HOLOMORPHIC MAPPINGS INTO SOME
DOMAIN IN A COMPLEX NORMED SPACE

TATSUHIRO HONDA

ABSTRACT. Let $D_1, D_2$ be convex domains in complex normed spaces $E_1, E_2$ respectively. When a mapping $f : D_1 \to D_2$ is holomorphic with $f(0) = 0$, we obtain some results like the Schwarz lemma. Furthermore, we discuss a condition whereby $f$ is linear or injective or isometry.

1. Introduction

Let $\Delta = \{z \in \mathbb{C}; |z| < 1\}$ be the unit disc in $\mathbb{C}$. The classical Schwarz lemma in one complex variable is as follows:

THE CLASSICAL SCHWARZ LEMMA. Let $f : \Delta \to \Delta$ be a holomorphic mapping with $f(0) = 0$. Then the following statements hold:

(i) $|f(z)| \leq |z|$ for any $z \in \Delta$,
(ii) if there exists $z_0 \in \Delta \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, or if $|f'(0)| = 1$, then there exists a complex number $\lambda$ of modulus 1 such that $f(z) = \lambda z$ and $f$ is an automorphism of $\Delta$.

It is natural to consider an extension of the above results to more general domains or higher dimensional spaces. However, condition (ii) in above no longer holds even for the bidisc $\Delta \times \Delta$. In fact, one can easily construct a holomorphic mapping $f : \Delta \times \Delta \to \Delta \times \Delta$ such that $f(0) = 0$ and $\|f(z)\| = \|z\|$ for $z$ in an open subset of $\Delta \times \Delta$, but $f$ is not an isometry (cf. J. P. Vigué [18]). Nevertheless, E. Vesentini [15], [16] showed that if $\|f(w)\| = \|w\|$ holds on $B_1$ and if every boundary point of the unit ball $B_2$ is a complex extreme point, then $f : B_1 \to B_2$ is a linear isometry, where $B_1, B_2$ are the open unit balls in normed spaces $E_1, E_2$ over $\mathbb{C}$ respectively. J. P. Vigué [18], [19] proved that if every boundary point of the unit ball $B$ for some norm in $\mathbb{C}^n$ is a complex extreme
point of $\overline{B}$ and if $\|f(w)\| = \|w\|$ holds on an open subset $U$ of $B$, then $f : B \to B$ is a linear automorphism of $\mathbb{C}^n$. H. Hamada [6] generalized the above classical Schwarz lemma to the case where $\|f(w)\| = \|w\|$ holds on some local complex submanifold of codimension 1. The author [10], [11] generalized to the case where $\|f(w)\| = \|w\|$ holds on a non-pluripolar subset. H. Hamada and the author [8] generalized to the case where $\|f(w)\| = \|w\|$ holds on a totally real submanifold.

In this paper, we consider some condition whereby a holomorphic mapping is linear or injective or isometric.

2. Notation and preliminaries

All topologies considered throughout this paper shall be Hausdorff. A vector space $E$ over $\mathbb{C}$ is said to be locally convex if $E$ is equipped with the Hausdorff topology defined by some family $\Pi$ of seminorms such that $\sup_{\alpha \in \Pi} \alpha(x) > 0$ for all $x \in E \setminus \{0\}$, that is, a fundamental system of neighborhoods of $x$ in this topology is made up of finite intersections of sets $x + \alpha^{-1}([0,a])$, $\alpha \in \Pi$, $0 < a < \infty$. Then all seminorms in $\Pi$ are continuous, but the family $cs(E)$ of all continuous semi-norm on $E$ is generally larger than $\Pi$. A sequence $\{z_n\}_{n \in \mathbb{N}}$ on a locally convex space $E$ is a Cauchy sequence in $E$ if for each $\alpha \in \Pi$ and each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\alpha(z_n - z_m) < \varepsilon$ for all $m, n \geq n_0$. A locally convex space $E$ is said to be sequentially complete if any Cauchy sequence converges.

Let $F$ be a locally convex space, let $E$ be a sequentially complete locally convex space. Let $U$ be an open subset in $F$, and let $f : U \to E$ be a holomorphic mapping. For $a \in U$, there uniquely exists a sequence of $n$-homogeneous polynomials $P_n : F \to E$ such that the expansion

$$f(a + z) = f(a) + \sum_{n=1}^{\infty} P_n(z)$$

holds for all $z$ in the largest balanced subset of $U - a$. This series is called the Taylor expansion of $f$ by $n$-homogeneous polynomials $P_n$ at $a$.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$, and let $\gamma(\lambda) = 1/(1 - |\lambda|^2)$. The Poincaré distance $\rho$ on $\Delta$ is defined for $z, w \in \Delta$ as follows:

$$\rho(z, w) = \frac{1}{2} \log \frac{1 + |(z - w)/(1 - z\bar{w})|}{1 - |(z - w)/(1 - z\bar{w})|}.$$
Let $D$ be a domain in a sequentially complete locally convex space $E$. The gauge $N_D$ of $D$ is defined for $z \in E$ as follows:

$$N_D(z) = \inf\{ \alpha > 0; z \in \alpha D\}.$$ 

The Carathéodory pseudodistance $C_D$ on $D$ is defined for $p, q \in D$ as follows:

$$C_D(p, q) = \sup\{ \rho(f(p), f(q)); f \in H(D, \Delta)\}.$$ 

The Lempert function $\delta_D$ of $D$ is defined for $p, q \in D$ as follows:

$$\delta_D(p, q) = \inf\{ \rho(\xi, \eta); \xi, \eta \in \Delta, ^3 \varphi \in H(\Delta, D)$$

such that $\varphi(\xi) = p, \varphi(\eta) = q\}.$

The Kobayashi pseudodistance $K_D$ on $D$ is defined for $p, q \in D$ as follows:

$$K_D(p, q) = \inf\left\{ \sum_{k=1}^{m} \delta_D(x_k, x_{k+1}); m \in \mathbb{N}, \{p = x_1, x_2, \ldots, x_{m+1} = q\} \subset D\right\}.$$ 

Then we have established between the various pseudodistances on a domain $D$:

$$C_D \leq K_D \leq \delta_D$$

on $D \times D$.

The infinitesimal Carathéodory pseudometric $c_D$ for $D$ is defined for $z \in D, v \in E$ as follows:

$$c_D(z, v) = \sup\{ |d\psi(z)(v)|; \psi \in H(D, \Delta)\}.$$ 

The infinitesimal Kobayashi pseudometric $\kappa_D$ for $D$ is defined for $z \in D, v, \psi \in E$ as follows:

$$(2.1) \quad \kappa_D(z, v) = \inf\{ \gamma(\lambda)|\alpha|; ^3 \varphi \in H(\Delta, D), ^3 \lambda \in \Delta$$

such that $\varphi(\lambda) = z, \alpha \varphi'(\lambda) = v\}.$

Then holomorphic mappings $\varphi \in H(\Delta, D)$ as in (2.1) certainly exist. In fact, if $R$ is the radius of the open disc $\{\lambda \in \mathbb{C}; \lambda v \in U(z)\}$, where $U(z)$ is a neighborhood of $z$, we may take the mapping

$$\varphi(\lambda) = z + \frac{\lambda}{\zeta} v$$

for $|\zeta| \geq 1/R$. Hence $\kappa_D(z, v) \leq 1/R$.

Moreover, for any $\psi \in H(D, \Delta)$ with $\psi(z) = 0$, we have $(\psi \circ \varphi)'(0) = d\psi(z)(\varphi'(0))$. It follows from this that

$$c_D \leq \kappa_D$$

on $D \times E$.

We use convexity to obtain the relationship among the pseudodistances or pseudometrics (S. Dineen [3], T. Franzoni and E. Vesentini [5], M. Hervé [9] etc).
**Proposition 2.1.** If $D$ is a balanced convex domain in a sequentially complete locally convex space $E$, then

(i) $C_D(0, x) = K_D(0, x) = \delta_D(0, x) = \rho(0, N_D(x))$ for any $x \in D$,
(ii) $c_D(0, v) = \kappa_D(0, v) = N_D(v)$ for any $v \in E$.

Let $D$ be a balanced pseudoconvex domain in a sequentially complete locally convex space $E$. Then we have the following proposition as the gauge $N_D$ is plurisubharmonic on $E$.

**Proposition 2.2.** If $D$ is a balanced pseudoconvex domain in a sequentially complete locally convex space $E$, then

(i) $K_D(0, x) = \delta_D(0, x) = \rho(0, N_D(x))$ for any $x \in D$,
(ii) $\kappa_D(0, v) = N_D(v)$ for any $v \in E$.

Using the above proposition, we obtain the following generalization of part (i) of the Schwarz lemma to balanced pseudoconvex domains in sequentially complete locally convex spaces.

**Proposition 2.3.** Let $E_j$ be a sequentially complete locally convex space and let $D_j$ be a balanced pseudoconvex domain in $E_j$ for $j = 1, 2$. Let $f : D_1 \to D_2$ be a holomorphic mapping with $f(0) = 0$. Then

$$N_{D_2} \circ f(z) \leq N_{D_1}(z).$$

**Proof.** By Proposition 2.2 (i), we have

$$\rho(0, N_{D_1}(z)) = \delta_{D_1}(0, z) = \delta_{D_2}(0, f(z)) = \rho(0, N_{D_2} \circ f(z)).$$

Since $\rho(0, r)$ is strictly increasing for $0 \leq r < 1$, we obtain this proposition.

The following definition of a complex geodesic due to E. Vesentini [15, 16, 17].

**Definition 2.4.** Let $D$ be a domain in a sequentially complete locally convex space $E$ endowed with a pseudodistance $d_D$. A holomorphic mapping $\varphi : \Delta \to D$ is said to be a complex $d_D$-geodesic for $(x, y)$ if

$$d_D(x, y) = \rho(\xi, \eta)$$

for any points $\xi, \eta \in \Delta$ such that $\varphi(\xi) = x$ and $\varphi(\eta) = y$.

A holomorphic mapping $\varphi : \Delta \to D$ is said to be a complex $c_D$-geodesic for $(z, v)$ if $c_D(z, v) = \gamma(\lambda)|\alpha|$ holds for any $\lambda \in \Delta$ and any $\alpha \in \mathbb{C}$ such that $\varphi(\lambda) = z$ and $\alpha \varphi'(\lambda) = v$. 
A holomorphic mapping $\varphi : \Delta \to D$ is said to be a complex $\kappa_D$-geodesic for $(z,v)$ if $\kappa_D(z,v) = |\gamma(\lambda)\alpha|$ holds for any $\lambda \in \Delta$ and $\alpha \in \mathbb{C}$ such that $\varphi(\lambda) = z$ and $\alpha \varphi'(\lambda) = v$.

The following results about a complex geodesic are well-known (cf. S. Dineen [3], T. Franzoni and E. Vesentini [5], M. Hervé [9] etc).

**Proposition 2.5.** Let $D$ be a domain in a sequentially complete locally convex space $E$ endowed with a pseudodistance $d_D$ or a pseudometric $\mu_D$. Then the following statements hold:

(i) a holomorphic mapping $\varphi : \Delta \to D$ is a complex $d_D$-geodesic for $(x,y)$ if and only if there exists only one pair $(\xi, \eta) \in \Delta^2$ with $(\xi \neq \eta)$ such that $\varphi(\xi) = x$, $\varphi(\eta) = y$ and

$$d_D(x,y) = \rho(\xi, \eta),$$

(ii) a holomorphic mapping $\varphi : \Delta \to D$ is a complex $\mu_D$-geodesic for $(z,v)$ if and only if there exists only one point $\lambda \in \Delta$ such that $\varphi(\lambda) = z$, $\alpha \varphi'(\lambda) = v$ and

$$\mu_D(\varphi(\lambda), \varphi'(\lambda)) = |\alpha| \gamma(\lambda).$$

A point $x$ of the closure $\overline{D}$ of $D$ is said to be a complex extreme point of $\overline{D}$ if $y = 0$ is the only vector in $E$ such that the function $: \zeta \mapsto x + \zeta y$ maps $\Delta$ into $D$. For example, $C^2$-smooth strictly convex boundary points are complex extreme points.

For a bounded balanced pseudoconvex domain $D$, the holomorphic mapping $\varphi(\zeta) = \zeta a/N_D(a)$ is a complex $\delta_D$-geodesic and $\kappa_D$-geodesic for $(0,a)$ for any $a \in D$ with $N_D(a) > 0$. In fact, M. Hervé [9] has given the following characterization of the uniqueness of complex geodesics (see e.g. E. Vesentini [15], [16], [17]).

**Proposition 2.6.** Let $D$ be a balanced convex domain in a sequentially complete locally convex space $E$. Let $a \in D$ be such that $N_D(a) > 0$, and let $\varphi : \Delta \to D$ be the holomorphic mapping defined by $\varphi(\zeta) = \zeta a/N_D(a)$. Then the following conditions are equivalent:

(i) the point $b = a/N_D(a)$ is a complex extreme point of $\overline{D}$;

(ii) $\varphi$ is the unique (modulo $\text{Aut}(\Delta)$) complex $C_D$-geodesic for $(0,a)$;

(iii) $\varphi$ is the unique (modulo $\text{Aut}(\Delta)$) complex $K_D$-geodesic for $(0,a)$;

(iv) $\varphi$ is the unique (modulo $\text{Aut}(\Delta)$) complex $\delta_D$-geodesic for $(0,a)$;

(v) $\varphi$ is the unique (modulo $\text{Aut}(\Delta)$) complex $c_D$-geodesic for $(0,a)$;

(vi) $\varphi$ is the unique (modulo $\text{Aut}(\Delta)$) complex $\kappa_D$-geodesic for $(0,a)$. 

Using the uniqueness of complex geodesics, we obtain the linearity of complex geodesics as in the following proposition.

Proposition 2.7. Let $D_j$ be a bounded balanced convex domain in complex normed spaces $E_j$ for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic mapping with $f(0) = 0$. Let $x \in D_1 \setminus \{0\}$ and let $\varphi(\zeta) = \zeta x / N_{D_1}(x)$. We assume that $f(x) / N_{D_2} \circ f(x)$ is a complex extreme point of $\overline{D_2}$. If one of the following conditions is satisfied, then $f \circ \varphi$ is a linear complex $\delta_{D_2}$-geodesic.

(i) $N_{D_2} \circ f(x) = N_{D_1}(x)$.
(ii) $\delta_{D_2}(f(0), f(x)) = \delta_{D_1}(0, x)$.
(iii) $K_{D_2}(f(0), f(x)) = K_{D_1}(0, x)$.
(iv) $C_{D_2}(f(0), f(x)) = C_{D_1}(0, x)$.

Proof. By Proposition 2.1 (i), the conditions (i), (ii), (iii) and (iv) are equivalent. Suppose that (i) is satisfied. By Proposition 2.1 (i),

$$
\delta_{D_2}(f \circ \varphi(0), f \circ \varphi \circ N_{D_1}(x)) = \delta_{D_2}(0, f(x)) = \delta_{D_1}(0, x) = \rho(0, N_{D_1}(x)).
$$

By Proposition 2.5 (i), $f \circ \varphi$ is a complex $\delta_{D_2}$-geodesic for $(0, f(x))$. By Proposition 2.6, we have

$$
f \circ \varphi(\zeta) = \zeta e^{i\theta} \frac{f(x)}{N_{D_2} \circ f(x)}
$$

for some $\theta \in \mathbb{R}$. \hfill \Box

3. Special versions of the Schwarz Lemma

Now we introduce the projective space $\mathbb{P}(E)$. Let $E$ be a locally convex space. Let $z$ and $z'$ be points in $E \setminus \{0\}$. $z$ and $z'$ are said to be equivalent if there exists $\lambda \in \mathbb{C}^*$ such that $z = \lambda z'$. We denote by $\mathbb{P}(E)$ the quotient space of $E \setminus \{0\}$ by this equivalence relation. Then $\mathbb{P}(E)$ is a Hausdorff space. The Hausdorff space $\mathbb{P}(E)$ is called the complex projective space induced by $E$. We denote by $Q$ the quotient map from $E \setminus \{0\}$ to $\mathbb{P}(E)$ (see M. Nishihara [14]).

Theorem 3.8. Let $E_j$ be a complex normed space, let $D_j$ be a bounded balanced convex domain in $E_j$ for $j = 1, 2$ and let $f : D_1 \to D_2$ be a holomorphic mapping with $f(0) = 0$. Let $X$ be a non-empty subset
of $D_1$ such that $X$ is mapped homeomorphically onto an open subset
$\Omega$ in the complex projective space $\mathbb{P}(E_1)$ by the quotient map $Q$ from
$E_1 \setminus \{0\}$ onto $\mathbb{P}(E_1)$. We assume that $f(x)/ND_1(f(x))$ is a complex
extreme point of $D_2$ for any $x \in X$ and that there exists $w_0 \in X$
such that $w_0/ND_1(w_0)$ is a complex extreme point of $D_1$. If one of the
following conditions is satisfied, then $f$ is linear and injective.
(i) $ND_2(f(x)) = ND_1(x)$ for any $x \in X$.
(ii) $C_{D_1}(f(0), f(x)) = C_{D_1}(0, x)$ for any $x \in X$.
(iii) $K_{D_1}(f(0), f(x)) = K_{D_1}(0, x)$ for any $x \in X$.
(iv) $\delta_{D_1}(f(0), f(x)) = \delta_{D_1}(0, x)$ for any $x \in X$.

Proof. By Proposition 2.1 (i), the conditions (i), (ii), (iii) and (iv) are
equivalent. Suppose that (i) is satisfied. We take a point $w \in X \setminus \{0\}$ and
set $\varphi(\zeta) = \zeta w/ND_1(w)$ for $\zeta \in \Delta$. Then $\varphi$ is a complex $\delta_{D_1}$-geodesic.
We have
$$
\delta_{D_2}(f \circ \varphi(0), f \circ \varphi(ND_1(w))) = \rho(0, ND_1(w)).
$$
By Proposition 2.7, $f \circ \varphi$ is a complex $\delta_{D_2}$-geodesic. It follows from this
that there exists a point $y \in D_2 \setminus \{0\}$ such that
$$
(3.1) \quad f \circ \varphi(\zeta) = \zeta y \frac{y}{ND_2(y)}.
$$

On the other hand, let $f(x) = \sum_{n=1}^{\infty} P_n(x)$ be the Taylor expansion of
$f$ by $n$-homogeneous polynomials $P_n$ in a neighborhood $V$ of $0$ in $E_1$.
Then we have
$$
(3.2) \quad f \circ \varphi(\zeta) = \sum_{n=1}^{\infty} P_n(\frac{\zeta}{ND_1(w)}) = \sum_{n=1}^{\infty} \left( \frac{\zeta}{ND_1(w)} \right)^n P_n(w)
$$
in a neighborhood of $0$ in $\Delta$. By (3.1) and (3.2), we obtain
$$
P_n(w) = 0 \quad \text{for } w \in X, n \geq 2.
$$

We take any point $y \in C^*X = \{tx; t \in C^*, x \in X\}$. Then there exist
t \in C^*$ and $x \in X$ such that $y = tx$. Hence
$$
P_n(y) = P_n(tx) = t^n P_n(x) = 0,$$
that is, $P_n \equiv 0$ on $C^*X \subset E_1$ for every $n \geq 2$. Since $Q$ is continuous,
the set $C^*X = Q^{-1}(\Omega)$ contains an open subset $U$ of $E$. By the identity theorem,
$$
P_n \equiv 0 \quad \text{on } E_1 \text{ for every } n \geq 2.
Therefore $f = P_1$, that is, $f$ is linear.

Next we show that $f$ is injective. Let $z$ be a point of $E_1$ with $f(z) = 0$. Since $f$ is linear, we have

$$N_{D_2} \circ f(tx) = N_{D_2}(tf(x)) = |t|N_{D_2} \circ f(x) = |t|N_{D_1}(x) = N_{D_1}(tx)$$

for every $t \in \mathbb{C}^*$, $x \in X$. It follows from this that

$$N_{D_2} \circ f(y) = N_{D_1}(y) \quad \text{for all } y \in \mathbb{C}^* X.$$

Since $\mathbb{C}^* X$ is open, there exists a positive number $r > 0$ such that $w_0 + \zeta z \in \mathbb{C}^* X$ for $\zeta \in \mathbb{C}, |\zeta| < r$. Then we have

$$N_{D_2} \circ f(w_0 + \zeta z) = N_{D_1}(w_0 + \zeta z). \quad (3.3)$$

On the other hand,

$$N_{D_2} \circ f(w_0 + \zeta z) = N_{D_2}(f(w_0) + \zeta f(z)) = N_{D_2} \circ f(w_0) = N_{D_1}(w_0). \quad (3.4)$$

By (3.3) and (3.4), we have

$$N_{D_1}(w_0 + \zeta z) = N_{D_1}(w_0).$$

Hence

$$N_{D_1} \left( \frac{w_0}{N_{D_1}(w_0)} + \frac{\zeta}{N_{D_1}(w_0)} z \right) = 1 \quad \text{for } |\zeta| < r.$$

Since $w_0/N_{D_1}(w_0)$ is a complex extreme point of $D_1$, we have $z = 0$.

Therefore $f$ is injective. \(\square\)

Since complex Hilbert spaces are endowed with the norm which is induced from its inner products, we have the following corollary.

**Corollary 3.9.** Let $H_j$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle_j$, let $B_j$ be the open unit ball of $H_j$ for the norm $\| \cdot \|_j = \langle \cdot, \cdot \rangle_j^{1/2}$ for $j = 1, 2$. Let $f : B_1 \to B_2$ be a holomorphic map with $f(0) = 0$. Let $X$ be a non-empty subset of $B_1$ such that $X$ is mapped onto an open subset $\Omega$ in the projective space $P(H_1)$ by the quotient map $Q$. If $\|w\|_1 = \|f(w)\|_2$ holds for every $w \in X$, then $f$ is a linear isometry.
If $H_1 = H_2 = \mathbb{C}^n$ with the Euclidean unit ball $B$, then $f$ is a linear automorphism of $B$.

Proof. Since every point of the boundary $\partial B_j = \{ z \in H_j ; \| z \|_j - 1 = 0 \}$ of $B_j$ is a complex extreme point of the closure $\overline{B}_j$ of $B_j$ for $j = 1, 2$, by Theorem 3.8, $f$ is linear and injective.

We consider a function

\[ g(z) = \| z \|_1^2 - \| f(z) \|_2^2 \]

for $z \in H_1$. By Proposition 2.3, we have $g \geq 0$ on $H_1$.

Since $\partial \overline{B}_j \geq 0$, the non-negative valued function $g$ is plurisubharmonic on $H_1$. Hence $\log g$ is plurisubharmonic on $H_1$. Since $\| w \|_1 = \| f(w) \|_2$ for every $w \in X$,

\[ \log g \equiv -\infty \]

on an open subset $\mathbb{C}^*X = Q^{-1}(\Omega)$. Therefore $f$ is a linear isometry. \qed

4. Infinitesimal pseudometrics

Proposition 4.10. Let $D_j$ be a bounded balanced convex domain in a complex normed space $E_j$ for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic mapping with $f(0) = 0$. Let $x \in D \setminus \{0\}$ and let $\varphi(\zeta) = \zeta x/N_{D_1}(x)$. We assume that $df(0)x/N_{D_2}(df(0)x)$ is a complex extreme point of $\overline{D}_2$. If one of the following conditions is satisfied, then $f \circ \varphi$ is a linear complex $\kappa_{D_2}$-geodesic.

(i) $N_{D_2} \circ f(x) = N_{D_1}(x)$.
(ii) $c_{D_2}(f(0), f(x)) = c_{D_1}(0, x)$.
(iii) $\kappa_{D_2}(f(0), f(x)) = \kappa_{D_1}(0, x)$.

Proof. By Proposition 2.1, (ii),

\[ \kappa_{D_2}(0, df(0)x) = \kappa_{D_1}(0, x) \]

\[ = N_{D_1}(x). \]

Since $N_{D_1}(x)(f \circ \varphi)'(0) = df(0)x$, $f \circ \varphi$ is a complex $\kappa_{D_2}$-geodesic for $(0, df(0)x)$. By Proposition 2.6, we have

\[ f \circ \varphi(\zeta) = \zeta e^{i\theta} \frac{df(0)x}{N_{D_2}(df(0)x)} \]

for some $\theta \in \mathbb{R}$. \qed
**Theorem 4.11.** Let $E_j$ be a complex normed space, let $D_j$ be a bounded balanced convex domain in $E_j$ for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic mapping. Let $V$ be a connected open neighborhood of the origin in $D_1$. We assume that $\kappa_{D_2} (0, df(0)x) = \kappa_{D_1} (0, x)$ for $x \in V$. If $f(0) = 0$ and $df(0)x/N_{D_1}(df(0)x)$ is a complex extreme point of $D_2$ for any $x \in V \setminus \{0\}$, and if there exists $w \in V \setminus \{0\}$ such that $w/N_{D_1}(w)$ is a complex extreme point of $D_1$, then $f$ is linear and injective.

**Proof.** Let $f(z) = \sum_{n=1}^{\infty} P_n(z)$ be the expansion of $f$ by $n$-homogeneous polynomials $P_n$ in a neighborhood of $0$ in $E_1$. Since $\kappa_{D_2} (f(0^2), df(0)v) = \kappa_{D_1} (0, v)$ for any $v \in V$, by Proposition 4.10, $f(\zeta x/N_{D_1}(x))$ is the restriction of a linear map for any $x \in V$. Then we have

$$P_n(x) = 0 \quad \text{on } V \quad \text{for } n \geq 2$$

as in the proof of Theorem 3.8. By the analytic continuation theorem, we have $P_n$ is identically $0$ for $n \geq 2$. Therefore $f$ is the restriction of a linear map.

Let $\varphi(\zeta) = \zeta w/N_{D_1}(w)$. By Proposition 2.6, $f \circ \varphi$ is a complex $\delta_{D_2}$-geodesic for $(0, f(v))$. By Proposition 2.1,

$$\rho(0, N_{D_2}(f(v))) = \delta_{D_2}(0, f(v)) = \rho(0, N_{D_1}(v)).$$

This implies that $N_{D_2}(f(v)) = N_{D_1}(v)$. The rest of the proof is same as Theorem 3.8. □

We note that the map $f$ is not necessarily a linear isometry under the assumption of the above theorem (cf. J. P. Vigué [18]). The following theorem was obtained by H. Cartan for bounded domain in $C^2$ (see T. Franzoni and E. Vesentini [5] etc).

**Theorem 4.12.** Let $D$ be a bounded domain in a complex normed space $E$, and let $f : D \to D$ be a holomorphic mapping. If there exists $x_0 \in D$ such that $f(x_0) = x_0$ and $df(x_0)$ is an identity, $f$ is the identity map.

Using the above theorem of Cartan, we obtain the following theorem.

**Theorem 4.13.** Let $E_j$ be a complex normed space, let $D_j$ be a bounded balanced convex domain in $E_j$ for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic mapping. Let $V$ be a connected open neighborhood of the origin in $D_1$. We assume that $\kappa_{D_2} (0, df(0)x) = \kappa_{D_1} (0, x)$ for $x \in V$. If the inverse $df(0)^{-1}$ exists, then $f(0) = 0$ and $f$ is the restriction of $df(0)$ to $D_1$. 
Proof. First we will show $f(0) = 0$. We assume that $a = f(0) \neq 0$. Since $a \in D$, there exists a point $v \in E_i$ such that $N_{D_1}(v) = 1$ and $df(0)v = a/N_{D_1}(a)$. Then we have
\[
\kappa_{D_2}(a, a/N_{D_2}(a)) = \kappa_{D_2}(f(0), df(0)v) \\
\leq \kappa_{D_1}(0, v) \\
= N_{D_1}(v) \\
= 1.
\]

Therefore
\[
(4.1) \quad \kappa_{D_2}(a, a) \leq N_{D_2}(a).
\]

On the other hand, we set $\varphi(\zeta) = \zeta a/N_{D_2}(a)$ for $\zeta \in \Delta$. Then $\varphi$ is a complex $\kappa_{D_2}$-geodesic for $(0, a)$. So we have
\[
\kappa_{D_2}(a, a) = \kappa_{D_2}(N_{D_2}(a), N_{D_2}(a)\varphi'(N_{D_2}(a))) \\
= \kappa_{\Delta}(N_{D_2}(a), N_{D_2}(a)) \\
= \frac{N_{D_2}(a)}{1 - \{N_{D_2}(a)\}^2}.
\]

Therefore $\kappa_{D_2}(a, a) > N_{D_2}(a)$. This contradicts with (4.1). We obtain $f(0) = 0$. By the assumptions, we have $N_{D_1}(df(0)^{-1}(w)) < 1$ for $w \in D_2$. Now we consider a holomorphic mapping $g = df(0)^{-1} \circ f$. Then $g$ is a holomorphic mapping from $D_1$ to $D_1$ such that $g(0) = 0$ and $dg(0)$ is identity. By Theorem 4.12, $g$ is identity. 

In the Hilbert space case, since every boundary point of the unit ball is a complex extreme point, by the proofs of Corollary 3.9, Theorem 4.11 and Theorem 4.13, we obtain the following corollary.

**Corollary 4.14.** Let $H_j$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle_j$, let $B_j$ be the open unit ball of $H_j$ for the norm $\| \cdot \|_j = \langle \cdot, \cdot \rangle_j^\frac{1}{2}$ for $j = 1, 2$. Let $f : B_1 \to B_2$ be a holomorphic map. Let $V$ be a connected open neighborhood of the origin in $B_1$. We assume that $\kappa_{B_2}(0, df(0)x) = \kappa_{B_1}(0, x)$ for $x \in V$. Then $f(0) = 0$ and $f$ is a linear isometry.

If $H_1 = H_2 = \mathbb{C}^n$ with the Euclidean unit ball $B$, then $f$ is a linear automorphism of $B$. 
References


Ariake National College of Technology,
Omata, Fukuoka, 836-8585, Japan
E-mail: honda@ariake-nct.ac.jp