SKewed POLYNOMIAL RINGS OVER $\sigma$-QUASI-BAER AND $\sigma$-PRINCIPALLY QUASI-BAER RINGS

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Abstract. Let $R$ be a ring $R$ and $\sigma$ be an endomorphism of $R$. $R$ is called $\sigma$-rigid (resp. reduced) if $a\sigma(a) = 0$ (resp. $a^2 = 0$) for any $a \in R$ implies $a = 0$. An ideal $I$ of $R$ is called a $\sigma$-ideal if $\sigma(I) \subseteq I$. $R$ is called $\sigma$-quasi-Baer (resp. right (or left) $\sigma$-p.q.-Baer) if the right annihilator of every $\sigma$-ideal (resp. right (or left) principal $\sigma$-ideal) of $R$ is generated by an idempotent of $R$. In this paper, a skew polynomial ring $A = R[x; \sigma]$ of a ring $R$ is investigated as follows: For a $\sigma$-rigid ring $R$, (1) $R$ is $\sigma$-quasi-Baer if and only if $A$ is quasi-Baer if and only if $A$ is $\tilde{\sigma}$-quasi-Baer for every extended endomorphism $\tilde{\sigma}$ on $A$ of $\sigma$; (2) $R$ is right $\sigma$-p.q.-Baer if and only if $R$ is $\sigma$-p.q.-Baer if and only if $A$ is right p.q.-Baer if and only if $A$ is p.q.-Baer if and only if $A$ is $\tilde{\sigma}$-p.q.-Baer if and only if $A$ is right $\tilde{\sigma}$-p.q.-Baer for every extended endomorphism $\tilde{\sigma}$ on $A$ of $\sigma$.

1. Introduction and some definitions

Throughout this paper, $R$ will denote an associative ring with identity, $\sigma$ will be an endomorphism of $R$, and $A$ will be the skew polynomial ring $R[x; \sigma]$, i.e., $A$ is a ring of polynomials over $R$ in an indeterminate $x$ with multiplication subject to the relation $xr = \sigma(r)x$ for all $r \in R$. When $\sigma$ is identity 1, we write $R[x]$ for $R[x; 1]$. In [11] Kaplansky introduced the Baer rings (i.e., rings in which the right annihilator of every nonempty subset is generated (as a right ideal) by an idempotent) to abstract various properties of rings of operators on Hilbert spaces. In [8], Clark introduced the quasi-Baer rings (i.e., rings in which the right annihilator of every right ideal is generated (as a right ideal) by an
idempotent) which are generalizations of Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. Further works on quasi-Baer rings appear in [12], [3], [4] and [5]. The study of Baer and quasi-Baer rings has its roots in functional analysis. Recently, in [6] Birkenmeier, Kim and Park defined a right (or left) principally quasi-Baer (simply, called right (or left) $p.q.$-Baer) ring as a generalization of quasi-Baer ring by the rings in which the right (or left) annihilator of every right (or left) principal ideal of $R$ is generated by an idempotent of $R$. $R$ is called a $p.q.$-Baer ring if it is both right p.q.-Baer and left p.q.-Baer. Another generalization of Baer ring is a p.p.-ring. A ring $R$ is called a right (resp. left) p.p.-ring if the right (resp. left) annihilator of any element of $R$ is generated by an idempotent of $R$. $R$ is called a p.p.-ring if it is both right and left p.p.-ring.

A subset $S$ of a ring $R$ is called a $\sigma$-set if $S$ is a $\sigma$-stable set, i.e., $\sigma(S) \subseteq S$. In particular, if a singleton set $S = \{a\}$ of $R$ is $\sigma$-set, i.e., $\sigma(a) = a$, then $a$ is called a $\sigma$-element of $R$. A left (right, two-sided) ideal $I$ of $R$ is called a left (right, two-sided) $\sigma$-ideal if $I$ is a $\sigma$-set. By analog, we can define a $\sigma$-Baer ring (resp. $\sigma$-quasi-Baer-ring) by the ring in which the right annihilator of every $\sigma$-set (resp. $\sigma$-ideal) is generated by an idempotent. We also define a right (or left) $\sigma$-p.q.-Baer ring (resp. right (or left) $\sigma$-p.p.-ring) by the ring in which the right (or left) annihilator of every right (or left) principal $\sigma$-ideal (resp. $\sigma$-element) is generated by an idempotent. $R$ is called a $\sigma$-p.q.-Baer ring (resp. $\sigma$-p.p.-ring) if it is both right $\sigma$-p.q.-Baer (resp. right $\sigma$-p.p.) and left $\sigma$-p.q.-Baer (resp. left $\sigma$-p.p.). In this paper, we denote the right (resp. left) annihilator of a subset $S$ of a ring $R$ by $r_R(S) = \{a \in R \mid Sa = 0\}$ (resp. $l_R(S) = \{a \in R \mid aS = 0\}$). We recall that $R$ is a $\sigma$-rigid (resp. reduced) ring if for some endomorphism $\sigma$ of $R$, $a\sigma(a) = 0$ (resp. $a^2 = 0$) implies that $a = 0$ for each $a \in R$. We can note that any $\sigma$-rigid ring is reduced and this endomorphism $\sigma$ is a monomorphism. Now we can observe the following implications: Baer (resp. quasi-Baer) $\Rightarrow$ Baer (resp. $\sigma$-quasi-Baer); right (or left) p.q.-Baer (resp. right (or left) p.p.) $\Rightarrow$ right (or left) $\sigma$-p.q.-Baer (resp. right (or left) $\sigma$-p.p.); $\sigma$-Baer $\Rightarrow$ $\sigma$-quasi-Baer $\Rightarrow$ $\sigma$-p.q.-Baer. All the implications are strict by the following examples:

Example 1. [9, Example 9] Let $Z$ be the ring of integers and consider the ring $Z \oplus Z$ with the usual addition and multiplication. Then the subring $R = \{(a, b) \in Z \oplus Z \mid a \equiv b \pmod{2}\}$ of $Z \oplus Z$ is a commutative reduced ring which has only two idempotents $(0, 0)$ and $(1, 1)$. Observe
that \( R \) is not p.p. (and then \( R \) is not Baer). Indeed, for \( a = (2, 0) \in R \), 
\( r_R(a) = (0) \oplus 2\mathbb{Z} \) which is not generated by an idempotent of \( R \). Since \( R \) is reduced, \( R \) is not p.q.-Baer and hence it is not quasi-Baer. Let 
\( \sigma : R \rightarrow R \) be a map defined by \( \sigma((a, b)) = (b, a) \) for all \((a, b) \in R\). 
Then \( \sigma \) is an endomorphism of \( R \). Note that all the \( \sigma \)-sets of \( R \) are 
\( S \oplus S \) for some subset \( S \) of \( \mathbb{Z} \). Let \( T = S \oplus S \). If \( T = (0) \), then 
\( r_R(T) = R = (1, 1)R \). If \( T \neq (0) \), then \( r_R(T) = (0) = (0, 0)R \). Hence \( R \) 
is \( \sigma \)-Baer, and so \( R \) is \( \sigma \)-quasi-Baer, \( \sigma \)-p.q.-Baer and \( \sigma \)-p.p.

**Example 2.** Let \( Z \) be the ring of integers. Let \( R = \begin{pmatrix} Z & Z \\ 0 & Z \end{pmatrix} \) be 
the upper \( 2 \times 2 \) triangular matrix ring over \( Z \). Since \( Z \) is quasi-Baer, \( R \) is quasi-Baer by [12, Proposition 9]. But it is neither left p.p. nor 
right p.p. by [7, Example 8.1] and hence it is not p.p.. Consider an 
endomorphism \( \sigma : R \rightarrow R \) given by 
\[
\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R.
\]

We claim that \( R \) is \( \sigma \)-p.p. but it is not \( \sigma \)-Baer. First, note that every 
\( \sigma \)-element of \( R \) is of the form

\[
\alpha = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.
\]

Let \( \beta = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in r_R(\alpha) \) be arbitrary. Then \( \alpha \beta = \begin{pmatrix} ax & ay \\ 0 & cz \end{pmatrix} = 0. \)

Consider the following four cases;
(i) If \( a \) and \( c \neq 0 \), then \( x = y = z = 0 \). Thus \( r_R(\alpha) = (0) \), which is 
generated by idempotent \( 0 \) of \( R \).
(ii) If \( a \neq 0 \) and \( c = 0 \), then \( x = y = 0 \) and \( z \) is arbitrary. Thus

\[
r_R(\alpha) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \in R \right\} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R,
\]
i.e., it is generated by an idempotent \( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) of \( R \).
(iii) If \( a = 0 \) and \( c \neq 0 \), then \( x, y \) are arbitrary and \( z = 0 \). Thus

\[
r_R(\alpha) = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in R \right\} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} R,
\]
i.e., it is generated by an idempotent \( \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \) of \( R \).

(iv) If \( a \) and \( c = 0 \), then \( x, y \) and \( z \) are arbitrary. Thus \( r_R(\alpha) = R \), which is generated by idempotent \( 1 \) of \( R \). Hence \( R \) is a right \( \sigma \)-p.p. ring. Similarly, we can show that \( R \) is a left \( \sigma \)-p.p. ring.

Consequently, \( R \) is a \( \sigma \)-p.p. ring.

**Example 3.** [6, Example 1.3] Let \( Z_2 \) be the field of two elements and consider \( R = \{(x_n) \in \prod_{i=1}^{\infty} Z_2 \mid x_n \text{ is eventually constant} \} \). Then \( R \) is a Boolean ring which is not self-injective. By [12, p.79, p.249 and p.250], \( R \) is not Baer and hence it is not quasi-Baer since \( R \) is reduced. But \( R \) is p.q.-Baer and hence it is p.p. since \( R \) is reduced.

1. Let \( \sigma_1 : R \to R \) be defined by \( \sigma_1((x_1, x_2, \ldots)) = (x_2, x_3, \ldots) \). Then \( \sigma_1 \) is an endomorphism of \( R \). Note that the \( \sigma_1 \)-ideals of \( R \) are only \( R \) and \( 0 \). Hence \( R \) is \( \sigma_1 \)-quasi-Baer.

2. Let \( \sigma_2 : R \to R \) be defined by \( \sigma_1((x_1, x_2, x_3, \ldots)) = (0, x_2, x_3, \ldots) \). Then \( \sigma_1 \) is an endomorphism of \( R \). Note that every ideal of \( R \) is a \( \sigma_2 \)-ideal of \( R \). Hence \( R \) is not \( \sigma_2 \)-quasi-Baer. But \( R \) is \( \sigma_2 \)-p.q.-Baer.

3. Let \( \sigma_3 : R \to R \) be defined by \( \sigma_3((x_1, x_2, x_3, \ldots)) = (x_2, x_1, x_3, \ldots) \) and consider a projection \( \pi : R \to R \) given by \( \pi((x_1, x_2, \ldots)) = (x_3, x_4, \ldots) \). Then \( \sigma_3 \) is an endomorphism of \( R \). Note that every ideal of \( R \) is not always \( \sigma_3 \)-ideal of \( R \), for example, \((0) \times Z_2 \times \pi(I) \) is an ideal of \( R \) for some ideal \( I \) of \( R \) but it is not \( \sigma_3 \)-ideal of \( R \). On the other hand, for any ideal \( I \) of \( R \), \( J = Z_2 \times Z_2 \times \pi(I) \) and \( K = (0) \times (0) \times \pi(I) \) are \( \sigma_3 \)-ideals of \( R \). Then \( r_R(J) = (0) \times (0) \times r_R(\pi(I)) \) and \( r_R(K) = Z_2 \times Z_2 \times r_R(\pi(I)) \). Since \( R \) is not quasi-Baer, \( \pi(R) \) is not quasi-Baer and so \( R \) is not \( \sigma_3 \)-quasi-Baer. But \( R \) is \( \sigma_3 \)-p.q.-Baer.

We begin with the following lemmas:

**Lemma 1.1.** Let \( R \) be a ring with an endomorphism \( \sigma \). Then

1. If \( I \) is a right \( \sigma \)-ideal of \( R \), then \( RI \) is a right \( \sigma \)-ideal of \( R \);
2. If \( I \) is a left \( \sigma \)-ideal of \( R \), then \( IR \) is a left \( \sigma \)-ideal of \( R \).

**Proof.** (1) Let \( I \) be a right \( \sigma \)-ideal of \( R \). Clearly, \( RI \) is a right ideal of \( R \). Let \( t \in RI \) be arbitrary. Then \( t = \sum_{i=1}^{n} a_i b_i \) for some \( a_i \in R, b_i \in I \) and some integer \( n \in \mathbb{Z}^{+} \). Since \( I \) is a right \( \sigma \)-ideal of \( R \), \( \sigma(I) \subseteq I \). For each \( i \), \( \sigma(a_i b_i) = \sigma(a_i)\sigma(b_i) \in RI \), and so \( \sigma(RI) \subseteq RI \). Hence \( RI \) is a right \( \sigma \)-ideal of \( R \).

(2) It follows from the similar argument given as in (1). \( \square \)
Lemma 1.2. Let $R$ be a ring with an endomorphism $\sigma$. Then $R$ is $\sigma$-quasi-Baer if and only if the right annihilator of every right $\sigma$-ideal of $R$ is generated by an idempotent.

Proof. For any right $\sigma$-ideal $I$ of $R$, $RI$ is a $\sigma$-ideal of $R$ and $r_R(I) = r_R(RI)$ since $R$ has an identity. \hfill \Box

Lemma 1.3. Let $R$ be a $\sigma$-rigid ring. Then $R$ is $\sigma$-Baer if and only if $R$ is $\sigma$-quasi-Baer.

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose that $R$ is $\sigma$-quasi-Baer. Let $S$ be any $\sigma$-set of $R$. Consider the right ideal $< S >$ of $R$ generated by $S$. Since $S$ is a $\sigma$-set of $R$, $< S >$ is a right $\sigma$-ideal of $R$. Since $R$ is $\sigma$-quasi-Baer, $r_R(< S >) = eR$ for some idempotent $e \in R$ by Lemma 1.2. We will show that $r_R(S) = r_R(< S >)$. Clearly, $r_R(< S >) \subseteq r_R(S)$. Let $b = \sum_{i=1}^{n} s_i x_i \in < S >$ be arbitrary. If $a \in r_R(S)$, then $s_i a = 0$ for all $s_i \in S$. Since $R$ is reduced, $s_i a = 0$ if and only if $as_i = 0$ and only if $s_i Ra = 0$. Then $0 = \sum_{i=1}^{n} (as_i) x_i = \sum_{i=1}^{n} (s_i x_i) a = ba$, and so $a \in r_R(< S >)$. Thus $r_R(S) = r_R(< S >) = eR$. Hence $R$ is $\sigma$-Baer. \hfill \Box

Corollary 1.4. Let $R$ be a reduced ring. Then $R$ is Baer if and only if $R$ is quasi-Baer.

Proof. It follows from Lemma 1.3 by letting $\sigma = 1$. \hfill \Box

Lemma 1.5. Let $R$ be a $\sigma$-rigid ring. Then the following statements are equivalent:

1. $R$ is a right $\sigma$-p.p.-ring;
2. $R$ is a $\sigma$-p.p.-ring;
3. $R$ is a right $\sigma$-p.q.-Baer ring;
4. $R$ is a $\sigma$-p.q.-Baer ring;
5. For any $\sigma$-element $a \in R$ and any positive integer $n$, $r_R(a^nR) = eR$ for some idempotent $e \in R$.

Proof. Since $R$ is $\sigma$-rigid, $r_R(a) = l_R(a) = r_R(aR) = l_R(Ra) = r_R(a^nR)$ for any $\sigma$-element $a \in R$ and any positive integer $n$. Hence we have the result. \hfill \Box

In [1], Armendariz has shown that if $R$ is reduced, then $R$ is a Baer ring if and only if the polynomial ring $R[x]$ is a Baer ring. In this paper, we will generalize the result by showing that if $R$ is $\sigma$-rigid, then $R$ is
a $\sigma$-quasi-Baer ring if and only if the skew polynomial ring $R[x; \sigma]$ is a quasi-Baer ring; $R$ is a right (or left) $\sigma$-p.q.-Baer ring if and only if the skew polynomial ring $R[x; \sigma]$ is a right (or left) p.q.-Baer ring.

**Lemma 1.6.** Let $R$ be a $\sigma$-rigid ring. Then for all $a, b, c, d \in R$,

1. $a\sigma(b) = 0$ if and only if $\sigma(b)a = 0$
2. If $ab = 0$ and $bc + da = 0$, then $bc = da = 0$
3. If $ab = 0$ and $ad + cb = 0$, then $ad = cb = 0$
4. If $ab = 0$, then $a\sigma(b) = \sigma(a)b = 0$
5. If $a\sigma^k(b) = 0$ for some positive integer $k$, then $ab = 0$

**Proof.** (1) is clear.

(2) If $ab = 0$ and $bc + da = 0$, then $0 = (bc + da)b = (bc)b + (da)b = bcb$, and so $bc = 0$. Hence $da = 0$.

(3) It is similar to the proof of (2).

(4) Suppose that $ab = 0$. Since $R$ is reduced, $ba = 0$. Thus

$$a\sigma(b)\sigma(a\sigma(b)) = a\sigma(ba)\sigma^2(b) = 0.$$  

Since $R$ is $\sigma$-rigid, $a\sigma(b) = 0$. Similarly, if $ab = 0$, then $\sigma(a)b = 0$.

(5) If $a\sigma^k(b) = 0$ for some positive integer $k$, then by using (4) repeatedly we have $\sigma^k(ab) = \sigma^k(a)\sigma^k(b) = 0$, and so $ab = 0$ because $\sigma$ is a monomorphism.  

For a ring $R$ with an endomorphism $\sigma$, there exists an endomorphism of $A = R[x; \sigma]$ which extends $\sigma$. For example, consider a map $\tilde{\sigma}$ on $A$ defined by $\tilde{\sigma}(f(x)) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n$ for all $f(x) = a_0 + a_1x + \cdots + a_nx^n \in A$. Then $\tilde{\sigma}$ is an endomorphism of $A$ and $\tilde{\sigma}(a) = \sigma(a)$ for all $a \in R$, which means that $\tilde{\sigma}$ is an extension of $\sigma$. We call the endomorphism of $A = R[x; \sigma]$ which extends $\sigma$ an extended endomorphism of $\sigma$. Let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A$ of $\sigma$. Note that $\Sigma_\sigma \neq \emptyset$ since $\tilde{\sigma} \in \Sigma_\sigma$.

**Lemma 1.7.** Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. Then

1. If $I$ is a $\sigma$-ideal of $R$, then $IA$ is a $\theta$-ideal of $A$ for all $\theta \in \Sigma_\sigma$;
2. If $I$ is a right principal $\sigma$-ideal of $R$, then $IA$ is a right principal $\theta$-ideal of $A$ for all $\theta \in \Sigma_\sigma$;
3. If $I$ is a left principal $\sigma$-ideal of $R$, then $AI$ is a left principal $\theta$-ideal of $A$ for all $\theta \in \Sigma_\sigma$.  

Proof. It is straitforward.

Lemma 1.8. Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. Then $R$ is $\sigma$-rigid if and only if $A$ is $\theta$-rigid for all $\theta \in \Sigma_\sigma$. In this case, $\sigma(e) = e$ for every idempotent $e \in R$.

Proof. Assume that $R$ is $\sigma$-rigid and $A$ is not $\theta$-rigid for some $\theta \in \Sigma_\sigma$. Then there exists a nonzero $f \in A$ such that $f \theta(f) = 0$. Since $R$ is $\sigma$-rigid, $f \notin R$. Let $f = \sum_{i=0}^{m} a_i x^i$ where $a_i \in R, a_m \neq 0$ for some $m \geq 1$. Since $f \theta(f) = 0$, $a_m \sigma^m(a_m) = 0$. Since $R$ is $\sigma$-rigid, $a_m^2 = 0$ by Lemma 1.6, and then $a_m = 0$ since $R$ is reduced, a contradiction. Hence $A$ is $\theta$-rigid for all $\theta \in \Sigma_\sigma$. The converse is true by the definition of extended endomorphism of $\sigma$. Let $e$ be any idempotent of $R$. In case that $A$ is $\theta$-rigid for each $\theta \in \Sigma_\sigma$ (and then $A$ is reduced). Hence $e$ is central idempotent in $A$, and thus $ex = xe = \sigma(e)x$, which implies that $\sigma(e) = e$.

Note that for a reduced ring $R$, $A = R[x; \sigma]$ is not necessarily reduced. Indeed, consider the reduced ring $R$ and $\sigma$ introduced in Example 1. Let $f = (0,2)x \in A$. Then $f^2 = (0,2)x(0,2)x = (0,2)\sigma(0,2)x^2 = (0,2)(2,0)x^2 = (0,0)x^2 = 0$. But $f \neq 0$. Hence $A$ is not reduced.

We need the following corollary as a special case of [9, Proposition 6].

Corollary 1.9. Let $R$ be a $\sigma$-rigid ring. Then for any

$$f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma],$$

$fg = 0$ if and only if $a_i b_j = 0$ for each $i, j$.

2. Skew polynomial rings over $\sigma$-quasi-Baer and $\sigma$-p.q.-Baer rings

We recall from [2] an idempotent $e \in R$ is left (resp. right) semicentral in $R$ if $eae = ae$ (resp. $eae = ea$), for all $a \in R$. Equivalently, an idempotent $e \in R$ is left (resp. right) semicentral if $eR$ (resp. $Re$) is an ideal of $R$. Since the right annihilator of a right $\sigma$-ideal is an ideal, we can note that the right annihilator of a right $\sigma$-ideal is generated by a left semicentral idempotent in a $\sigma$-quasi-Baer ring. Observe that
if \( e_1, e_2, \ldots, e_m \) are left (or right) semicentral idempotents of \( R \), then
\[ e = e_1 e_2 \cdots e_m \]
is an idempotent of \( R \). Thus we can obtain the following lemma;

**Lemma 2.1.** Let \( R \) be a ring with an endomorphism \( \sigma \). Then \( R \) is a right (resp. left) \( \sigma \)-p.q.-Baer if and only if the right (resp. left) annihilator of every finitely generated right (resp. left) \( \sigma \)-ideal of \( R \) is generated by an idempotent of \( R \).

**Proof.** It is enough to show the left-handed version because the right-handed version is similarly proved. Suppose that \( R \) is right \( \sigma \)-p.q.-Baer and let \( I = \sum_{i=1}^{m} a_i R \) be any finitely generated right \( \sigma \)-ideal of \( R \). Then \( r_R(I) = \bigcap_{i=1}^{m} e_i R \) where \( r_R(a_i R) = e_i R \). By the above observation, \( r_R(I) \) is an ideal of \( R \) and \( e_i \) is a left semicentral idempotent of \( R \). Since each \( e_i \) is left semicentral idempotents of \( R \), \( e = e_1 e_2 \cdots e_m \) is idempotent of \( R \), and so \( r_R(I) = eR \). The converse is clear.

**Lemma 2.2.** Let \( R \) be a \( \sigma \)-rigid ring. If \( e \in R \) is a left semicentral idempotent, then \( e \) is also a left semicentral idempotent in \( R[x; \sigma] \).

**Proof.** Let \( f = \sum_{i=0}^{m} a_i x^i \in R[x; \sigma] \) be arbitrary. Since \( R \) is \( \sigma \)-rigid, \( \sigma(e) = e \) for any idempotent \( e \in R \) by Lemma 1.8. Since \( e \) is a left semicentral idempotent, \( ea_i e = a_i e \) for each \( i \). Then \( fe = \sum_{i=0}^{m} a_i \sigma_i(e)x^i = \sum_{i=0}^{m} a_i e x^i = \sum_{i=0}^{m} ea_i e x^i = efe \). Hence \( e \) is a left semicentral idempotent in \( R[x; \sigma] \).

**Theorem 2.3.** Let \( R \) be a ring with an endomorphism \( \sigma \) and let \( \sigma \) be the set of all extended endomorphisms on \( A = R[x; \sigma] \) of \( \sigma \). If \( R \) is \( \sigma \)-rigid, then the following are equivalent:

1. \( R \) is \( \sigma \)-quasi-Baer;
2. \( A \) is quasi-Baer;
3. \( A \) is \( \theta \)-quasi-Baer for all \( \theta \in \sigma \).

**Proof.** (1) \( \Rightarrow \) (2). Suppose that \( R \) is \( \sigma \)-quasi-Baer. Let \( I \) be an arbitrary ideal of \( A \). If \( g \in r_A(I) \), then \( fg = 0 \) for all \( f \in I \). Let \( f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j \). Then by Corollary 1.9, \( a_i b_j = 0 \) for all \( i, j \). Consider the set \( I_c \) of all coefficients of polynomials in \( I \). Then \( I_c \) is an ideal of \( R \) and \( b_0, b_1, \ldots, b_n \in r_R(I_c) \). We can observe that \( I_c \) is an \( \sigma \)-ideal of \( R \). Indeed, for any \( f = \sum_{i=0}^{m} a_i x^i \in I, xf = \sum_{i=0}^{m+1} \sigma(a_i)x^i \), and so \( \sigma(a_i) \in I_c \) for each \( i \). Thus \( I_c \) is a \( \sigma \)-ideal of \( R \). Since \( R \) is \( \sigma \)-quasi-Baer and \( I_c \) is a \( \sigma \)-ideal of \( R \), \( r_R(I_c) = eR \) for some idempotent \( e \in R \). Thus \( g = ge \) and hence \( r_A(I) \subseteq eA \). Now \( I_c e = 0 \). Since \( \sigma(e) =...
by Lemma 1.8, we have $Ie = 0$ so $eA \subseteq r_A(I)$. Therefore $r_A(I) = eA$. Hence $A$ is quasi-Baer.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Suppose that $A$ is $\theta$-quasi-Baer for all $\theta \in \Sigma_\sigma$. Let $I$ be any $\sigma$-ideal of $R$. Then by Lemma 1.7, $IA$ is a $\theta$-ideal of $A$. Since $A$ is $\theta$-quasi-Baer, $r_A(IA) = eA$ for some semicentral idempotent $e \in A$. Since $A$ is $\theta$-rigid (and so $A$ is reduced) by Lemma 1.8, $e$ is a central idempotent in $A$, and hence $e$ is an idempotent in $R$ by [10, Theorem 3.15]. Since $r_R(I) = r_A(IA) \cap R = eR$, $R$ is $\sigma$-quasi-Baer.

**Remark.** (1) If $\sigma$ is an automorphism, we can check the condition “$R$ is $\sigma$-rigid” does not need by using a similar method in the proof of Theorem 1.2 in [6]. (2) there is an example of a $\sigma$-quasi-Baer ring $R$ and an endomorphism $\sigma$ of $R$ such that $R[x; \sigma]$ is not quasi-Baer (refer Example 1.4 in [6]).

**Corollary 2.4.** Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. If $R$ is $\sigma$-rigid, then the following are equivalent:

1. $R$ is $\sigma$-Baer;
2. $A$ is Baer;
3. $A$ is $\theta$-quasi-Baer for all $\theta \in \Sigma_\sigma$.

**Proof.** It follows from Lemma 1.3 and Theorem 2.3.

**Corollary 2.5.** [1, Theorem A] Let $R$ be a reduced ring and let $A = R[x]$. Then $R$ is Baer if and only if $R[x]$ is Baer.

**Proof.** It follows from Corollary 1.4 and Corollary 2.4.

**Theorem 2.6.** Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_\sigma$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. If $R$ is $\sigma$-rigid, then the following are equivalent:

1. $R$ is right $\sigma$-p.q.-Baer;
2. $R$ is $\sigma$-p.q.-Baer;
3. $A$ is right p.q.-Baer;
4. $A$ is p.q.-Baer;
5. $A$ is $\theta$-p.q.-Baer for all $\theta \in \Sigma_\sigma$;
6. $A$ is right $\theta$-p.q.-Baer for all $\theta \in \Sigma_\sigma$. 

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 1.5. (3) $\Leftrightarrow$ (4) also follows from Lemma 1.5 by letting $\sigma = 1$. (4) $\Rightarrow$ (5) $\Rightarrow$ (6) is clear. It remains to show that (1) $\Rightarrow$ (3) and (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). Suppose that $R$ is right $\sigma$-p.q.-Baer. Let $I$ be any right principal ideal of $A$ generated by $h = \sum_{k=0}^{n} a_{k}x^{k}$. If $g \in r_{A}(I)$, then $fg = 0$ for all $f \in I$. Let $f = \sum_{i=0}^{l} c_{i}x^{i}, g = \sum_{j=0}^{m} b_{j}x^{j}$. Then by Lemma 1.6, $c_{i}b_{j} = 0$ for all $i, j$. Let $I_{c}$ be the set of all coefficients of all $f \in I$. Note that $I_{c}$ is a right $\sigma$-ideal of $R$ and $b_{0}, b_{1}, \ldots, b_{n} \in r_{R}(I_{c})$ as given in the proof of Theorem 2.3. Since $I$ is a right principal ideal of $A$, $I_{c}$ is a right finitely generated ideal of $R$ with a generating set $\{a_{0}, \ldots, a_{n}\}$. Since $R$ is right $\sigma$-p.q.-Baer and $I_{c}$ is a right finitely generated $\sigma$-ideal of $R$, $r_{R}(I_{c}) = eR$ for some idempotent $e$ of $R$ by Lemma 2.1. Hence $r_{A}(I) = eA$, and so $A$ is right $\sigma$-p.q.-Baer.

(6) $\Rightarrow$ (1). Suppose that $A$ is right $\theta$-p.q.-Baer for all $\theta \in \Sigma_{\sigma}$. Let $I$ be any right principal $\sigma$-ideal of $R$. Then by Lemma 1.1, $IA$ is a right principal $\theta$-ideal of $A$. Since $A$ is $\theta$-p.q.-Baer, $r_{A}(IA) = eA$ for some semicentral idempotent $e \in A$. Since $A$ is $\theta$-rigid (and so reduced) by Lemma 1.8, $e$ is a central idempotent in $A$, and hence $e$ is an idempotent in $R$ by [10, Theorem 3.15]. Since $r_{R}(I) = r_{A}(IA) \cap R = eR$, $R$ is right $\sigma$-p.q.-Baer.

Corollary 2.7. Let $R$ be a ring with an endomorphism $\sigma$ and let $\Sigma_{\sigma}$ be the set of all extended endomorphisms on $A = R[x; \sigma]$ of $\sigma$. If $R$ is $\sigma$-rigid, then the following are equivalent:

1. $R$ is right $\sigma$-p.p.;
2. $R$ is $\sigma$-p.p.;
3. $A$ is right $p.p.$;
4. $A$ is $p.p.$;
5. $A$ is $\theta$-p.p. for all $\theta \in \Sigma_{\sigma}$;
6. $A$ is right $\theta$-p.p. for all $\theta \in \Sigma_{\sigma}$.

Proof. It follows from the Lemma 1.5 and Theorem 2.6. □

Corollary 2.8. [1, Theorem B] Let $R$ be a reduced. Then $R$ is p.p.-Baer if and only if $R[x]$ is p.p.-Baer;

Proof. It follows from the Lemma 1.5 (by letting $\sigma = 1$) and Corollary 2.7. □

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References


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