} \) then \( p^a \prod_{n=1}^{\infty}(1 + p^n)(1 + p^{2n-1}) \) and \( p^a \prod_{n=1}^{\infty}(1 + p^n)(1 + p^{2n-1}) \) are transcendental numbers.

(c) If \( b \in \mathbb{Q} - \{\frac{1}{24}(a - 1)\} \) then \( p^b \prod_{n=1}^{\infty}(\sum_{m=0}^{a-1} p^{mn}) \) is a transcendental number with \( a \in \mathbb{Z}^+ \).

(d) Assume that \( g_2(\tau) \) and \( g_3(\tau) \) are nonzero. If \( \varphi(\frac{x}{2}) \) is transcendental, so are \( \varphi(\frac{x+1}{2}), \varphi(\frac{x}{2}), g_2(\tau), g_3(\tau) \) and \( \Delta(\tau) \). And if \( \varphi(\frac{x}{2}) \) is algebraic then so are \( \varphi(\frac{x+1}{2}), \varphi(\frac{x}{2}), g_2(\tau), g_3(\tau) \) and \( \Delta(\tau) \).

If \( f(\tau), g(\tau) \in \overline{\mathbb{Q}} \) \( (\varphi(\frac{x}{2}), \varphi(\frac{x+1}{2}), \varphi(\frac{x}{2}), g_2(\tau), g_3(\tau), \Delta(\tau)) \) and these are of the same even weight, then \( \frac{f(\tau)}{g(\tau)} \) is an algebraic number.

**Proof.** (a), (b), (c) Immediate from Proposition 4, (10) and (11).

(d) Since \( g_3(\tau) \) is nonzero, we see by (a), (b), (c), and (e) of Theorem 2 that \( \varphi(\frac{x+1}{2}), \varphi(\frac{3}{2}) \) and \( \varphi(\frac{1}{2}) \) are nonzero. By (1), (2), (3) and Proposition 4, we derive that

\[
(1') \quad \frac{\varphi(\frac{x+1}{2})}{\varphi(\frac{x}{2})} = \frac{2^x - 1}{2 - \rho}, \quad \frac{\varphi(\frac{3}{2})}{\varphi(\frac{1}{2})} = \frac{1 + \rho}{-2 + \rho}
\]

are algebraic numbers, and so is \( \frac{\varphi(\frac{3}{2})}{\varphi(\frac{x+1}{2})} \). By the assumption, (4), (5)
and \( (1') \), we get that

\[
g_2(\tau) = -4 \left[ \wp \left( \frac{1}{2} \right) \wp \left( \frac{\tau}{2} \right) + \wp \left( \frac{\tau + 1}{2} \right) \wp \left( \frac{\tau}{2} \right) + \wp \left( \frac{1}{2} \right) \wp \left( \frac{\tau + 1}{2} \right) \right]
\]

\[
= -4 \left[ \frac{1 + \rho}{-2 + \rho} \wp \left( \frac{\tau}{2} \right)^2 + \frac{2 \rho - 1}{2 - \rho} \wp \left( \frac{\tau}{2} \right)^2 + 2 \rho - 1 \cdot \frac{1 + \rho}{-2 + \rho} \wp \left( \frac{\tau}{2} \right)^2 \right]
\]

\[
= 12 \rho^2 - \rho + 1 \left( \frac{\tau}{2} \right)^2,
\]

\[
g_3(\tau) = -4 \left( \frac{2 \rho - 1}{\rho - 2} \right)^2 \wp \left( \frac{\tau}{2} \right)^3,
\]

\[
\Delta(\tau) = \frac{432}{(\rho - 2)^6} \{4(\rho^2 - \rho + 1)^3 - (\rho - 2)^2(2 \rho - 1)^2(\rho + 1)^2 \wp \left( \frac{\tau}{2} \right)^6
\]

\[
= 2^4 \cdot 3^6 \left( \frac{\rho - 1}{\rho - 2} \right)^2 \wp \left( \frac{\tau}{2} \right)^6.
\]

Using the above relation, we can readily check the final statement. \( \Box \)

Let

\[
\phi(\tau) := \phi(e^{\pi i \tau}) = \frac{\eta \left( \frac{\tau + 1}{2} \right)^2}{\eta(\tau + 1)} = \prod_{n=1}^{\infty} \left( 1 + \rho^{2n - 1} \right)^2 (1 - \rho^{2n}) = \theta_3(0, \tau).
\]

Here we refer to \([8, p.86]\) for the last equality. Berndt, Chan and Zhang showed in \([4]\) the following proposition by using three of Ramanujan’s modular equations, values for certain class invariants of Ramanujan, representations for quotients of values of \( \phi \) in terms of class invariants and the theta-transformation formula.

**Proposition 6.** ([4]) Let \( m \) and \( n \) be positive integers. Then \( \phi(mn) / \phi(ni) \) is algebraic. Furthermore, if \( m \) is odd, then \( \sqrt{2m} \phi(mn) / \phi(ni) \) is an algebraic integer dividing \( 2 \sqrt{m} \), while if \( m \) is even, then \( 2 \sqrt{m} \phi(mn) / \phi(ni) \) is an algebraic integer dividing \( 4 \sqrt{m} \).

In \([13]\) we derived certain analogues of these results purely in terms of infinite products, which would be a generalization in case \( m \) even.
PROPOSITION 7. ([13]) Let $\tau$ be any imaginary quadratic and $r, s, u, v$ be positive integers such that $(r, s) = (u, v) = 1$. Then $4\sqrt{rv} \frac{\phi(r, \tau)}{\phi(u, \tau)}$ is an algebraic integer. Furthermore, $2\sqrt{v} \frac{\phi(r, u \cdot \tau)}{\phi(u, v \cdot \tau)}$ and $2\sqrt{v} \frac{\phi(u, r \cdot \tau)}{\phi(u, v \cdot \tau)}$ are algebraic integers.

The same argument used in [13] to prove that $4\sqrt{rv} \frac{\phi(r, \tau)}{\phi(u, \tau)}$ is an algebraic integer can be applied to the following theorem.

THEOREM 8. Let $\tau \in k \cap \mathfrak{h}$.

(a) Let $\theta, \theta_2, \theta_3$ be the Jacobi theta functions ([8], [9]). Then $\theta_3(\tau) = \alpha \eta(\tau)$, $\theta(\tau) = \alpha' \eta(\tau)$, $\theta_2(\tau) = \eta(2\tau) \alpha'' = \frac{\eta(\tau)}{\sqrt{2}} \alpha''$, where $\alpha, \alpha'$, $\alpha'', \alpha'''$ are algebraic integers. Furthermore, we see that $a\sqrt{d} \frac{\theta(a^2 + b^2)}{\theta(\tau)}$, $\sqrt{a'd'} \frac{\theta(\tau)}{\theta(a^2 + b^2)}$, $a'd' \frac{\theta(a^2 + b^2)}{\theta(\tau)}$, $a'd' \frac{\theta(a^2 + b^2)}{\theta(\tau)}$ are algebraic integers for $a, a'd', d', \alpha'$ positive integers and $b, b'$ integers.

(b) Let us define

\[ a(\tau) := a(p^2) = \sum_{n,m=-\infty}^{\infty} p^2(n^2 + nm + m^2), \]
\[ b(\tau) := b(p^2) = \sum_{n,m=-\infty}^{\infty} \omega^{n-m} p^2(n^2 + nm + m^2), \]
\[ c(\tau) := c(p^2) = \sum_{n,m=-\infty}^{\infty} p^2((n + \frac{1}{2})^2 + (n + \frac{1}{2})(m + \frac{1}{2}) + (m + \frac{1}{2})^2), \]

where $\omega := e^{\frac{2\pi i}{d}}$.

Then $a(\tau) = \frac{\beta}{\beta'} b(\tau)$, $c(\tau) = \frac{\beta''}{\beta'''} b(\tau)$, $a(\tau) = \beta'' c(\tau)$ and $b(\tau) = \beta''' c(\tau)$, where $\beta, \beta', \beta'', \beta'''$ are algebraic integers. Furthermore, $m\sqrt{md} \frac{\beta^{(m', d, d')}}{b(\tau)}$, $d'\sqrt{m'd'} \frac{b(\tau)}{b^{(m', d, d')}}$, $md' \sqrt{mm'd'd'} \frac{b^{(m', d, d')}}{b^{(m', d, d')}}$, $md' \sqrt{mm'd'd'} \frac{c^{(m', d, d')}}{c^{(m', d, d')}}$ are algebraic integers with $m, m', d, d'$ positive integers and $n, n'$ integers.
Proof. (a) We know from [8] that \( \theta_3(\tau) := \theta_3(p^2) = \prod_{n=1}^{\infty} \frac{\eta(\frac{\tau+1}{2})}{\eta(\tau+1)} \). Since
\[
\frac{\theta_3(\tau)}{\eta(\tau)} = \prod_{n=1}^{\infty} \frac{(1 + p^{2n-1})^2(1 - p^{2n})}{p^{1/12}(1 - p^{2n})} = p^{-1/2} \prod_{n=1}^{\infty} (1 + p^{2n-1})^2,
\]
we get that there exists an algebraic integer \( \alpha \) satisfying \( \theta_3(\tau) = \alpha \eta(\tau) \).

By [9], we have the identities
\[
(12) \quad \theta(\tau) := \theta(p^2) = \frac{\eta(\tau)^2}{\eta(2\tau)} \quad \text{and} \quad \theta_2(\tau) := \theta_2(p^2) = \frac{2\eta(4\tau)^2}{\eta(2\tau)}.
\]

And we deduce from (9) and (12) that
\[
\frac{\theta(\tau)}{\eta(\tau)} = \frac{\eta(\tau)^2}{\eta(2\tau)\eta(\tau)} = \frac{\eta(\tau)}{\eta(2\tau)},
\]
\[
\frac{\theta_2(\tau)}{\eta(2\tau)} = \frac{2\eta(4\tau)^2}{\eta(2\tau)^2} = \left( \frac{\sqrt{2}\eta(4\tau)}{\eta(2\tau)} \right)^2,
\]
\[
\sqrt{2} \frac{\theta_3(\tau)}{\eta(\tau)} = 2\sqrt{2} \frac{\eta(4\tau)^2}{\eta(\tau)\eta(2\tau)} = \left( 2 \frac{\eta(4\tau)}{\eta(\tau)} \right) \left( \sqrt{2} \frac{\eta(4\tau)}{\eta(2\tau)} \right)
\]
are algebraic integers. Also, it follows from (12) that
\[
\sqrt[\alpha d']^\frac{\theta(a\tau+b)}{\eta(\tau)} = \left( \begin{array}{cccc}
a & b & \frac{1}{2} & \eta
d & 0 & \frac{1}{2} & \frac{\eta\left(\frac{b}{d}(\frac{\tau}{2})\right)}{\eta(\tau)}
da & 0 & \frac{1}{2} & \frac{\eta\left(\frac{a\tau+b}{d}\right)}{\eta(\tau)}
\end{array} \right)^2
\]
\[
\sqrt[\alpha d']^\frac{\theta(\tau)}{\eta(a\tau+b)} = \left( \begin{array}{cccc}
a' & 2b' & \frac{1}{2} & \eta\left(\frac{a'\tau+2b'}{d'}\right)
d' & 0 & \frac{1}{2} & \eta(2\tau)
da' & 0 & \frac{1}{2} & \eta(\tau)
\end{array} \right)^2
\]
\[
= \left( \begin{array}{cccc}
d' & -2b' & \frac{1}{2} & \eta\left(\frac{a'\tau+2b'}{d'}\right)
da & 0 & \frac{1}{2} & \frac{\eta\left(\frac{b'}{d'}\right)}{\eta(\tau)}
da' & 0 & \frac{1}{2} & \eta\left(\frac{a\tau+b}{d}\right)
\end{array} \right)^2.
\]
are algebraic integers.

Similarly, we conclude that 
\[ ad' \sqrt{a'd'} \frac{\theta_2(a\tau + b\tau_2)}{\theta_2(a\tau_2 + b\tau)} \]

is an algebraic integer.

(b) J. Borwein, P. Borwein and Garvan showed in [7] that 
\[ b(\tau) = \frac{n^3(\tau)}{n(3\tau)}, \quad c(\tau) = \frac{n^3(3\tau)}{n(\tau)} \]

and \( a(\tau)^3 = b(\tau)^3 + c(\tau)^3 \). Thus we get
\[ \frac{b(\tau)}{c(\tau)} = \left( \frac{n(3\tau)}{n(\tau)} \right)^4 \quad \text{and} \quad \frac{c(\tau)}{b(\tau)} = \frac{1}{9} \left( \frac{\sqrt{3}n(3\tau)}{n(\tau)} \right)^4. \]

Put \( \alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \) and \( \alpha_4 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \). By using (7), we obtain that 
\[ \frac{b(\tau)}{c(\tau)} \] and \( \frac{9a(\tau)}{b(\tau)} \) are algebraic integers. Since \( a(p^2)^3 = b(p^2)^3 + c(p^2)^3 \), we conclude that \( \frac{a(\tau)}{c(\tau)} \) and \( \frac{9a(\tau)}{b(\tau)} \) are algebraic integers. Also, we get that
\[ m\sqrt{md} \frac{b(\tau)}{b(\tau)} = \left( \frac{\sqrt{m}n(\tau)}{n(\tau)} \right)^3 \cdot \left( \frac{\sqrt{d}n(3\tau)}{n(3\tau)} \right)^3, \]
\[ d'\sqrt{md'} \frac{b(\tau)}{b(\tau)} = \left( \frac{\sqrt{m}n(\tau)}{n(\tau)} \right)^3 \cdot \left( \frac{\sqrt{d}n(3\tau)}{n(3\tau)} \right)^3, \]

are algebraic integers.

Duverney, Ke. Nishioka, Ku. Nishioka, and Shiokawa proved in [10] that the Rogers-Ramanujan continued fraction \( RR(z) \) is transcendental for any algebraic number \( z \) with \( 0 < |z| < 1 \). Meanwhile, we consider in Theorem 9 some examples of the Rogers-Ramanujan continued fraction for the case of transcendental numbers \( z \).

Let
\[ F_1(p^2) := \frac{p^2}{1 + 1 + \cdots}, \]
\[ F_2(p^2) := \frac{p^2}{1 + p^2 + \cdots}, \]
\[ F_3(p^2) := \frac{p^2}{1 + p^6 + \cdots}, \]
\[ F_4(p^2) := \frac{p^2 + p^2}{1 + 1 + \cdots} \]
be the continued fractions as in [10].
THEOREM 9. If \( l_1 \neq -\frac{4}{5}, \ l_2 \neq \frac{1}{4}, \ l_3 \neq 0, \ l_4 \neq \frac{2}{3} \) are rational numbers then \( p^{l_1} \left( \frac{1}{F_1(p^2)} - F_1(p^2) - 1 \right), \ p^{l_2} F_2(p^2), \ p^{l_3} \left( \frac{1}{F_3(p^2)} + F_3(p^2) \right) \), \( p^{l_4} F_4(p^2) \) are transcendental numbers.

Furthermore,

\[
p^{-\frac{1}{2}} \left( \frac{1}{F_1(p^2)} - F_1(p^2) - 1 \right), \ 2p^{\frac{1}{4}} F_2(p^2), \ 2 \left( \frac{1}{F_3(p^2)} + F_3(p^2) \right)
\]

and \( 2\sqrt{2} p^{\frac{1}{2}} F_4(p^2) \) are algebraic integers.

Proof. First, we consider the value of continued fraction \( F_1(p^2) \). It follows from \([3]\) and \([10]\) that \( \frac{1}{F_1(p^2)} - F_1(p^2) - 1 = p^{\frac{1}{2}} \frac{\eta(\frac{1}{5}\tau)}{\eta(5\tau)} \).

Set \( \alpha = \begin{pmatrix} 1 & 0 \\ 0 & 25 \end{pmatrix} \). Then we obtain by (9) that

\[
p^{-\frac{1}{2}} \left( \frac{1}{F_1(p^2)} - F_1(p^2) - 1 \right) = \frac{\eta(\frac{1}{5}\tau)}{\eta(5\tau)}
\]

is an algebraic integer. By (11) and (13) we are led to the fact that

\[ p^{l_1} \left( \frac{1}{F_1(p^2)} - F_1(p^2) - 1 \right) \]

is a transcendental number except for the case \( l_1 = -\frac{4}{5} \).

Secondly, it follows from \([3, p.221], [8], [9], [10], [12, p.186]\) that

\[
F_2(p^2) = \frac{\theta_2(p)}{2p^{\frac{1}{4}} \theta_3(p^2)} = \frac{1}{2p^{\frac{1}{4}}} \frac{2\eta(2\tau)^2 \eta(4\tau)^2 \eta(\tau)^2}{\eta(2\tau)^5} = \frac{1}{p^{\frac{1}{4}}} \frac{\eta(4\tau)^2 \eta(\tau)}{\eta(2\tau)^3} = \frac{1}{2p^{\frac{1}{4}}} \left( \frac{\sqrt{2} \eta(4\tau)}{\eta(2\tau)} \right)^2 \frac{\eta(\tau)}{\eta(2\tau)}.
\]

Thus, by (11) and (14) we derive that \( 2p^{\frac{1}{4}} F_2(p^2) \) is an algebraic integer and \( p^{l_2} F_2(p^2) \) is a transcendental number provided that \( l_2 \neq \frac{1}{4} \).
In a similar way we get that
\[
\frac{1}{F_3(p^2)} + F_3(p^2) = \frac{2\theta_3(p^2)}{\theta_2(p^4)} \quad ([3, \text{p.221}, \text{[10]})
\]
\[
= \frac{2\theta_3(p^2)}{\theta_2(p^4)} = \frac{2\eta(2\tau)^5 \eta(4\tau)}{\eta(4\tau)^2 \eta(\tau)^2 2\eta(8\tau)^2}
\]
\[
= \eta(2\tau)^5 \eta(4\tau) \eta(\tau)^2 \eta(8\tau)^2
\]
\[
= \frac{1}{2} \left( \frac{\eta(2\tau)}{\eta(4\tau)} \right)^2 \left( \frac{\eta(2\tau)}{\eta(\tau)} \right)^2.
\]
So \(2 \left( \frac{1}{F_3(p^2)} + F_3(p^2) \right)\) is an algebraic integer; hence \(p^{l_3} \left( \frac{1}{F_3(p^2)} + F_3(p^2) \right)\) is a transcendental number unless \(l_3 = 0\).

Finally, we deduce from [3, p.345] and [10] that
\[
F_4(p^2) = \frac{\eta(\tau)\eta(6\tau)^3}{p^3 \eta(2\tau)\eta(3\tau)^3} = \frac{1}{2\sqrt{2}p^3} \left( \frac{\eta(\tau)}{\eta(2\tau)} \right) \left( \sqrt{2} \frac{\eta(6\tau)}{\eta(3\tau)} \right)^3.
\]
Hence we conclude by (9) and (11) that \(2\sqrt{2}p^{3/2} F_4(p^2)\) is an algebraic integer and \(p^{l_4} F_4(p^2)\) is a transcendental number when \(l_4 \neq \frac{2}{3}\) \(\Box\)

Since \(j(\tau)\) is an algebraic integers
\[
(15) \quad \frac{1}{27} - \frac{64}{j(\tau)} = \frac{g_3(\tau)^2}{g_2(\tau)^3}
\]
is an algebraic number.

**Theorem 10.** Let “η” denote the derivation \(z \frac{d}{dz}\) and \(J(p^2) = J(\tau)\). Assume that \(g_2(\tau)\) and \(g_3(\tau)\) are nonzero. Then \(\eta(\tau)^4 \frac{Q(p^2)}{R(p^2)}\) and \(\frac{J'(\tau)}{\eta(\tau)^4}\) are algebraic numbers.
Proof. By Proposition 4, we note that

\[
\frac{Q(p^2)}{R(p^2)} = \frac{1}{\eta(\tau)^4} \frac{\frac{3}{4\pi^4} \frac{g_2(\tau)}{\eta(\tau)^6}}{\frac{27}{8\pi^6} \frac{g_3(\tau)}{\eta(\tau)^{12}}}
\]

Since \(\frac{3}{4\pi^4} \frac{g_2(\tau)}{\eta(\tau)^6}\) and \(\frac{27}{8\pi^6} \frac{g_3(\tau)}{\eta(\tau)^{12}}\) are algebraic numbers, \(\eta(\tau)^4 \frac{Q(p^2)}{R(p^2)}\) is an algebraic number. We see from [9] that \(\frac{J'(\tau)}{J(\tau)} = -\frac{R(p^2)}{Q(p^2)}\).

By the above, we get that

\[
\frac{J'(\tau)}{\eta(\tau)^2} = -\left(\frac{\eta(\tau)^4 \frac{Q(p^2)}{R(p^2)}}{\eta(\tau)^2}\right)^{-1} J(\tau)
\]

is an algebraic number.

\[\square\]

§4. Elliptic curves and infinite products

In this section, we assume that \(j = j(\tau) \neq 0\) and \(R(p^2) \neq 0\). It follows from (4), (5) and the definition of \(j(\tau)\) that there exists a polynomial

\[
M(x) = x^5 - 3x^5 + \left(6 - \frac{j}{256}\right)x^4 + (-7 + \frac{j}{128})x^3
\]

\[+ \left(6 - \frac{j}{256}\right)x^2 - 3x + 1,
\]

with \(j = j(\tau)\) such that \(M \left(\prod_{n=1}^{\infty} \left(\frac{1 - p^{2n-1}}{1 + p^{2n-1}}\right)^8\right) = 0\). We then derive by (16) that

\[
[\mathbb{Q}\left(\prod_{n=1}^{\infty} \left(\frac{1 - p^{2n-1}}{1 + p^{2n-1}}\right)^8\right) : \mathbb{Q}(j(\tau))] \leq 6.
\]

Since

\[
\prod_{n=1}^{\infty} \left(\frac{1 - p^{2n-1}}{1 + p^{2n-1}}\right) = \frac{\theta_4^2(0, \tau)}{\theta_4^2(0, \tau)}
\]

([2, p.362]), we have \([\mathbb{Q}(\theta_4^2(0, \tau)) : \mathbb{Q}(j(\tau))] \leq 6\).
Put $\delta = \rho + \frac{1}{\rho}$. Then we obtain
\[
M(\rho) = \rho^3 \left( \delta^3 - 3\delta^2 + (3 - \frac{j}{256})\delta - 1 + \frac{j}{128} \right) = 0.
\]
From the Jacobi's relation we have
\[
\prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16} = 16^2 p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{16} + \frac{2}{\prod_{n=1}^{\infty} (1 + p^{2n})^8}
\]
and hence
\[
16^2 p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{24}
\]
\[
= \left( \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16} \right) \prod_{n=1}^{\infty} (1 + p^{2n})^8 - 2
\]
\[
= \left( \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16} \right) \prod_{n=1}^{\infty} (1 + p^{2n})^8 - 2
\]
\[
= (\rho + \frac{1}{\rho}) - 2.
\]
Let $\kappa = 16^2 p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{24}$.
Since $M(\rho) = 0$, there exists a polynomial
\[
N(z) = z^3 + 3z^2 + (3 - \frac{j}{256})z + 1
\]
with $j = j(\tau)$ satisfying $N(\kappa) = 0$. This implies that
\[
[\mathbb{Q}(p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{24} : \mathbb{Q}(j)] \leq 3.
\]
Hence we can construct an elliptic curve
\[
(16') \quad E : y^2 = x^3 + 3x^2 + (3 - \frac{j}{256})x + 1
\]
with $j = j(\tau)$ satisfying $P = (16^2 p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{24}, 0) \in E$. Here, the discriminant of $E$ is
\[
\Delta(E) = \frac{j^2(-1728 + j)}{262144} \quad \text{and} \quad j(E) = \frac{1728j}{-1728 + j}.
\]
We summarize the above as follows.
THEOREM 11. Let $R(p^2) \neq 0$.

(a) $[\mathbb{Q}(\prod_{n=1}^{\infty} \left(1 - \frac{p^{2n-1}}{1 + p^{2n-1}} \right)^8 : \mathbb{Q}(j(\tau)))] \leq 6$ and $[\mathbb{Q}(p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{24} : \mathbb{Q}(j))] \leq 3$.

(b) Let $E : y^2 = x^3 + 3x^2 + (3 - \frac{j}{256})x + 1$ be an elliptic curve. Then $P = (16^2 p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{24}, 0) \in E$, $\Delta(E) = \frac{r^3(-1728 + j)}{262144}$ and $j(E) = \frac{1728j}{-1728 + j}$ with $j = j(\tau)$.

§5. Other examples with infinite products

In general, by (16) we find the values of $\prod_{n=1}^{\infty} (1 - \frac{p^{2n-1}}{1 + p^{2n-1}})^8$ when $j(\tau)$ is known.

Put $\prod_{n=1}^{\infty} (1 - \frac{p^{2n-1}}{1 + p^{2n-1}})^8 = \rho$. By (1'), it follows that

$$\frac{\varphi(\frac{\tau^1 + 1}{2})}{\varphi(\frac{\tau}{2})} = \frac{2\rho - 1}{2 - \rho} \quad \text{and} \quad \frac{\varphi(\frac{1}{2})}{\varphi(\frac{5}{2})} = \frac{1 + \rho}{-2 + \rho}.$$

Let

$$f_0(z; \tau) = -27^3 \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)} \varphi(z, \Lambda_{\tau})$$

be the first Weber function for $\tau \in \mathfrak{h}$ and $z \in \mathbb{C}$. For a fixed integer $N > 1$ and $r, s$ in $\mathbb{Z}$ not both divisible by $N$, let

$$f_{r,s}(\tau) = f_0\left(\frac{r\tau + s}{N}; \tau\right)$$

be the Fricke function.

By (1), (4) and (5) we obtain the identity for $f_{1,0}$:

$$f_{1,0}$$

$$= f_0\left(\frac{\tau}{2}; \tau\right)$$

$$= -27^3 \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)} \varphi\left(\frac{\tau}{2}\right)$$

$$= -\frac{1}{2p^2} \left\{ \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16} \right.$$  

$$- \prod_{n=1}^{\infty} (1 + p^{2n-1})^8(1 - p^{2n-1})^8 \right\} \cdot \left\{ -2 \prod_{n=1}^{\infty} (1 + p^{2n-1})^{24} - 2 \prod_{n=1}^{\infty} (1 - p^{2n-1})^{24} \right.$$
+ 3 \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} (1 - p^{2n-1})^8 \\
+ 3 \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 (1 - p^{2n-1})^{16} \}
\cdot \left\{ -2 \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 + \prod_{n=1}^{\infty} (1 - p^{2n-1})^8 \right\}.

By the Jacobi relation, we derive that
\[
\frac{1}{p^3} = 16^2 \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} \Bigg/ \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} \\
- 2 \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 (1 - p^{2n-1})^8 + \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16}.
\]

Since \( \prod_{n=1}^{\infty} (1 + p^{2n-1})(1 - p^{2n-1})(1 + p^n) = 1 \), we end up with the following:

\[
f_{1,0} = -2^7 \left[ \left( \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16} \right.ight.
\left. - \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 (1 - p^{2n-1})^8 \right]
\cdot \left\{ -2 \prod_{n=1}^{\infty} (1 + p^{2n-1})^{24} - 2 \prod_{n=1}^{\infty} (1 - p^{2n-1})^{24} \\
+ 3 \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} (1 - p^{2n-1})^8 \\
+ 3 \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 (1 - p^{2n-1})^{16} \}
\left. \cdot \left\{ -2 \prod_{n=1}^{\infty} (1 + p^{2n-1})^8 + \prod_{n=1}^{\infty} (1 - p^{2n-1})^8 \right\} \right] / \\
\left[ \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16} (\prod_{n=1}^{\infty} (1 + p^{2n-1})^8 \\
- \prod_{n=1}^{\infty} (1 - p^{2n-1})^8)^2 \right].\]
We are also able to express \( j(\tau) \) as

\[
j(\tau) = 2^8 \left[ \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} + \prod_{n=1}^{\infty} (1 - p^{2n-1})^{16} \right. \\
\left. \quad - \prod_{n=1}^{\infty} (1 + p^{2n-1})^{8} (1 - p^{2n-1})^{8} \right] / \left[ \prod_{n=1}^{\infty} (1 + p^{2n-1})^{16} (1 - p^{2n-1})^{16} \right. \\
\left. \quad \cdot \left\{ \prod_{n=1}^{\infty} (1 + p^{2n-1})^{8} - \prod_{n=1}^{\infty} (1 - p^{2n-1})^{8} \right\}^2 \right].
\]

We can apply (1), (4), and (5) to \( \frac{f_{1,0}}{f_{0,1}} = -2 \cdot 3^2 \frac{\wp}{g_2} \wp (\frac{\tau}{2}) \). Then we can use

\[
\frac{f_{1,0}}{f_{0,1}} = \frac{\wp (\frac{\tau}{2})}{\wp (\frac{\tau}{4})}, \quad \frac{f_{1,0}}{f_{1,1}} = \frac{\wp (\frac{\tau}{2})}{\wp (\frac{\tau+1}{2})}.
\]

Thus we get the following:

\[
\begin{align*}
 f_{1,0} &= -\frac{1}{2} \frac{(-2 + 3\rho + 3\rho^2 - 2\rho^3)(-2 + \rho)}{(1 - \rho + \rho^2)^2} \cdot j, \\
 f_{0,1} &= -\frac{1}{2} \frac{(-2 + 3\rho + 3\rho^2 - 2\rho^3)(1 + \rho)}{(1 - \rho + \rho^2)^2} \cdot j, \\
 f_{1,1} &= -\frac{1}{2} \frac{(-2 + 3\rho + 3\rho^2 - 2\rho^3)(1 - 2\rho)}{(1 - \rho + \rho^2)^2} \cdot j.
\end{align*}
\]

We denote by \( F_{N,C} \) the field of modular functions of level \( N \). As is known([16]),

\[
F_{1,C} = \mathbb{C}(j) \quad \text{and} \quad F_{N,C} = F_{1,C}(f_{r,s})_{\text{all } r,s} = \mathbb{C}(j, f_{r,s})_{\text{all } r,s}.
\]

From the identities of \( j(\tau) \), \( f_{1,0} \), \( f_{0,1} \) and \( f_{1,1} \) we see that

\[
F_{2,C} = \mathbb{C}(p^{-\frac{3}{2}} \prod_{n=1}^{\infty} (1 + p^{2n-1})^8, p^{-\frac{3}{2}} \prod_{n=1}^{\infty} (1 - p^{2n-1})^8).
\]

We summarize the above in the following theorem.

**Theorem 12.** (a) \[
\frac{f_{1,1}}{f_{1,0}} = \frac{\wp (\frac{\tau+1}{2})}{\wp (\frac{\tau}{2})} = \frac{2\rho - 1}{2 - \rho},
\]

\[
\frac{f_{0,1}}{f_{1,0}} = \frac{\wp (\frac{\tau}{2})}{\wp (\frac{\tau}{4})} = \frac{1 + \rho}{-2 + \rho}.
\]
(b)

\[ f_{1,0} = \frac{1}{2} \frac{(-2 + 3\rho + 3\rho^2 - 2\rho^3)(-2 + \rho)}{(1 - \rho + \rho^2)^2} \cdot j, \]

\[ f_{0,1} = \frac{1}{2} \frac{(-2 + 3\rho + 3\rho^2 - 2\rho^3)(1 + \rho)}{(1 - \rho + \rho^2)^2} \cdot j, \]

\[ f_{1,1} = \frac{1}{2} \frac{(-2 + 3\rho + 3\rho^2 - 2\rho^3)(1 - 2\rho)}{(1 - \rho + \rho^2)^2} \cdot j. \]

(c) \( \mathbb{F}_{2,\mathbb{C}} = \mathbb{C}(p^{-\frac{1}{3}} \prod_{n=1}^{\infty} (1 + p^{2n-1})^8, p^{-\frac{1}{3}} \prod_{n=1}^{\infty} (1 - p^{2n-1})^8). \)

**Example 13.** We consider the cases of complex multiplication. We know from [21] all elliptic curves defined over \( \mathbb{Q} \) with complex multiplication by an order \( R = \mathbb{Z} + fR_k \) of conductor \( f \) in a quadratic imaginary field \( K = \mathbb{Q}(\sqrt{-D}) \) of discriminant \( -D \). By using (15), (16) and (16'), we get the following:

**Case** \( D = 7, f = 1: \) \( j(E) = -3^35^3, M(x) = x^6 - 3x^5 + \frac{4911}{256}x^4 - \frac{4271}{256}x^3 + \frac{4911}{256}x^2 - 3x + 1, E : y^2 = x^3 + 3x^2 + \frac{4143}{256}x + 1. \)

By using Mathematica 4.0, we write the value of \( 162p^2 \prod_{n=1}^{\infty} (1 + p^{2n})^{24} \) with \( p = e^{\sqrt{-1}(\frac{1+\sqrt{7}}{2})} \) as follows: \( -\frac{1}{16} \) or \( \frac{1}{32}(47 - 45\sqrt{-7}) \) or \( \frac{1}{32}(47 + 45\sqrt{-7}). \)

And by using Theorem 12, we get the following:

<table>
<thead>
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<th>( D )</th>
<th>( f )</th>
<th>( j(E) )</th>
<th>( \rho )</th>
<th>( f_{1,0} )</th>
<th>( f_{0,1} )</th>
<th>( f_{1,1} )</th>
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<td>( -8505+1215\sqrt{-7} )</td>
<td>( -8505-1215\sqrt{-7} )</td>
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<tr>
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<td></td>
<td>( \frac{1+\sqrt{-7}}{2} )</td>
<td>( -8505+1215\sqrt{-7} )</td>
<td>( -8505-1215\sqrt{-7} )</td>
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<td>( -8505+1215\sqrt{-7} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \frac{1+3\sqrt{-7}}{2} )</td>
<td>8505</td>
<td>( -8505+1215\sqrt{-7} )</td>
<td>( -8505-1215\sqrt{-7} )</td>
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**References**


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