HOMOMORPHISMS BETWEEN $C^*$-ALGEBRAS ASSOCIATED WITH THE TRIF FUNCTIONAL EQUATION AND LINEAR DERIVATIONS ON $C^*$-ALGEBRAS

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Abstract. It is shown that every almost linear mapping $h : A \to B$ of a unital $C^*$-algebra $A$ to a unital $C^*$-algebra $B$ is a homomorphism under some condition on multiplication, and that every almost linear continuous mapping $h : A \to B$ of a unital $C^*$-algebra $A$ of real rank zero to a unital $C^*$-algebra $B$ is a homomorphism under some condition on multiplication.

Furthermore, we are going to prove the generalized Hyers-Ulam-Rassias stability of $*$-homomorphisms between unital $C^*$-algebras, and of $\mathbb{C}$-linear $*$-derivations on unital $C^*$-algebras.

1. Introduction

Let $X$ and $Y$ be Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Rassias [7] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2p} \|x\|^p$$

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Recently, Trif [8] proved the following: let $q := \frac{l(l-1)}{d-l}$, $r := -\frac{l}{d-l}$.

Denote by $\varphi : X^d \to [0, \infty)$ a function such that

$$\tilde{\varphi}(x_1, \ldots, x_d) = \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \ldots, q^j x_d) < \infty$$

for all $x_1, \ldots, x_d \in X$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|d_{d-2C_{l-2}} f(\frac{x_1 + \cdots + x_d}{d}) + d_{-2C_{l-1}} \sum_{j=1}^{d} f(x_j)$$

$$-l \sum_{1 \leq j_1 < \cdots < j_l \leq d} f(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) \|$$

$$\leq \varphi(x_1, \ldots, x_d)$$

for all $x_1, \ldots, x_d \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{l \cdot d_{-1}C_{l-1}} \tilde{\varphi}(qx, rx, \ldots, rx)$$

$d_{-1}$ times

for all $x \in X$. And Park [6] applied the Trif’s result to the Trif functional equation in Banach modules over a $C^*$-algebra.

B. E. Johnson [3, Theorem 7.2] also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras: Suppose that $U$ and $B$ are Banach $*$-algebras which satisfy the conditions of [3, Theorem 3.1]. Then for each positive $\epsilon$ and $K$ there is a positive $\delta$ such that if $T \in L(U, B)$ with $\|T\| < K$, $\|T^*\| < \delta$ and $\|T(x^*)^* - T(x)\| \leq \delta \|x\|$ (where $x \in U$) then there is a $*$-homomorphism $T' : U \to B$ with $\|T - T'\| < \epsilon$. Here $L(U, B)$ is the space of bounded linear maps from $U$ into $B$, and $T^*(x, y) = T(xy) - T(x)T(y)$ ($x, y \in U$). See [3] for details.

Throughout this paper, let $A$ be a unital $C^*$-algebra with norm $\|\cdot\|$ and unit $e$, and $B$ a unital $C^*$-algebra with norm $\|\cdot\|$. Let $U(A)$ be the set of unitary elements in $A$, $A_{sa} = \{x \in A \mid x = x^* \}$, and $I_1(A_{sa}) = \{v \in A_{sa} \mid \|v\| = 1, v \text{ is invertible}\}$. Let $q = \frac{l(l-1)}{d-l}$ and $r = -\frac{l}{d-l}$ for integers $l, d$ with $2 \leq l \leq d - 1$.

In this paper, we prove that every almost linear mapping $h : A \to B$ is a homomorphism when $h(q^nu)h(q^ny) = h(q^n)h(y)$ holds for all $u \in U(A)$, all
Homomorphisms between $C^*$-algebras

$y \in A$, and all $n = 0, 1, 2, \ldots$, and that for a unital $C^*$-algebra $A$ of real rank zero (see [1]), every almost linear continuous mapping $h : A \to B$ is a homomorphism when $h(q^n u y) = h(q^n u)h(y)$ holds for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \ldots$.

Furthermore, we are going to prove the generalized Hyers-Ulam-Rassi-as stability of $*$-homomorphisms between unital $C^*$-algebras, and of $\mathbb{C}$-linear $*$-derivations on unital $C^*$-algebras.

2. $*$-homomorphisms between unital $C^*$-algebras

We are going to investigate $*$-homomorphisms between unital $C^*$-algebras.

**Theorem 1.** Let $h : A \to B$ be a mapping satisfying $h(0) = 0$ and $h(q^n u y) = h(q^n u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \ldots$, for which there exists a function $\varphi : A^d \to [0, \infty)$ such that

(i) $\widetilde{\varphi}(x_1, \ldots, x_d) := \sum_{j=0}^{\infty} q^{-j}\varphi(q^j x_1, \ldots, q^j x_d) < \infty$,

$$
\|d \cdot d - 2C_{l-2} h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d}\right) + d - 2C_{l-2} \sum_{j=1}^{d} \mu h(x_j)
$$

(ii) $- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) \leq \varphi(x_1, \ldots, x_d)$,

(iii) $\|h(q^n u^*) - h(q^n u)^*\| \leq \varphi(q^n u, \ldots, q^n u)$

for all $\mu \in T^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, all $u \in U(A)$, all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d \in A$. Assume that (iv) $\lim_{n \to \infty} h(q^n e)$ is invertible. Then the mapping $h : A \to B$ is a $*$-homomorphism.

**Proof.** Put $\mu = 1 \in T^1$. It follows from the Trif theorem [8, Theorem 3.1] that there exists a unique additive mapping $\Theta : A \to B$ such that

(j) $\|h(x) - \Theta(x)\| \leq \frac{1}{l \cdot d - 1C_{l-1}} \widetilde{\varphi}(q^x, r x_1, \ldots, r x)$

for all $x \in A$. The additive mapping $\Theta : A \to B$ is given by

$$
\Theta(x) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n x)
$$
for all $x \in \mathcal{A}$.

Put $x_1 = \cdots = x_d = x$ in (ii). For each $\mu \in \mathbb{T}^1$,

$$
\|d \cdot d_{\mathcal{C} \mathcal{L}_2(h(\mu x) - \mu h(x))}\| \leq \varphi(x, \ldots, x) \quad \text{d times}
$$

for all $x \in \mathcal{A}$. So

$$
q^{-n}\|d \cdot d_{\mathcal{C} \mathcal{L}_2(h(\mu q^n x) - \mu h(q^n x))}\| \leq q^{-n}\varphi(q^n x, \ldots, q^n x) \quad \text{d times}
$$

for all $x \in \mathcal{A}$. By (i),

$$
q^{-n}\|d \cdot d_{\mathcal{C} \mathcal{L}_2(h(\mu q^n x) - \mu h(q^n x))}\| \to 0
$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus

$$
q^{-n}\|h(\mu q^n x) - \mu h(q^n x)\| \to 0
$$

as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$
(1) \quad \Theta(\mu x) = \lim_{n \to \infty} h(q^n \mu x) = \lim_{n \to \infty} \frac{\mu h(q^n x)}{q^n} = \mu \Theta(x)
$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and $M$ an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [4, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. So by (1)

$$
\Theta(\lambda x) = \Theta(M \frac{1}{3} \cdot 3 \frac{\lambda}{M} x) = M \cdot \Theta(\frac{1}{3} \cdot 3 \frac{\lambda}{M} x) = \frac{M}{3} \Theta(3 \frac{\lambda}{M} x)
$$

$$
= \frac{M}{3} \Theta(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3} (\Theta(\mu_1 x) + \Theta(\mu_2 x) + \Theta(\mu_3 x))
$$

$$
= \frac{M}{3} (\mu_1 + \mu_2 + \mu_3) \Theta(x) = \frac{M}{3} \cdot 3 \frac{\lambda}{M} \Theta(x) = \lambda \Theta(x)
$$

for all $x \in \mathcal{A}$. Hence

$$
\Theta(\zeta x + \eta y) = \Theta(\zeta x) + \Theta(\eta y) = \zeta \Theta(x) + \eta \Theta(y)
$$

for all $\zeta, \eta \in \mathbb{C}$ ($\zeta, \eta \neq 0$) and all $x, y \in \mathcal{A}$. And $\Theta(0x) = 0 = 0\Theta(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.
By (i) and (iii), we get
\[
\Theta(u^*) = \lim_{n \to \infty} \frac{h(q^n u^*)}{q^n} = \lim_{n \to \infty} \frac{h(q^n u)^*}{q^n} = \left( \lim_{n \to \infty} \frac{h(q^n u)}{q^n} \right)^* = \Theta(u)^*
\]
for all \( u \in U(A) \). Since \( \Theta \) is \( \mathbb{C} \)-linear and each \( x \in A \) is a finite linear combination of unitary elements (see [5, Theorem 4.1.7]), i.e.,
\[
x = \sum_{j=1}^{m} \lambda_j u_j \quad (\lambda_j \in \mathbb{C}, u_j \in U(A)),
\]

\[
\Theta(x^*) = \Theta\left(\sum_{j=1}^{m} \lambda_j u_j^*\right) = \sum_{j=1}^{m} \lambda_j \Theta(u_j^*) = \sum_{j=1}^{m} \lambda_j \Theta(u_j)^*
\]

\[
= \left( \sum_{j=1}^{m} \lambda_j \Theta(u_j) \right)^* = \Theta\left(\sum_{j=1}^{m} \lambda_j u_j\right)^* = \Theta(x)^*
\]
for all \( x \in A \).

Since \( h(q^n uy) = h(q^n u)h(y) \) for all \( u \in U(A) \), all \( y \in A \), and all \( n = 0, 1, 2, \ldots \),

\[
(2) \quad \Theta(uy) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n uy) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n u)h(y) = \Theta(u)h(y)
\]
for all \( u \in U(A) \) and all \( y \in A \). By the additivity of \( \Theta \) and (2),
\[
q^n \Theta(uy) = \Theta(q^n uy) = \Theta(u(q^n y)) = \Theta(u)h(q^n y)
\]
for all \( u \in U(A) \) and all \( y \in A \). Hence

\[
(3) \quad \Theta(uy) = \frac{1}{q^n} \Theta(u)h(q^n y) = \Theta(u) \frac{1}{q^n} h(q^n y)
\]
for all \( u \in U(A) \) and all \( y \in A \). Taking the limit in (3) as \( n \to \infty \), we obtain

\[
(4) \quad \Theta(uy) = \Theta(u)\Theta(y)
\]
for all \( u \in U(A) \) and all \( y \in A \). Since \( \Theta \) is \( \mathbb{C} \)-linear and each \( x \in A \) is a finite linear combination of unitary elements, i.e.,
\[
x = \sum_{j=1}^{m} \lambda_j u_j \quad (\lambda_j \in \mathbb{C}, u_j \in U(A)),
\]
it follows from (4) that
\[
\Theta(xy) = \Theta\left(\sum_{j=1}^{m} \lambda_j u_j y\right) = \sum_{j=1}^{m} \lambda_j \Theta(uy)
\]

\[
= \sum_{j=1}^{m} \lambda_j \Theta(u_j)\Theta(y) = \Theta\left(\sum_{j=1}^{m} \lambda_j u_j\right)\Theta(y)
\]

\[
= \Theta(x)\Theta(y)
\]
for all \( x, y \in \mathcal{A} \).

By (2) and (4),
\[
\Theta(e)\Theta(y) = \Theta(ey) = \Theta(e)h(y)
\]
for all \( y \in \mathcal{A} \). Since \( \lim_{n \to \infty} \frac{h(\frac{q^n}{e})}{q^n} = \Theta(e) \) is invertible,
\[
\Theta(y) = h(y)
\]
for all \( y \in \mathcal{A} \).

Therefore, the mapping \( h : \mathcal{A} \to \mathcal{B} \) is a \(*\)-homomorphism, as desired. \( \square \)

**Corollary 2.** Let \( h : \mathcal{A} \to \mathcal{B} \) be a mapping satisfying \( h(0) = 0 \) and \( h(q^nu) = h(q^n)h(y) \) for all \( u \in \mathcal{U}(\mathcal{A}) \), all \( y \in \mathcal{A} \), and all \( u, v, w, x, y \in \mathcal{A} \).

Assume that \( \lim_{n \to \infty} \frac{h(\frac{q^n}{e})}{q^n} \) is invertible. Then the mapping \( h : \mathcal{A} \to \mathcal{B} \) is a \(*\)-homomorphism.

**Proof.** Define \( \varphi(x_1, \ldots, x_d) = \theta(\sum_{j=1}^d ||x_j||^p) \), and apply Theorem 1. \( \square \)

**Theorem 3.** Let \( h : \mathcal{A} \to \mathcal{B} \) be a mapping satisfying \( h(0) = 0 \) and \( h(q^nu) = h(q^n)h(y) \) for all \( u \in \mathcal{U}(\mathcal{A}) \), all \( y \in \mathcal{A} \), and all \( u, v, w, x, y \in \mathcal{A} \),

for which there exists a function \( \varphi : \mathcal{A}^d \to [0, \infty) \) satisfying (i), (ii), and (iv) such that

\[
\|d \cdot d^{-2}C_{d-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d}) + d^{-2}C_{d-1} \sum_{j=1}^d \mu h(x_j)
\]

\[
- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) \| \leq \varphi(x_1, \ldots, x_d)
\]

for \( \mu = 1, i, \) and all \( x_1, \ldots, x_d \in \mathcal{A} \). If \( h(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in \mathcal{A} \), then the mapping \( h : \mathcal{A} \to \mathcal{B} \) is a \(*\)-homomorphism.
Proof. Put $\mu = 1$ in (v). By the same reasoning as the proof of Theorem 1, there exists a unique additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (†). By the same reasoning as the proof of [7, Theorem], the additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (v). By the same method as the proof of Theorem 1, one can obtain that

$$\Theta(ix) = \lim_{n \to \infty} \frac{h(q^nix)}{q^n} = \lim_{n \to \infty} \frac{ih(q^nix)}{q^n} = i\Theta(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\Theta(\lambda x) = \Theta(sx + itx) = s\Theta(x) + t\Theta(ix)$$

$$= s\Theta(x) + it\Theta(x) = (s + it)\Theta(x)$$

$$= \lambda \Theta(x)$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$\Theta(\zeta x + \eta y) = \Theta(\zeta x) + \Theta(\eta y) = \zeta \Theta(x) + \eta \Theta(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as the proof of Theorem 1. \qed

From now on, assume that $\mathcal{A}$ is a unital $C^*$-algebra of real rank zero, where "real rank zero" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [1]).

Now we are going to investigate continuous $*$-homomorphisms between unital $C^*$-algebras.

**Theorem 4.** Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(q^nuy) = h(q^n)h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying (i), (ii), (iii), and (iv). Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.

**Proof.** By the same reasoning as the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear involution $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (†).

Since $h(q^nuy) = h(q^n)h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$,

$$\Theta(uy) = \lim_{n \to \infty} \frac{1}{q^n} h(q^nuy) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n)h(y) = \Theta(u)h(y)$$
for all \( u \in I_1(A_{sa}) \) and all \( y \in A \). By the additivity of \( \Theta \) and (5),

\[
q^n \Theta(uy) = \Theta(q^n uy) = \Theta(u(q^n y)) = \Theta(u) h(q^n y)
\]

for all \( u \in I_1(A_{sa}) \) and all \( y \in A \). Hence

\[
\Theta(uy) = \frac{1}{q^n} \Theta(u) h(q^n y) = \Theta(u) \frac{1}{q^n} h(q^n y)
\]

for all \( u \in I_1(A_{sa}) \) and all \( y \in A \). Taking the limit in (6) as \( n \to \infty \), we obtain

\[
\Theta(uy) = \Theta(u) \Theta(y)
\]

for all \( u \in I_1(A_{sa}) \) and all \( y \in A \).

By (5) and (7),

\[
\Theta(e) \Theta(y) = \Theta(ey) = \Theta(e) h(y)
\]

for all \( y \in A \). Since \( \lim_{n \to \infty} \frac{h(q^n e)}{q^n} = \Theta(e) \) is invertible,

\[
\Theta(y) = h(y)
\]

for all \( y \in A \). So \( \Theta : A \to B \) is continuous. But by the assumption that \( A \) has real rank zero, it is easy to show that \( I_1(A_{sa}) \) is dense in \( \{ x \in A_{sa} | ||x|| = 1 \} \). So for each \( w \in \{ z \in A_{sa} | ||z|| = 1 \} \), there is a sequence \( \{ \kappa_j \} \) such that \( \kappa_j \to w \) as \( j \to \infty \) and \( \kappa_j \in I_1(A_{sa}) \). Since \( \Theta : A \to B \) is continuous, it follows from (7) that

\[
\Theta(wy) = \Theta(\lim_{j \to \infty} \kappa_j y) = \lim_{j \to \infty} \Theta(\kappa_j y) = \lim_{j \to \infty} \Theta(\kappa_j) \Theta(y) = \Theta(\lim_{j \to \infty} \kappa_j) \Theta(y)
\]

\[
= \Theta(w) \Theta(y)
\]

for all \( w \in \{ z \in A_{sa} | ||z|| = 1 \} \) and all \( y \in A \).

For each \( x \in A \), \( x = \frac{x+x^*}{2} + i \frac{x-x^*}{2i} \), where \( x_1 := \frac{x+x^*}{2} \) and \( x_2 := \frac{x-x^*}{2i} \) are self-adjoint.
First, consider the case that \( x_1 \neq 0, x_2 \neq 0 \). Since \( \Theta : \mathcal{A} \to \mathcal{B} \) is C-linear, it follows from (8) that

\[
\Theta(xy) = \Theta(x_1 y + ix_2 y) = \Theta(||x_1|| \frac{x_1}{||x_1||} y + i||x_2|| \frac{x_2}{||x_2||} y) \\
= ||x_1|| \Theta\left( \frac{x_1}{||x_1||} y \right) + i||x_2|| \Theta\left( \frac{x_2}{||x_2||} y \right) \\
= ||x_1|| \Theta\left( \frac{x_1}{||x_1||} \right) \Theta(y) + i||x_2|| \Theta\left( \frac{x_2}{||x_2||} \right) \Theta(y) \\
= \{ \Theta(||x_1|| \frac{x_1}{||x_1||}) + i\Theta(||x_2|| \frac{x_2}{||x_2||}) \} \Theta(y) = \Theta(x_1 + ix_2) \Theta(y) \\
= \Theta(x) \Theta(y)
\]

for all \( y \in \mathcal{A} \).

Next, consider the case that \( x_1 \neq 0, x_2 = 0 \). Since \( \Theta : \mathcal{A} \to \mathcal{B} \) is C-linear, it follows from (8) that

\[
\Theta(xy) = \Theta(x_1 y) = \Theta(||x_1|| \frac{x_1}{||x_1||} y) = ||x_1|| \Theta\left( \frac{x_1}{||x_1||} y \right) \\
= ||x_1|| \Theta\left( \frac{x_1}{||x_1||} \right) \Theta(y) = \Theta(||x_1|| \frac{x_1}{||x_1||}) \Theta(y) = \Theta(x_1) \Theta(y) \\
= \Theta(x) \Theta(y)
\]

for all \( y \in \mathcal{A} \).

Finally, consider the case that \( x_1 = 0, x_2 \neq 0 \). Since \( \Theta : \mathcal{A} \to \mathcal{B} \) is C-linear, it follows from (8) that

\[
\Theta(xy) = \Theta(ix_2 y) = \Theta(i||x_2|| \frac{x_2}{||x_2||} y) = i||x_2|| \Theta\left( \frac{x_2}{||x_2||} y \right) \\
= i||x_2|| \Theta\left( \frac{x_2}{||x_2||} \right) \Theta(y) = \Theta(i||x_2|| \frac{x_2}{||x_2||}) \Theta(y) = \Theta(ix_2) \Theta(y) \\
= \Theta(x) \Theta(y)
\]

for all \( y \in \mathcal{A} \). Hence

\[\Theta(xy) = \Theta(x) \Theta(y)\]

for all \( x, y \in \mathcal{A} \).

Therefore, the mapping \( h : \mathcal{A} \to \mathcal{B} \) is a *-homomorphism, as desired. \( \square \)
COROLLARY 5. Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(q^n uy) = h(q^n u)h(y)$ for all $u \in I_1(A_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$
\|d \cdot d_{2C_{l-2}} h(\frac{\mu x_1 + \cdots + \mu x_d}{d}) + d_{2C_{l-1}} \sum_{j=1}^{d} \mu h(x_j)
- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) \| \leq \theta(\sum_{j=1}^{d} \|x_j\|^p),
$$

$$
\|h(q^n u^*) - h(q^n u)^*\| \leq dq^np \theta
$$

for all $\mu \in T^1$, all $u \in I_1(A_{sa})$, all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d \in \mathcal{A}$. Assume that $\lim_{n \to \infty} \frac{h(q^n e)}{q^n}$ is invertible. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.

Proof. Define $\varphi(x_1, \ldots, x_d) = \theta(\sum_{j=1}^{d} \|x_j\|^p)$, and apply Theorem 4. \hfill \Box

THEOREM 6. Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(q^n uy) = h(q^n u)h(y)$ for all $u \in I_1(A_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \ldots$, for which there exists a function $\varphi : A^d \to [0, \infty)$ satisfying (i), (iii), (iv), and (v). Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism.

Proof. By the same reasoning as the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality $(\dagger)$. The rest of the proof is the same as the proofs of Theorems 1 and 4. \hfill \Box

3. Stability of $*$-homomorphisms between unital $C^{**}$-algebras

We are going to show the generalized Hyers-Ulam-Rassias stability of $*$-homomorphisms between unital $C^{**}$-algebras.

THEOREM 7. Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which
there exists a function $\varphi : \mathcal{A}^{d+2} \to [0, \infty)$ such that
\[(vi)\]
$$\varphi(x_1, \ldots, x_d, z, w) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \ldots, q^j x_d, q^j z, q^j w) < \infty,$$

$$\|d \cdot d^{-2} C_{l-2} h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot d^{-2} C_{l-2}}\right) + d^{-2} C_{l-1} \sum_{j=1}^{d} \mu h(x_j)\|
\]

\[(vii)\]
$$- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) - h(z)h(w)\|
\leq \varphi(x_1, \ldots, x_d, z, w),$$

\[(viii)\]
$$\|h(q^n u^*) - h(q^n u)^*\| \leq \varphi(q^n u, \ldots, q^n u, 0, 0)
\]
d times

for all $\mu \in T^1$, all $u \in \mathcal{U}(\mathcal{A})$, all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d, z, w \in \mathcal{A}$. Then there exists a unique $*$-homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$ such that
\[(ix)\]
$$\|h(x) - \Theta(x)\| \leq \frac{1}{l \cdot d^{-1} C_{l-1}} \varphi(qx, rx, \ldots, rx, 0, 0)
\]
d times

for all $x \in \mathcal{A}$.

Proof. Put $z = w = 0$ and $\mu = 1 \in T^1$ in (vii). By the same reasoning as the proof of Theorem 1, there exists a unique $C$-linear involution $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (ix). The $C$-linear mapping $\Theta : \mathcal{A} \to \mathcal{B}$ is given by
\[(9)\]
$$\Theta(x) = \lim_{n \to \infty} \frac{1}{q^n} h(q^n x)$$

for all $x \in \mathcal{A}$.

It follows from (9) that
\[(10)\]
$$\Theta(x) = \lim_{n \to \infty} \frac{h(q^{2n} x)}{q^{2n}}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_d = 0$ in (vi). Then we get
$$\|d \cdot d^{-2} C_{l-2} h\left(\frac{zw}{d \cdot d^{-2} C_{l-2}}\right) - h(z)h(w)\| \leq \varphi(0, \ldots, 0, z, w)$$
d times
for all $z, w \in A$. Since
\[
\frac{1}{q^{2n}} \varphi(0, \ldots, 0, q^n z, q^n w) \leq \frac{1}{q^n} \varphi(0, \ldots, 0, q^n z, q^n w),
\]

\[
\frac{1}{q^{2n}} \|d \cdot d_{-2} C_{l-2} h\big(\frac{1}{d \cdot d_{-2} C_{l-2}} q^n z \cdot q^n w\big) - h(q^n z)h(q^n w)\|
\]

\[(11)\]

\[
\leq \frac{1}{q^{2n}} \varphi(0, \ldots, 0, q^n z, q^n w) \leq \frac{1}{q^n} \varphi(0, \ldots, 0, q^n z, q^n w)
\]

for all $z, w \in A$. By (vi), (10), and (11),
\[
d \cdot d_{-2} C_{l-2} \Theta\big(\frac{zw}{d \cdot d_{-2} C_{l-2}}\big) = \lim_{n \to \infty} \frac{d \cdot d_{-2} C_{l-2} h\big(\frac{1}{d \cdot d_{-2} C_{l-2}} q^{2n} zw\big)}{q^{2n}}
\]

\[
= \lim_{n \to \infty} \frac{d \cdot d_{-2} C_{l-2} h\big(\frac{1}{d \cdot d_{-2} C_{l-2}} q^n z \cdot q^n w\big)}{q^n \cdot q^n}
\]

\[
= \lim_{n \to \infty} \left( \frac{h(q^n z)}{q^n} \cdot \frac{h(q^n w)}{q^n} \right)
\]

\[
= \lim_{n \to \infty} \frac{h(q^n z)}{q^n} \cdot \lim_{n \to \infty} \frac{h(q^n w)}{q^n}
\]

\[
= \Theta(z)\Theta(w)
\]

for all $z, w \in A$. But since $\Theta$ is $\mathbb{C}$-linear,
\[
\Theta(zw) = d \cdot d_{-2} C_{l-2} \Theta\big(\frac{zw}{d \cdot d_{-2} C_{l-2}}\big) = \Theta(z)\Theta(w)
\]

for all $z, w \in A$. Hence the $\mathbb{C}$-linear mapping $\Theta : A \to B$ is a $*$-homomorphism satisfying the inequality (ix), as desired. \hfill \Box

**Corollary 8.** Let $h : A \to B$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that
\[
\|d \cdot d_{-2} C_{l-2} h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot d_{-2} C_{l-2}}\right) + d_{-2} C_{l-1} \sum_{j=1}^{d} \mu h(x_j)
\]

\[- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) - h(z)h(w)\|
\]

\[
\leq \theta \left( \sum_{j=1}^{d} ||x_j||^p + ||z||^p + ||w||^p \right),
\]

\[
\|h(q^n u^*) - h(q^n u)^*\| \leq dq^{np}\theta
\]
for all $\mu \in \mathbb{T}^1$, all $u \in U(\mathcal{A})$, all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d, z, w \in \mathcal{A}$. Then there exists a unique $*$-homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$ such that

$$
\| h(x) - \Theta(x) \| \leq \frac{q^{1-p}(q^p + (d-1)r^p)\theta}{l \cdot d-1C_{l-1}(q^{1-p} - 1)} \|x\|^p
$$

for all $x \in \mathcal{A}$.

**Proof.** Define $\varphi(x_1, \ldots, x_d, z, w) = \theta(\sum_{j=1}^d \|x_j\|^p + \|z\|^p + \|w\|^p)$, and apply Theorem 7. □

**Theorem 9.** Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying (vi) and (viii) such that

$$
\| d \cdot d-2C_{l-2}h\left(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot d-2C_{l-2}}\right) + d-2C_{l-1} \sum_{j=1}^d \mu h(x_j) \\
-l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(x_{j_1} + \cdots + x_{j_l}) - h(z)h(w) \|
\leq \varphi(x_1, \ldots, x_d, z, w)
$$

for $\mu = 1, i$, and all $x_1, \ldots, x_d, z, w \in \mathcal{A}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique $*$-homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (ix).

**Proof.** By the same reasoning as the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $\Theta : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (ix).

The rest of the proof is the same as the proofs of Theorems 1 and 7. □

**4. Stability of linear $*$-derivations on unital $C^*$-algebras**

We are going to show the generalized Hyers-Ulam-Rassias stability of linear $*$-derivations on unital $C^*$-algebras.

**Theorem 10.** Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying (vi) and (viii) such
that
\[
\|d \cdot d^{-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot d^{-2}C_{l-2}})
+ d^{-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j)
- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - zh(w) - wh(z)\|
\]
\[
\leq \varphi(x_1, \ldots, x_d, z, w)
\]
for all \(\mu \in \mathbb{T}^1\) and all \(x_1, \ldots, x_d, z, w \in \mathcal{A}\). Then there exists a unique \(C\)-linear \(*\)-derivation \(D : \mathcal{A} \to \mathcal{A}\) such that
\[
\|h(x) - D(x)\| \leq \frac{1}{l \cdot d^{-1}C_{l-1}} \tilde{\varphi}(qx, rx, \ldots, rx, 0, 0)
d \text{times}
\]
for all \(x \in \mathcal{A}\).

Proof. Put \(z = w = 0\) and \(\mu = 1 \in \mathbb{T}^1\) in (x). By the same reasoning as the proof of Theorem 1, there exists a unique \(C\)-linear involution \(D : \mathcal{A} \to \mathcal{A}\) satisfying the inequality (xi). The \(C\)-linear mapping \(D : \mathcal{A} \to \mathcal{A}\) is given by
\[
D(x) = \lim_{n \to \infty} \frac{1}{q^n}h(q^n x)
\]
for all \(x \in \mathcal{A}\).

It follows from (12) that
\[
D(x) = \lim_{n \to \infty} \frac{h(q^{2n} x)}{q^{2n}}
\]
for all \(x \in \mathcal{A}\). Let \(x_1 = \cdots = x_d = 0\) in (x). Then we get
\[
\|d \cdot d^{-2}C_{l-2}h(\frac{zw}{d \cdot d^{-2}C_{l-2}}) - zh(w) - wh(z)\| \leq \varphi(0, \ldots, 0, z, w)
d \text{times}
\]
for all \(z, w \in \mathcal{A}\). Since
\[
\frac{1}{q^{2n}} \varphi(0, \ldots, 0, q^n z, q^n w) \leq \frac{1}{q^n} \varphi(0, \ldots, 0, q^n z, q^n w),
d \text{times}
\]
\[ \frac{1}{q^{2n}} \|d \cdot d_{-2}C_{l-2}h(\frac{1}{d \cdot d_{-2}C_{l-2}}q^n z \cdot q^n w) \\
- q^n zh(q^n w) - q^n wh(q^n z)\| \leq \frac{1}{q^{2n}} \varphi(0, \ldots, 0, q^n z, q^n w) \leq \frac{q^n}{d} \varphi(0, \ldots, 0, q^n z, q^n w) \]

(14)

for all \(z, w \in \mathcal{A}\).

By (x), (13), and (14),

\[
d \cdot d_{-2}C_{l-2}D(\frac{zw}{d \cdot d_{-2}C_{l-2}}) = \lim_{n \to \infty} d \cdot d_{-2}C_{l-2}h(\frac{1}{d \cdot d_{-2}C_{l-2}}q^{2n} zw) \frac{q^{2n}}{q^{2n}} = \lim_{n \to \infty} d \cdot d_{-2}C_{l-2}h(\frac{1}{d \cdot d_{-2}C_{l-2}}q^n z \cdot q^n w) \frac{q^n}{q^n} = \lim_{n \to \infty} (q^n z \cdot h(q^n w) + q^n w \cdot h(q^n z)) \frac{q^n}{q^n} = zD(w) + wD(z)
\]

for all \(z, w \in \mathcal{A}\). But since \(D\) is \(\mathbb{C}\)-linear,

\[D(zw) = d \cdot d_{-2}C_{l-2}D(\frac{zw}{d \cdot d_{-2}C_{l-2}}) = zD(w) + wD(z)\]

for all \(z, w \in \mathcal{A}\). Hence the \(\mathbb{C}\)-linear mapping \(D : \mathcal{A} \to \mathcal{A}\) is a \(\ast\)-derivation satisfying the inequality (xi), as desired.

\[\square\]

**Corollary 11.** Let \(h : \mathcal{A} \to \mathcal{A}\) be a mapping with \(h(0) = 0\) for which there exist constants \(\theta \geq 0\) and \(p \in [0, 1)\) such that

\[
\|d \cdot d_{-2}C_{l-2}h(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot d_{-2}C_{l-2}}) + d_{-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j) \\
- l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) - zh(w) - wh(z)\| \leq \theta\left(\sum_{j=1}^{d} \|x_j\|^p + \|z\|^p + \|w\|^p\right),
\]

\[
\|h(q^n u^*) - h(q^n w^*)\| \leq dq^np \theta
\]
for all $\mu \in T^1$, all $u \in \mathcal{U}(A)$, all $n = 0, 1, 2, \ldots$, and all $x_1, \ldots, x_d, z, w \in A$. Then there exists a unique $C$-linear $*$-derivation $D : A \to A$ such that
\[
\|h(x) - D(x)\| \leq \frac{q^{1-p}(q^p + (d - 1)r^p)\theta}{l \cdot d \cdot C_{l-1}(q^{1-p} - 1)} \|x\|^p
\]
for all $x \in A$.

Proof. Define $\varphi(x_1, \ldots, x_d, z, w) = \theta\left(\sum_{j=1}^d x_j^p + \|z\|^p + \|w\|^p\right)$, and apply Theorem 10. \qed

**Theorem 12.** Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^{d+2} \to [0, \infty)$ satisfying (vi) and (viii) such that
\[
\|d \cdot d^{l-2} C_{l-1} \frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{zw}{d \cdot d^{l-2} C_{l-1}} + d^{l-2} C_{l-1} \sum_{j=1}^d \mu h(x_j) - l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \frac{\mu h(x_{j_1} + \cdots + x_{j_l})}{l} - wh(z)\|
\leq \varphi(x_1, \ldots, x_d, z, w)
\]
for $\mu = 1, i$, and all $x_1, \ldots, x_d, z, w \in A$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique $C$-linear $*$-derivation $D : A \to A$ satisfying the inequality (xi).

Proof. By the same reasoning as the proof of Theorem 3, there exists a unique $C$-linear mapping $D : A \to A$ satisfying the inequality (xi).

The rest of the proof is the same as the proofs of Theorems 1 and 10. \qed

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