FREE LIE SUPERALGEBRAS AND THE REPRESENTATIONS OF $\mathfrak{gl}(m, n)$ AND $\mathfrak{q}(n)$

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Abstract. Let $\mathcal{L}$ be the free Lie superalgebra generated by a $\mathbb{Z}_2$-graded vector space $V$ over $\mathbb{C}$. Suppose that $\mathfrak{g}$ is a Lie superalgebra $\mathfrak{gl}(m, n)$ or $\mathfrak{q}(n)$. We study the $\mathfrak{g}$-module structure on the $k$th homogeneous component $\mathcal{L}_k$ of $\mathcal{L}$ when $V$ is the natural representation of $\mathfrak{g}$. We give the multiplicities of irreducible representations of $\mathfrak{g}$ in $\mathcal{L}_k$ by using the character of $\mathcal{L}_k$. The multiplicities are given in terms of the character values of irreducible (projective) representations of the symmetric groups.

1. Introduction

Let $\mathcal{L}$ be the free Lie algebra generated by a vector space $V$ over a field $k$. If $V$ is a representation of a group $G$ (finite or infinite), then $\mathcal{L}$ becomes a representation of $G$, and its homogeneous component $\mathcal{L}_k(k \geq 1)$ is a submodule of $\mathcal{L}$. Hence, it is natural to ask how $\mathcal{L}$ (or $\mathcal{L}_k$) decomposes into irreducible representations of $G$ whenever it is semisimple.

Consider $V$ as a representation of its full linear group. For simplicity, assume that $k = \mathbb{C}$. Let $V \cong \mathbb{C}^m$ be an $m$-dimensional vector space over $\mathbb{C}$, which is the natural representation of $GL(m)$, or $\mathfrak{gl}(m)$. The $k$-fold tensor product of $V$ is a $GL(m)$-module and decomposes into irreducible polynomial representations which are parameterized by the partitions $\lambda$ of $k$ with length $\ell(\lambda) \leq m$. Let $V^\lambda$ be the corresponding irreducible representation. It is well-known that the multiplicity of $V^\lambda$ in $V^{\otimes k}$ is given by the dimension of the irreducible representation $S^\lambda$ of the symmetric group $S_k$ corresponding to $\lambda$ ([30, 36]).

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Let \( \mathcal{L} \) be the free Lie algebra generated by \( V \) and \( \mathcal{L}_k(k \geq 1) \) its \( k \)th homogeneous component. Then \( \mathcal{L}_k \) is a finite dimensional \( GL(m) \)-submodule of \( V^\otimes k \). In [5], Brandt gave the character of \( \mathcal{L}_k \) (i.e. the trace of \( \text{diag}(x_1, \cdots, x_m) \) on \( \mathcal{L}_k \))

\[
ch \mathcal{L}_k = \frac{1}{k} \sum_{d \mid k} \mu(d)p_d(x)^{k/d}
\]

where \( p_d(x) = x_1^d + \cdots + x_m^d \) is the \( d \)th power symmetric function. Then from (1.1) and the Frobenius formula, the multiplicity of \( V^\lambda \) in \( \mathcal{L}_k \) is given by

\[
\frac{1}{k} \sum_{d \mid k} \mu(d) \chi^\lambda_{S_k} (\sigma_{(d^k/d)}),
\]

(which was first given by Wever [37]) where \( \chi^\lambda_{S_k} \) is the character of \( S^\lambda \) and \( \sigma_{(d^k/d)} \) is an element of cycle type \( (d^k/d) \) in \( S_k \) (see also [20, 21]). Also, various module structures of free Lie algebras have been studied in more general cases where \( V \) is a representation of an arbitrary group and the base field may have positive characteristic (see for example [6, 7, 9]).

In this paper, we will study the module structures of free Lie superalgebras generated by a representation of a Lie superalgebra. (see [16] for a general exposition on Lie superalgebras): The main interests in this paper are the super-analogues of (1.1) and (1.2). Let \( V \) be a \( \mathbb{Z}_2 \)-graded vector space over \( \mathbb{C} \), and \( \mathcal{L} \) the free Lie superalgebra generated by \( V \). We study the module structure of the \( k \)th homogeneous component \( \mathcal{L}_k \) when \( V \) is the natural representation of a Lie superalgebra \( \mathfrak{gl}(m,n) \) or \( \mathfrak{q}(n) \). As the main results, we describe the multiplicity of each irreducible representation in \( \mathcal{L}_k \).

This paper is organized as follows. In section 2, we introduce the character of \( \mathcal{L}_k \) given in [26]. We give here an alternative proof for the character of \( \mathcal{L}_k \) using the homological methods (cf. [19]). The character of \( \mathcal{L}_k \) is written as a linear combination of power super symmetric functions (Theorem 2.1). In section 3, we review some basic facts on the (super) symmetric functions and the irreducible characters of \( S_k \) and its double cover \( \tilde{S}_k \). Then, in section 4, we derive the multiplicities of irreducible representations in \( \mathcal{L}_k \), combining the character of \( \mathcal{L}_k \) and the expansion of power (super) symmetric functions into hook Schur functions and Schur \( Q \)-functions respectively (Theorem 4.1 and 4.3). The character values of the symmetric groups appear naturally as in the case of (1.2). In section 5, we consider the case when \( \mathcal{L} \) is the
free Lie algebra generated by $V = \mathbb{C}^n$. We describe the multiplicities of each irreducible polynomial representation of $\mathfrak{sp}(n)$ (n : even) and $\mathfrak{so}(n)$ in $L_k$ for $1 \leq k \leq n$, using the character values of the Brauer algebras (Proposition 5.3).

Throughout this paper, we assume that the ground field is $\mathbb{C}$.

2. Free Lie superalgebras and characters

In this section, we will derive the character of the homogeneous component of a free Lie superalgebra.

A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is a Lie superalgebra if there is a bilinear map $[,] : L \times L \rightarrow L$ such that

(i) $[L_a, L_b] \subset L_{a+b}$
(ii) $[x, y] = -(-1)^{ab}[y, x]$
(iii) $[x, [y, z]] = [[x, y], z] + (-1)^{ab}[y, [x, z]]$

for $a, b \in \mathbb{Z}_2$ and $x \in L_a$, $y \in L_b$.

Let $\Gamma$ be a countable abelian semigroup (under addition). We assume that every element in $\Gamma$ can be written as a sum of elements in $\Gamma$ in only finitely many ways, which we call the finiteness condition on $\Gamma$.

Let $V = \bigoplus_{\alpha \in \Gamma} V_\alpha$ be a $\Gamma$-graded vector space where each $V_\alpha = V_{(\alpha, 0)} \oplus V_{(\alpha, 1)}$ is a finite dimensional $\mathbb{Z}_2$-graded vector space with $\dim V_{(\alpha, 0)} = m_\alpha$ and $\dim V_{(\alpha, 1)} = n_\alpha$. Note that $V = \bigoplus_{(a, b) \in \Gamma \times \mathbb{Z}_2} V_{(a, b)}$ is also $(\Gamma \times \mathbb{Z}_2)$-graded. Set $V_a = \bigoplus_{n \in \Gamma} V_{(a, n)}$ for $a \in \mathbb{Z}_2$.

Consider the free Lie superalgebra $L = L_0 \oplus L_1$ generated by $V$. The universal enveloping algebra of $L$ is isomorphic to the tensor algebra $T(V) = \bigoplus_{k \geq 0} V \otimes_k V$ generated by $V$ (as $\mathbb{Z}_2$-graded algebras), and $L$ can be embedded into $T(V)$ in such a way that $[x, y] = x \otimes y - (-1)^{ab} y \otimes x$ for $x \in V_a$ and $y \in V_b$ ($a, b \in \mathbb{Z}_2$). Since $T(V)$ is $\Gamma$-graded, $L$ is also $\Gamma$-graded, that is, $L = \bigoplus_{\alpha \in \Gamma} L_\alpha$. Note that the dimension of $L_\alpha$ ($\alpha \in \Gamma$) is finite from the finiteness condition on $\Gamma$.

For each $\alpha \in \Gamma$, we set

$$L_{(\alpha, 0)} = L_\alpha \cap L_0,$$

$$L_{(\alpha, 1)} = L_\alpha \cap L_1,$$

(2.1)

Since $L_\alpha = L_{(\alpha, 0)} \oplus L_{(\alpha, 1)}$, $L$ is a $(\Gamma \times \mathbb{Z}_2)$-graded Lie superalgebra.

Set $G = \prod_{\alpha \in \Gamma} G_\alpha \subset GL(V)$ where $G_\alpha = GL(V_{(\alpha, 0)}) \times GL(V_{(\alpha, 1)})$. For each $(\alpha, a) \in \Gamma \times \mathbb{Z}_2$, $L_{(\alpha, a)}$ is a $G$-module where the action is given
by
\[ g \cdot [v_1[v_2[\cdots [v_{k-1}, v_k] \cdots]]] = [gv_1[gv_2[\cdots [gv_{k-1}, gv_k] \cdots]]] \]
for \( g \in G \) and \( v_i \in V \) \((1 \leq i \leq k)\).

For each \( \gamma \in \Gamma \), consider the following variables
\[ x^\gamma = (x_1^\gamma, \ldots, x_{m_\gamma}^\gamma) \in (\mathbb{C}^\times)^{m_\gamma}, \]
\[ y^\gamma = (y_1^\gamma, \ldots, y_{n_\gamma}^\gamma) \in (\mathbb{C}^\times)^{n_\gamma}. \]

For \((\alpha, a) \in \Gamma \times \mathbb{Z}_2\), we define the character of \( \mathcal{L}_{(\alpha, a)} \)
\[ \text{ch} \mathcal{L}_{(\alpha, a)} = \text{tr}((\text{diag}(x^\gamma, y^\gamma))_{\gamma \in \Gamma}| \mathcal{L}_{(\alpha, a)}). \]
From the finiteness condition on \( \Gamma \), \( \text{ch} \mathcal{L}_{(\alpha, a)} \) is a polynomial in \( x^\gamma, y^\gamma \)'s.
The character of \( \mathcal{L}_\alpha \) can be defined in the same way, and it is given by
\[ \text{ch} \mathcal{L}_\alpha = \text{ch} \mathcal{L}_{(\alpha, 0)} + \text{ch} \mathcal{L}_{(\alpha, 1)}. \]

Next, we define the set of partitions of \( \alpha \) \((\alpha \in \Gamma)\) to be
\[ P(\alpha) = \{ s = (s_\gamma)_{\gamma \in \Gamma} \mid s_\gamma \in \mathbb{Z}_{\geq 0}, \sum_{\gamma \in \Gamma} s_\gamma = \alpha \}. \]
It is a finite set from the finiteness condition on \( \Gamma \). For \( s \in P(\alpha) \), we write \(|s| = \sum s_\gamma\) and \( s! = \prod s_\gamma! \).

Now, we can state the formula for the character of \( \mathcal{L}_\alpha \) (see [26]). We give here an alternative proof based on the homological method used in [19].

**Theorem 2.1.** ([26]) For \( \alpha \in \Gamma \), we have
\[ \text{ch} \mathcal{L}_\alpha = \sum_{d \geq 0, d_\beta = \alpha} \frac{1}{d!} \mu(d) \sum_{s \in P(\beta)} \frac{(|s| - 1)!}{s!} \prod_{\gamma \in \Gamma} p_d(x^\gamma, y^\gamma)^{s_\gamma}, \]
where \( \mu \) is the Möbius function and
\[ p_d(x^\gamma, y^\gamma) = \sum_{i=1}^{m_\gamma} (x_i^\gamma)^d - \sum_{j=1}^{n_\gamma} (-y_j^\gamma)^d \]
for \( d \geq 1 \).

**Proof.** First, we will compute \( \text{ch} \mathcal{L}_{(\alpha, a)} \) for \((\alpha, a) \in \Gamma \times \mathbb{Z}_2\). It is already given in [19] as the special case of a more general formula (see (2.17) in section 2). But for the reader’s convenience and self-containedness, we give a proof restricting the arguments in [19] to the case of free Lie superalgebras.
For $k \geq 0$, let $C_k(\mathcal{L}) = \bigoplus_{p+q=k} \Lambda^p(\mathcal{L}_0) \otimes S^q(\mathcal{L}_1)$ where $\Lambda^p(\mathcal{L}_0)$ is the $p$th alternating space of $\mathcal{L}_0$ and $S^q(\mathcal{L}_1)$ is the $q$th symmetric space of $\mathcal{L}_1$ (note that $C_0(\mathcal{L}) = \mathbb{C}$). The homology modules $H_k(\mathcal{L}) = H_k(\mathcal{L}, \mathbb{C})$ are determined by the following complex:

(2.9)

$$\cdots \rightarrow C_k(\mathcal{L}) \xrightarrow{d_k} C_{k-1}(\mathcal{L}) \xrightarrow{d_{k-1}} \cdots \rightarrow C_1(\mathcal{L}) \xrightarrow{d_1} C_0(\mathcal{L}) \rightarrow 0,$$

where the differentials $d_k : C_k(\mathcal{L}) \rightarrow C_{k-1}(\mathcal{L})$ are given by

(2.10)

$$d_k((x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q))$$

$$= \sum_{1 \leq s < t \leq p} (-1)^{s+t} ([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_p)$$

$$\otimes (y_1 \cdots y_q)$$

$$+ \sum_{s=1}^p \sum_{t=1}^q (-1)^s (x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_p) \otimes ([x_s, y_t]y_1 \cdots \hat{y}_t \cdots y_q)$$

$$- \sum_{1 \leq s < t \leq q} ([y_s, y_t] \wedge x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \hat{y}_s \cdots \hat{y}_t \cdots y_q)$$

for $k \geq 2$, $x_i \in \mathcal{L}_0$, $y_j \in \mathcal{L}_1$ and $d_1 = 0$ (cf. [8],[11]).

From the $(\Gamma \times \mathbb{Z}_2)$-grading of $\mathcal{L}$, $C_k(\mathcal{L})$ and $H_k(\mathcal{L})$ are $(\Gamma \times \mathbb{Z}_2)$-graded vector spaces, and for each $(\alpha, a) \in \Gamma \times \mathbb{Z}_2$, we can define $\text{ch}C_k(\mathcal{L})_{(\alpha, a)}$ and $\text{ch}H_k(\mathcal{L})_{(\alpha, a)}$ as in (2.4). Set

(2.11)

$$\text{ch}C_k(\mathcal{L}) = \sum_{(\alpha, a) \in \Gamma \times \mathbb{Z}_2} \text{ch}C_k(\mathcal{L})_{(\alpha, a)} u^a v^a,$$

$$\text{ch}H_k(\mathcal{L}) = \sum_{(\alpha, a) \in \Gamma \times \mathbb{Z}_2} \text{ch}H_k(\mathcal{L})_{(\alpha, a)} u^a v^a,$$

where $u^\alpha$ is a formal variable satisfying $u^\alpha u^\beta = u^{\alpha+\beta}$ ($\alpha, \beta \in \Gamma$), and $v$ is a variable commuting with $u^\alpha$ satisfying $v^2 = 1$. By the Euler-Poincaré principle, we have

(2.12)

$$\sum_{k \geq 0} (-1)^k \text{ch}C_k(\mathcal{L}) = \sum_{k \geq 0} (-1)^k \text{ch}H_k(\mathcal{L}).$$
On the other hand, the left hand side of the above equation can be written as follows.

\[
\sum_{k \geq 0} (-1)^k \text{ch} C_k(\mathcal{L})
\]

\[
= \prod_{(\alpha, a) \in \Gamma \times \mathbb{Z}_2} \exp \left( -\sum_{r \geq 1} \frac{(-1)^a}{r} \text{tr} (g^r | \mathcal{L}_{(\alpha, a)}) u^{ra} (-v)^{-ra} \right),
\]

where \( g = (\text{diag}(x^\gamma, y^\gamma))_{\gamma \in \Gamma} \in G \). Since \( H_k(\mathcal{L}) = V \) when \( k = 1 \) and \( H_k(\mathcal{L}) = 0 \) otherwise (see Corollary 3.2 in [19]), we obtain the twisted denominator identity of \( \mathcal{L} \):

\[
\prod_{(\alpha, a) \in \Gamma \times \mathbb{Z}_2} \exp \left( -\sum_{r \geq 1} \frac{(-1)^a}{r} \text{tr} (g^r | \mathcal{L}_{(\alpha, a)}) u^{ra} (-v)^{-ra} \right)
\]

\[= 1 - \sum_{\beta \in \Gamma} (p_1(x^\beta) u^\beta + p_1(y^\beta) u^\beta v),\]

where \( p_1(x^\beta) = \sum_{i=1}^{m_{\beta}} x_i^\beta \) and \( p_1(y^\beta) = \sum_{j=1}^{n_{\beta}} y_j^\beta \).

For each \((\beta, b) \in \Gamma \times \mathbb{Z}_2\), we set

\[(2.15) \quad P(\beta, b) = \{ s = (s_{\gamma, a})_{(\gamma, a) \in \Gamma \times \mathbb{Z}_2} \mid s_{\gamma, a} \in \mathbb{Z}_{\geq 0}, \sum s_{\gamma, a}(\gamma, a) = (\beta, b) \} \]

and write \(|s| = \sum s_{\gamma, a} \) and \( s! = \prod s_{\gamma, a}! \).

Taking the formal logarithm on the inverse of (2.14), we get

\[
\sum_{(\alpha, a) \in \Gamma \times \mathbb{Z}_2} \sum_{r \geq 1} \frac{(-1)^a}{r} \text{tr} (g^r | \mathcal{L}_{(\alpha, a)}) u^{ra} (-v)^{-ra}
\]

\[= \sum_{(\beta, b) \in \Gamma \times \mathbb{Z}_2} W_g(\beta, b) u^\beta (-v)^b,\]

where

\[
(2.17) \quad W_g(\beta, b) = \sum_{s \in P(\beta, b)} \frac{|s| - 1)!}{s!} \prod_{\gamma \in \Gamma} p_1(x^\gamma)^{s_{\gamma, 0}} (-p_1(y^\gamma))^{s_{\gamma, 1}}.
\]

By comparing the coefficients of \( u^a (-v)^b \) on both sides and applying the Möbius inversion, we obtain

\[
(-1)^a \text{ch} \mathcal{L}_{(\alpha, a)} = \sum_{d > 0} \frac{1}{d} \mu(d) W_g(d, \beta, b).
\]
Now, we have
\[
\text{ch.} \mathcal{L}_{(\alpha,0)} = \sum_{d\beta=\alpha \atop d \text{ even}} \frac{1}{d} \mu(d) \{ W_{g^d}(\beta,0) + W_{g^d}(\beta,1) \}
\]
(2.19)
\[
+ \sum_{d\beta=\alpha \atop d \text{ odd}} \frac{1}{d} \mu(d) W_{g^d}(\beta,0),
\]
and
\[
\text{ch.} \mathcal{L}_{(\alpha,1)} = - \sum_{d\beta=\alpha \atop d \text{ odd}} \frac{1}{d} \mu(d) W_{g^d}(\beta,1).
\]
(2.20)

For any $\beta \in \Gamma$ and $d \geq 1$, it is straightforward to check that
\[
W_{g^d}(\beta,0) + (-1)^d W_{g^d}(\beta,1)
\]
(2.21)
\[
= \sum_{s \in P(\beta)} \frac{(s| - 1)!}{s!} \prod_{\gamma \in \Gamma} p_d(x^\gamma, y^\gamma)^{s^\gamma},
\]
where $p_d(x^\gamma, y^\gamma) = \sum_{i=1}^m (x_i^\gamma)^d - \sum_{j=1}^n (-y_j^\gamma)^d$.

Therefore, we get
\[
\text{ch.} \mathcal{L}_\alpha = \text{ch.} \mathcal{L}_{(\alpha,0)} + \text{ch.} \mathcal{L}_{(\alpha,1)}
\]
(2.22)
\[
= \sum_{d\beta=\alpha} \frac{1}{d} \mu(d) \{ W_{g^d}(\beta,0) + (-1)^d W_{g^d}(\beta,1) \}
\]
\[
= \sum_{d\beta=\alpha} \frac{1}{d} \mu(d) \sum_{s \in P(\beta)} \frac{(s| - 1)!}{s!} \prod_{\gamma \in \Gamma} p_d(x^\gamma, y^\gamma)^{s^\gamma}.
\]

\[\square\]

If we take $g$ as the identity element in $G$, then the character of $\mathcal{L}_\alpha$ yields the dimension of $\mathcal{L}_\alpha$:
\[
\sum_{d\beta=\alpha} \frac{1}{d} \mu(d) \sum_{s \in P(\beta)} \frac{(s| - 1)!}{s!} \prod_{\gamma \in \Gamma} (m_\gamma - (-1)^d n_\gamma)^{s_\gamma},
\]
(2.23)
which is a generalization of the Witt’s dimension formula for free Lie algebras (see [25, 26], also compare with the formula in [18]).

In particular, suppose that $\Gamma = \mathbb{N}$ and $V_k = 0$ for $k \geq 2$, that is, $V = V_1$. Let $\dim V_{(1,0)} = m$ and $\dim V_{(1,1)} = n$. Then $\mathcal{L} = \bigoplus_{k \geq 1} \mathcal{L}_k$ is an $\mathbb{N}$-graded free Lie superalgebra. For $n \geq 1$, we have $P(n) = \{ n \}$. The character of $\mathcal{L}_k$ is the trace of $\text{diag}(x, y) \in GL(m) \times GL(n)$ on $\mathcal{L}_k$. 
for \( x = (x_1, \cdots, x_m) \in (\mathbb{C}^\times)^m \) and \( y = (y_1, \cdots, y_m) \in (\mathbb{C}^\times)^n \). From (2.7), we have

\[
\text{ch} \mathcal{L}_k = \frac{1}{k} \sum_{d \mid k} \mu(d) p_d(x, y)^{k/d},
\]

where \( p_d(x, y) = \sum_{i=1}^m x_i^d - \sum_{j=1}^n (-y_j)^d \) for \( d \geq 1 \). We will use the above formula for the main results in this paper.

Remark 2.2. Though the proof of (2.7) in [26] uses only PBW theorem, the importance of the homological method (or Euler-Poincaré principle) we used here lies in the fact that it can be used when we consider the characters of other class of Lie (super)algebras, for example, generalized Kac-Moody (super)algebras. In fact, when \( \mathcal{L} \) is a graded Lie algebra with a group action, Kac and Kang gave a formula for characters of \( \mathcal{L} \) (or traces on \( \mathcal{L} \)) by using the homology of Lie algebras [17]. This was also generalized to the case of graded Lie superalgebras by Kang and the author in [19]. See also [22].

3. Symmetric functions and characters of the symmetric groups

In this section, we give a brief review on the (projective) representations of the symmetric groups, and their characters which are closely related with the theory of symmetric functions (see [13, 24] for a general and detailed exposition).

A partition of \( k \) \((k \geq 1)\) is a finite non-increasing sequence \( \lambda = (\lambda_1, \cdots, \lambda_r) \) of positive integers whose sum is \( k \). We write \( \lambda \vdash k \), and denote by \( \mathcal{P}(k) \), the set of partitions of \( k \). Each \( \lambda_i \) is called a part of \( \lambda \), and \( r \) is called the length of \( \lambda \), denoted by \( \ell(\lambda) \). We also write \( \lambda = (1^{m_1}, 2^{m_2}, \cdots) \) where \( m_i \) is the number of the parts of \( \lambda \) equal to \( i \) \((i \geq 1)\). The conjugate of \( \lambda \) is the partition \( \lambda' = (\lambda'_1, \lambda'_2, \cdots) \) where \( \lambda'_i \) is the number of parts in \( \lambda \) which are no less than \( i \).

3.1. Characters of the symmetric groups

Fix an integer \( m \geq 1 \). Let \( S_m \) be the symmetric group. It is generated by the transposition \( \sigma_i = (i \ i + 1) \) for \( 1 \leq i \leq m - 1 \). Let \( x_1, \cdots, x_m \) be variables. \( S_m \) acts on \( \mathbb{Z}[x_1, \cdots, x_m] \) by permuting the indices of the variables. Consider \( \Lambda_m = \mathbb{Z}[x_1, \cdots, x_m]^{S_m} \) the ring of symmetric functions in \( m \) variables. For each partition \( \lambda \) with \( \ell(\lambda) \leq m \), the Schur
function corresponding to $\lambda$ is the polynomial $s_\lambda(x_1, \cdots, x_m) = s_\lambda(x)$ given by

\begin{equation}
(3.1) \quad s_\lambda(x) = \det(h_{\lambda_i - i + j}(x))_{1 \leq i, j \leq m},
\end{equation}

where $h_k(x)$ is defined by $\sum_{k \geq 0} h_k(x) t^k = \prod_{i=1}^m (1 - x_i t)^{-1}$ (we assume that $\lambda_k = 0$ for $k > \ell(\lambda)$). Then $\{s_\lambda(x) | \ell(\lambda) \leq m\}$ is a $\mathbb{Z}$-basis of $\Lambda_m$. If we multiply two Schur functions, we can write it as a linear combination of Schur functions again

\begin{equation}
(3.2) \quad s_\mu(x)s_\nu(x) = \sum_{\lambda} N^\lambda_{\mu\nu} s_\lambda(x),
\end{equation}

where the coefficients $N^\lambda_{\mu\nu}$ are called the Littlewood-Richardson coefficients.

For each $k \geq 1$, set $p_k(x_1, \cdots, x_m) = p_k(x) = x_1^k + \cdots + x_m^k$. The power symmetric function corresponding to a partition $\lambda$, is the polynomial $p_\lambda(x_1, \cdots, x_m) = p_\lambda(x)$ given by

\begin{equation}
(3.3) \quad p_\lambda(x) = p_{\lambda_1}(x) \cdots p_{\lambda_r}(x).
\end{equation}

It is known that $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_m = \mathbb{Q}[x_1, \cdots, x_m]^S_m$ is generated by $p_k(x)$ ($k \geq 1$) and hence spanned by $\{p_\lambda(x) | \lambda : \text{a partition}\}$.

Note that the irreducible representations of $S_k$ are parameterized by $\mathcal{P}(k)$. For $\lambda \in \mathcal{P}(k)$, let $\chi^\lambda_{S_k}$ be the irreducible character corresponding to $\lambda$. For $\mu = (\mu_1, \cdots, \mu_r) \in \mathcal{P}(k)$, let $s_i$ ($1 \leq i \leq r$) be a cycle of length $\mu_i$ for $1 \leq i \leq r$, which are mutually disjoint. Then $\sigma_\mu = s_1 \cdots s_r$ is a permutation of cycle type $\mu$. Since a character is determined by the values at the conjugacy classes of $S_k$, it suffices to know $\chi^\lambda_{S_k}(\sigma_\mu)$. These values are determined by the coefficient of $s_\lambda(x)$ in the expansion of $p_\mu(x)$ into Schur functions, which is known as the Frobenius formula:

**Theorem 3.1.** ([10]) For each $\mu \in \mathcal{P}(k)$, we have

\begin{equation}
(3.4) \quad p_\mu(x) = \sum_{\lambda: \ell(\lambda) \leq m} \chi^\lambda_{S_k}(\sigma_\mu) s_\lambda(x).
\end{equation}

Fix $n \geq 1$, and let $y_1, \cdots, y_n$ be variables. Let $\Lambda_{m/n}$ be the ring of polynomials $f(x, y)$ in $m + n$ variables $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ with integral coefficients satisfying

1. $f$ is invariant under the action of $S_m \times S_n$,
2. when we put $x_m = -y_n = t$, the resulting polynomial is independent of $t$. 


We call \( \Lambda_{m/n} \) the ring of super symmetric functions in \( m + n \) variables \( x_1, \cdots, x_m \) and \( y_1, \cdots, y_m \). For each partition \( \lambda = (\lambda_1 \cdots, \lambda_r) \), the hook Schur function corresponding to \( \lambda \) is the polynomial \( h_{s\lambda}(x, y) \) given by

\[
(3.5) \quad h_{s\lambda}(x, y) = \det(h_{\lambda_i-i+j}(x, y))_{1 \leq i, j \leq r},
\]

where \( h_k(x, y) \) is defined by \( \sum_{k \geq 0} h_k(x, y)t^k = \prod_{i=1}^{m}(1-x_it)^{-1}\prod_{j=1}^{n}(1+y_jt) \). It is known that \( h_{s\lambda}(x, y) \neq 0 \) if and only if \( \lambda \) is \((m, n)\)-hook shaped, that is, \( \lambda_{m+1} \leq n \). Note that \( \{ h_{s\lambda}(x, y) \mid \lambda : (m, n)\)-hook shaped \} is a \( \mathbb{Z} \)-basis of \( \Lambda_{m/n} \) (see [24, 27]).

Set \( p_k(x, y) = \sum_{i=1}^{m}x_i^k - \sum_{j=1}^{n}(-y_j)^k \) \( (k \geq 1) \). For each partition \( \lambda = (\lambda_1 \cdots, \lambda_r) \), define

\[
(3.6) \quad p_{\lambda}(x, y) = p_{\lambda_1}(x, y) \cdots p_{\lambda_r}(x, y),
\]

which is called the power super symmetric function corresponding to \( \lambda \). In [34], it was shown that \( \mathbb{Q} \otimes \mathbb{Z} \Lambda_{m/n} \) generated by \( p_k(x, y) \) \( (k \geq 1) \), and hence spanned by \( \{ p_{\lambda}(x, y) \mid \lambda : \text{a partition} \} \).

Then, we have the same relation between \( \{ h_{s\lambda}(x, y) \mid \lambda : (m,n)\)-hook shaped \} and \( \{ p_{\lambda}(x, y) \mid \lambda : \text{a partition} \} \), which can be proved in a standard way.

**Proposition 3.2.** For \( \lambda, \mu \in \mathcal{P}(k) \), we have

\[
(3.7) \quad p_{\mu}(x, y) = \sum_{\lambda \vdash k} \chi_{\lambda}^\lambda(\sigma_{\mu})h_{s\lambda}(x, y).
\]

**Proof.** Let \( z_1, \cdots, z_r \) be variables where \( r \) is sufficiently large. Following the arguments in [24] (see Section 4), we obtain the identities

\[
(3.8) \quad \sum_{\lambda} h_{s\lambda}(x, y)s_{\lambda}(z) = \frac{\prod_{k=1}^{m} \prod_{j=1}^{n}(1+y_jz_k)}{\prod_{k=1}^{m} \prod_{i=1}^{n}(1-x_iz_k)} = \sum_{\lambda} \frac{1}{z_{\mu}} p_{\lambda}(x, y)p_{\lambda}(z),
\]

where the sum is taken over all partitions and \( z_{\mu} \) is the number of elements in the centralizer of \( \sigma_{\mu} \) in \( S_k \) \( (\mu \vdash k) \). Note that the above equation can be viewed as a linear combination of \( s_{\lambda}(z) \) and \( p_{\lambda}(z) \) over the ring \( \Lambda_{m/n} \).

From (3.4) and the orthogonality of characters, we have for \( \lambda \in \mathcal{P}(k) \)

\[
(3.9) \quad s_{\lambda}(z) = \sum_{\mu} \frac{1}{z_{\mu}} \chi_{\lambda}^\lambda(\sigma_{\mu})p_{\mu}(z).
\]

Hence, we have

\[
(3.10) \quad \sum_{\mu} \frac{1}{z_{\mu}} (\sum_{\lambda} \chi_{\lambda}^\lambda(\sigma_{\mu})h_{s\lambda}(x, y))p_{\mu}(z) = \sum_{\lambda} \frac{1}{z_{\lambda}} p_{\lambda}(x, y)p_{\lambda}(z),
\]

\[
\sum_{\mu} \frac{1}{z_{\mu}} (\sum_{\lambda} \chi_{\lambda}^\lambda(\sigma_{\mu})h_{s\lambda}(x, y))p_{\mu}(z) = \sum_{\lambda} \frac{1}{z_{\lambda}} p_{\lambda}(x, y)p_{\lambda}(z),
\]
where $\lambda, \mu \vdash k$. Since $p_{\mu}(z)$ are linearly independent for $r \gg k$, we obtain (3.7).

**Remark 3.3.** The formula (3.4) and (3.7) can be interpreted from the Schur–Weyl duality [4, 32, 31, 36]. For example, when $V = V_0 = \mathbb{C}^m$, there is a right $S_k$-action on $V^\otimes k$ given by

$$v_1 \otimes \cdots \otimes v_k \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

for $v_i \in V$ and $\sigma \in S_k$. It commutes with a left $\mathfrak{gl}(m)$-action on $V^\otimes k$, and $\mathfrak{gl}(m)$ and $S_k$ give full centralizers of each other. By taking a trace of an element $\sigma^\mu \times \text{diag}(x_1, \cdots, x_m)$ on $V^\otimes k$, we can recover (3.4). For a general review on the duality theorems of various algebras and their relations with the Frobenius formula, the readers are referred to [1].

### 3.2. Spin characters of the double cover of the symmetric groups

For a given partition $\lambda$, we say that $\lambda$ is *strict* if all parts of $\lambda$ are distinct, and $\lambda$ is *even* (resp. *odd*) if all parts of $\lambda$ are even (resp. odd).

Let $\Gamma_m$ be the subring of $\Lambda_m$ generated by $q_k(x)$ ($k \geq 1$) where $q_k(x)$ is defined by $\sum_{k>0} q_k(x)t^k = \prod_{i=1}^m (1-tx_i)^{-1}(1+tx_i)$.

For each partition $\lambda = (\lambda_1, \cdots, \lambda_m)$, the Schur $Q$-function corresponding to $\lambda$ is the polynomial $Q_\lambda(x)$ given by

$$Q_\lambda(x) = 2^{\ell(\lambda)} \sum_{\sigma \in S_m/S_m^\lambda} \sigma \left( x_1^{\lambda_1} \cdots x_m^{\lambda_m} \prod_{\lambda_i > \lambda_j} \frac{x_i + x_j}{x_i - x_j} \right),$$

where $S_m^\lambda$ is the subgroup of permutations $\sigma$ such that $\lambda_{\sigma(i)} = \lambda_i$ for all $1 \leq i \leq m$. Then $Q_\lambda(x) = 0$ unless $\lambda$ is strict, and $\{ Q_\lambda(x) \mid \lambda : \text{strict}, \ell(\lambda) \leq m \}$ forms a Z-basis of $\Gamma_m$. Also, $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_m$ is generated by $p_k(x)$ for $k = 1, 3, 5, \cdots$, and hence spanned by $\{ p_\lambda(x) \mid \lambda : \text{odd} \}$.

Let $A_k$ be the associative algebra generated by $\tau_i$'s ($1 \leq i \leq k - 1$) satisfying the following relations:

$$\tau_i^2 = -1, \quad (1 \leq i \leq k - 1),$$

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad (1 \leq i \leq k - 2),$$

$$\tau_i \tau_j = -\tau_j \tau_i \quad (1 \leq i, j \leq k - 1, |i - j| > 1).$$

Let $\tilde{S}_k \subset A_k$ be the group generated by $-1$ and $\tau_i$ ($1 \leq i \leq k - 1$). There exists a surjective homomorphism $\pi : \tilde{S}_k \to S_k$ given by $\pi(\tau_i) = \sigma_i$ and $\pi(-1) = 1$ with $\ker \pi = \{1, -1\}$ which is central. Hence $\tilde{S}_k$ is a double cover of $S_k$ and $A_k$ is a twisted group algebra of $S_k$. A
representation of $A_k$ corresponds to a projective representation of $S_k$, which is also called a spin representation of $S_k$.

Set $\mathcal{OP}(k) = \{ \lambda \in \mathcal{P}(k) \mid \lambda : \text{odd} \}$ and $\mathcal{DP}(k) = \{ \lambda \in \mathcal{P}(k) \mid \lambda : \text{strict} \}$. Let $\mathcal{DP}^+(k)$ (resp. $\mathcal{DP}^-(k)$) be the set of strict partitions of $k$ such that the number of even parts is even (resp. odd).

We consider $A_k$ as a $\mathbb{Z}_2$-graded algebra (or superalgebra) with $\deg(\tau_i) = 1$ ($1 \leq i \leq k - 1$). Then for each $\lambda \in \mathcal{DP}(k)$, there exists a $\mathbb{Z}_2$-graded irreducible representation $T^\lambda$ of $A_k$, whose characters $\chi^\lambda_{A_k}$ give all the irreducible characters of $A_k$ (see [15, 38] for a more detailed exposition).

For $\mu = (\mu_1, \cdots, \mu_r) \in \mathcal{P}(k)$, set $\tau_\mu = t_1 \cdots t_r$ where

$$t_i = \tau_{\mu_1+\cdots+\mu_{i-1}+1} \cdots \tau_{\mu_1+\cdots+\mu_i+1}$$

for $1 \leq i \leq r$. Since $\chi_{A_k}^\lambda(\tau_\mu) = 0$ unless $\mu \in \mathcal{OP}(k)$, it suffices to know $\chi_{A_k}^\lambda(\tau_\mu)$ for $\mu \in \mathcal{OP}(k)$, and it is given by the coefficient of $Q^\lambda(x)$ in the expansion of $p_\mu(x)$ ($\mu \in \mathcal{OP}(k)$) into Schur $Q$-functions:

**Theorem 3.4** ([15, 30]). For each partition $\mu \in \mathcal{OP}(k)$, we have

$$p_\mu(x) = \sum_{\lambda \in \mathcal{OP}(k)} \frac{1}{\sqrt{2}}^{\ell(\mu)+\ell(\lambda)+\varepsilon(\lambda)} \chi_{A_k}^\lambda(\tau_\mu)Q^\lambda(x),$$

where $\varepsilon(\lambda) = (1 \mp 1)/2$ if $\lambda \in \mathcal{DP}^\pm(k)$.

**Remark 3.5.** The above formula is originally due to Schur ([30]), and our description of his result in the language of $\mathbb{Z}_2$-graded algebras is due to [15]. In [33], Sergeev established a duality relation of the Lie superalgebra $\mathfrak{q}(n)$ and the twisted group algebra of the hyperoctahedral group $H_k$. From this, we have a similar formula where the spin characters of $S_k$ are replaced by the characters of the twisted group algebra of $H_k$. The formula (3.14) can also be deduced from the duality relation of $\mathfrak{q}(n)$ and the twisted group algebra of $S_k$ ([38]).

4. **Multiplicities of irreducible representations in $L_n$**

Let $L$ be a Lie superalgebra. A $\mathbb{Z}_2$-graded vector space $V$ is called an $L$-module if there exists a bilinear map $L \times V \rightarrow V$, $(x, v) \mapsto x \cdot v$ such that

1. $x \cdot v \in V_{a+b}$ for $x \in L_a$, $v \in V_b$ ($a, b \in \mathbb{Z}_2$)
2. $[x, y] \cdot v = x \cdot (y \cdot v) - (-1)^{ab}y \cdot (x \cdot v)$ for $x \in L_a$ and $y \in L_b$.
Let $V$ and $W$ be $\mathbb{Z}_2$-graded vector spaces. Then $V \otimes W$ is also a $\mathbb{Z}_2$-graded vector space where $(V \otimes W)_a = \bigoplus_{b+c=a} V_b \otimes W_c$. Moreover, if they are $\mathcal{L}$-modules, then $V \otimes W$ becomes an $\mathcal{L}$-module where the action of $\mathcal{L}$ is given by

\[(4.1) \quad x \cdot (v \otimes w) = (x \cdot v) \otimes w + (-1)^{ab} v \otimes (x \cdot w),\]

where $x \in \mathcal{L}_a$, $v \in V_b$ and $w \in W$.

### 4.1. Decomposition as a $\mathfrak{gl}(m, n)$-module

Suppose that $V$ is a $\mathbb{Z}_2$-graded vector space with $V_0 = \mathbb{C}^m$ and $V_1 = \mathbb{C}^n$.

Let $\mathfrak{gl}(m, n)$ be the space of all $(m+n) \times (m+n)$ matrices. We may view an element of $\mathfrak{gl}(m, n)$ as an endomorphism of $V$. For $a \in \mathbb{Z}_2$, set

\[(4.2) \quad \mathfrak{gl}(m, n)_a = \{ X \in \mathfrak{gl}(m, n) \mid X(V_b) \subset V_{a+b} \text{ for } b \in \mathbb{Z}_2 \} \}

Then $\mathfrak{gl}(m, n) = \mathfrak{gl}(m, n)_0 \oplus \mathfrak{gl}(m, n)_1$ is a $\mathbb{Z}_2$-graded Lie superalgebra, called the general linear Lie superalgebra, with the superbracket defined by

\[(4.3) \quad [X, Y] = XY - (-1)^{ab} YX

for $X \in \mathfrak{gl}(m, n)_a$, $Y \in \mathfrak{gl}(m, n)_b$, and $a, b \in \mathbb{Z}_2$.

By left multiplication, $V$ becomes a $\mathfrak{gl}(m, n)$-module, which is called the natural representation. For $k \geq 1$, $V^\otimes k$ is a $\mathfrak{gl}(m, n)$-module with the action given by

\[(4.4) \quad x \cdot (v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} (-1)^{a_i} (\sum_{j < i} a_j) v_1 \otimes \cdots \otimes (X \cdot v_i) \otimes \cdots \otimes v_k,

for $X \in \mathfrak{gl}(m, n)_a$ and $v_i \in V_{a_i}$ ($1 \leq i \leq k$).

Let $\mathcal{L}$ be the free Lie superalgebra generated by $V$ and $\mathcal{L}_k$ the $k$th homogeneous component. Note that $\mathcal{L}_k$ is a subspace of $V^\otimes k$ and it is a $\mathfrak{gl}(m, n)$-submodule of $V^\otimes k$ where the action of $\mathfrak{gl}(m, n)$ is induced from $V^\otimes k$.

\[(4.5) \quad x \cdot [v_1, [v_2, \cdots [v_{k-1}, v_k] \cdots]]

= \sum_{i=1}^{k} (-1)^{a_i (\sum_{j < i} a_j)} [v_1, \cdots [X \cdot v_i, \cdots [v_{k-1}, v_k] \cdots]],

for $X \in \mathfrak{gl}(m, n)_a$ and $v_i \in V_{a_i}$ ($1 \leq i \leq k$).

For $k \geq 1$, it is known that $V^\otimes k$ is completely reducible as a $\mathfrak{gl}(m, n)$-module and its irreducible components are parameterized by the $(m, n)$-hook shaped partitions of $k$. For each $(m, n)$-hook shaped partition
\( \lambda \) of \( k \), let \( V^\lambda \) be the corresponding irreducible representation. Let \( x = (x_1, \cdots, x_m) \in (\mathbb{C}^*)^m \) and \( y = (y_1, \cdots, y_n) \in (\mathbb{C}^*)^n \) be variables. The character of \( V^\lambda \), i.e. the trace of \( \text{diag}(x, y) \) on \( V^\lambda \), is

\[
(4.6) \quad \text{ch} V^\lambda = h s_\lambda(x, y),
\]

where \( h s_\lambda(x, y) \) is the hook Schur function corresponding to \( \lambda \) (see [4] for the above arguments).

Note that \( \text{ch} V^{\otimes k} = p_1(x, y)^k = p_{(1^k)}(x, y) \). From (3.7), we have

\[
(4.7) \quad \text{ch} V^{\otimes k} = \sum_{\lambda} \chi^\lambda_{S_k} (1) h s_\lambda(x, y),
\]

where the sum is over all \((m, n)\)-hook shaped partitions of \( k \). Hence, the multiplicity of \( V^\lambda \) in \( V^{\otimes k} \) is equal to \( \chi^\lambda_{S_k}(1) \).

**Theorem 4.1.** For \( k \geq 1 \), let \( L_k \) be the \( k \)th homogeneous component of the free Lie superalgebra generated by the natural representation \( V \) of \( \mathfrak{gl}(m, n) \). As a \( \mathfrak{gl}(m, n) \)-module, \( L_k \) is completely reducible, and for each \((m, n)\)-hook shaped partition \( \lambda \) of \( k \), the multiplicity of \( V^\lambda \) in \( L_k \) is equal to

\[
(4.8) \quad \frac{1}{k} \sum_{d \mid k} \mu(d) \chi^\lambda_{S_k} (\sigma_{(d^k/d)}).
\]

**Proof.** Since \( L_k \) is a \( \mathfrak{gl}(m, n) \)-submodule of \( V^{\otimes k} \), it is completely reducible. In terms of characters, we have

\[
(4.9) \quad \text{ch} L_k = \sum_{\lambda} m_\lambda h s_\lambda(x, y),
\]

where the sum is over all \((m, n)\)-hook shaped partitions of \( k \), and \( m_\lambda \) is the multiplicity of \( V^\lambda \) in \( L_k \). On the other hand, we have

\[
(4.10) \quad \text{ch} L_k = \frac{1}{k} \sum_{d \mid k} \mu(d) p_d(x, y)^{k/d} \quad \text{by (2)}
\]

\[
= \frac{1}{k} \sum_{d \mid k} \mu(d) \left( \sum_{\lambda \vdash k} \chi^\lambda_{S_k} (\sigma_{(d^k/d)}) h s_\lambda(x, y) \right) \quad \text{by (3.7)}
\]

\[
= \sum_{\lambda \vdash k} \left( \frac{1}{k} \sum_{d \mid k} \mu(d) \chi^\lambda_{S_k} (\sigma_{(d^k/d)}) \right) h s_\lambda(x, y).
\]

Since \( \{ h s_\lambda(x, y) \mid \lambda : (m, n)\)-hook shaped \} \) is linearly independent in \( \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{m/n} \), we obtain the result by comparing (4.9) and (4.10). \( \square \)
Consider the action of $S_k$ on $V^\otimes k$ given by
\begin{equation}
(v_1 \otimes \cdots \otimes v_k) \cdot \sigma_i = (-1)^{\alpha_i, \alpha_{i+1}} (v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k),
\end{equation}
where $v_j \in V_{a_j}$ $(1 \leq j \leq k)$. It defines a right $S_k$-module structure on $V^\otimes k$ and commutes with the left action of $\mathfrak{gl}(m,n)$. With these two commuting actions, Berele and Regev established the Schur-Weyl duality ([4]). On the other hand, let $\text{Ind}_{C_k}^{S_k} \theta$ be the induced representation of a faithful representation $\theta$ of a cyclic subgroup $C_k$ of order $k$. Then, for each $(m,n)$-hook shaped partition $\lambda$ of $k$, it is not difficult to see that the multiplicity of the Specht module $S^\lambda$ in $\text{Ind}_{C_k}^{S_k} \theta$ is equal to (4.8). Therefore, from the Schur-Weyl duality, we have
\begin{equation}
\mathcal{L}_k \cong V^\otimes k \otimes_{C[S_k]} \text{Ind}_{C_k}^{S_k} \theta,
\end{equation}
as $\mathfrak{gl}(m,n)$-modules. When $\mathcal{L}$ is a free Lie algebra (or $V_1 = 0$), this was given by Klyachko ([20]).

Remark 4.2. In [19], the multiplicity $m_\lambda$ is given in a recursive form in terms of character values of the symmetric groups, and hence expressed in a rather complicated way. But in this paper, we express the character of $\mathcal{L}_k$ in terms of power super symmetric functions directly (2), and use the Frobenius formula (3.7) to obtain a closed form of the multiplicities. Some generalizations of the formula (1.2) using the theory of symmetric functions can be found in [9, 14].

4.2. Decomposition as a $q(n)$-module

In this subsection, we assume that $m = n$, i.e. $V = \mathbb{C}^n \oplus \mathbb{C}^n$. Let $q(n)$ be the Lie subsuperalgebra of $\mathfrak{gl}(n,n)$ consisting of all matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where $A$ and $B$ are $n \times n$ matrices. Then $V$ is a $q(n)$-module called the natural representation.

For $k \geq 1$, let us consider the $k$-fold tensor product of $V$. Then, as in the case of $\mathfrak{gl}(m,n)$, $V^\otimes k$ is completely reducible as a $q(n)$-module, and its irreducible components are parameterized by $\mathcal{P}(k)$.

For each $\lambda \in \mathcal{P}(k)$, let $U^\lambda$ be the corresponding irreducible representation. The character of $U^\lambda$, i.e. the trace of $\text{diag}(x,x)$ $(x \in (\mathbb{C}^\times)^n)$ on $U^\lambda$, is given by
\begin{equation}
\text{ch} U^\lambda = (\sqrt{2})^{d(\lambda) - \ell(\lambda)} Q_\lambda(x),
\end{equation}
where $Q_\lambda(x)$ is the Schur $Q$-function corresponding to $\lambda$ and $d(\lambda) = (1 - (-1)^{\ell(\lambda)})/2$ (see [33] for the above arguments).
Note that the trace of $\text{diag}(x, x)$ ($x \in (\mathbb{C}^\times)^n$) on $V^\otimes k$ is equal to $2^k p_1(x)^k = 2^k p_{(1^k)}(x)$. From (3.14), we have

$$2^k p_{(1^k)}(x) = \sum_{\lambda \in \mathcal{P}(k)} (\sqrt{2})^{k - d(\lambda) - \varepsilon(\lambda)} \chi_{\mathcal{A}_k}^\lambda(1) Q_\lambda(x),$$

which implies that the multiplicity of $U^\lambda$ in $V^\otimes k$ is $(\sqrt{2})^{k - d(\lambda) - \varepsilon(\lambda)} \chi_{\mathcal{A}_k}^\lambda(1)$.

**Theorem 4.3.** For $k \geq 1$, let $\mathcal{L}_k$ be the $k$th homogeneous component of the free Lie superalgebra generated by the natural representation $V$ of $\mathfrak{q}(n)$. As a $\mathfrak{q}(n)$-module, $\mathcal{L}_k$ is completely reducible, and for each $\lambda \in \mathcal{P}(k)$, the multiplicity of $U^\lambda$ in $\mathcal{L}_k$ is equal to

$$\frac{1}{k} \sum_{\substack{d|k \times d: \text{odd}}} \mu(d) \chi_{\mathcal{A}_k}^\lambda(\tau_{(d^{k/d})}(\sqrt{2})^{(k/d) - d(\lambda) - \varepsilon(\lambda)}).$$

**Proof.** Since $\mathcal{L}_k$ is a $\mathfrak{q}(n)$-submodule of $V^\otimes k$, it is completely reducible. In terms of characters, we have

$$\text{ch} \mathcal{L}_k = \sum_{\lambda \in \mathcal{P}(k)} m_\lambda \text{ch} U^\lambda,$$

where $m_\lambda$ is the multiplicity of $U^\lambda$ in $\mathcal{L}_k$.

On the other hand, we have

$$\text{ch} \mathcal{L}_k = \frac{1}{k} \sum_{d|k} \mu(d) p_d(x, x)^{k/d}$$

$$= \frac{1}{k} \sum_{d|k \times d: \text{odd}} \mu(d) 2^{k/d} p_d(x)^{k/d} \quad \text{by (2.24)}$$

$$= \frac{1}{k} \sum_{d|k \times d: \text{odd}} \mu(d) \left( \sum_{\lambda \in \mathcal{P}(k)} (\sqrt{2})^{(k/d) - d(\lambda) - \varepsilon(\lambda)} \chi_{\mathcal{A}_k}^\lambda(\tau_{(d^{k/d})}) \text{ch} U^\lambda \right) \quad \text{by (3.14)}$$

$$= \sum_{\lambda \in \mathcal{P}(k)} \left( \frac{1}{k} \sum_{d|k \times d: \text{odd}} \mu(d) (\sqrt{2})^{(k/d) - d(\lambda) - \varepsilon(\lambda)} \chi_{\mathcal{A}_k}^\lambda(\tau_{(d^{k/d})}) \right) \text{ch} U^\lambda.$$

Since $\{ Q_\lambda(x) \mid \lambda \in \mathcal{P}(k) \}$ is linearly independent in $\mathbb{Q} \otimes_\mathbb{Z} \Gamma_m$, we obtain the result by comparing (4.16) and (4.17). \qed
5. Remarks on free Lie algebras

Suppose that $V = \mathbb{C}^n$. Let $\mathfrak{L}$ be the free Lie algebra generated by $V$. Let $\mathfrak{g} = \mathfrak{sp}(n)$ (even) or $\mathfrak{so}(n)$ be a subalgebra of $\mathfrak{gl}(n)$. For each $\lambda$ with $\ell(\lambda) \leq n$, the irreducible $\mathfrak{gl}(n)$-module $V^\lambda$ decomposes into irreducible $\mathfrak{g}$-modules with the multiplicities given in terms of the Littlewood-Richardson coefficients (see (5.3)). Hence, by (1.2), the multiplicity of each irreducible $\mathfrak{g}$-module in $\mathcal{L}_k$ is given in terms of the character values of $S_k$ and the Littlewood-Richardson coefficients.

In this section, we show that the multiplicity of each irreducible $\mathfrak{g}$-module in $\mathcal{L}_k$ for $1 \leq k \leq n$ can be simplified in terms of the character values of the Brauer algebras, which is analogous to (1.2).

5.1. Characters of the Brauer algebras

Fix $f \geq 1$. Consider two rows each of which consists of $f$ vertices. An $f$-diagram is a graph with the above $2f$ vertices and $f$ edges where each vertex belongs to exactly one edge. For example, the following is a 4-diagram.

```
  • ———— • ———— •
  |       |       |
  • ———— • ———— •
  |       |       |
```

Let $z$ be an indeterminate. Given two $f$-diagrams $d_1$ and $d_2$, we associate an $f$-diagram $d$ obtained by (i) placing $d_2$ below $d_1$, (ii) identifying $f$ vertices in the bottom row of $d_1$ with $f$ vertices in the top row in $d_2$. Then we define $d_1d_2$ to be the $d$ multiplied by $z^c$ where $c$ is the number of cycles appearing in the middle row. For example, if

```
d_1 =  • ———— • ———— •  
    • ———— • ———— •
           |       |       |
         • ———— • ———— •
           |       |       |

d_2 =  • ———— • ———— •
    • ———— • ———— •
           |       |       |
         • ———— • ———— •
           |       |       |

then $d_1d_2 = z$
```
The Brauer algebra $D_f(z)$ is an associative $\mathbb{C}(z)$-algebra spanned by all $f$-diagrams whose multiplication is described above. Note that the multiplicative identity is an $f$-diagram consisting of exactly $f$ vertical edges. Then $D_f(z)$ is a semisimple $\mathbb{C}(z)$-algebra and the irreducible representations of $D_f(z)$ are indexed by $\bigcup_{0 \leq k \leq [f/2]} \mathcal{P}(f-2k)$ (see [35]).

For $d \in D_f(z)$ and $d' \in D_{f'}(z)$, by a natural embedding of $D_f(z) \otimes D_{f'}(z)$ into $D_{f+f'}(z)$, we may view $d \otimes d'$ as an element in $D_{f+f'}(z)$. Let $e$ be a $2$-diagram with $2$ horizontal edges. For $k \geq 2$, let $\gamma_k$ be the $k$-diagram of the following form

![Diagram](image)

and we set $\gamma_1$ to be the identity in $D_1(z)$. Note that the symmetric group $S_f$ can be embedded into $D_f(z)$ where $\sigma_i$ ($1 \leq i \leq f-1$) corresponds to the element $\gamma_1^{\otimes i-1} \otimes \gamma_2 \otimes \gamma_1^{\otimes f-i-1}$.

For each partition $\mu = (\mu_1, \cdots, \mu_r)$, let $\gamma_\mu = \gamma_{\mu_1} \otimes \cdots \otimes \gamma_{\mu_r}$. In [28], Ram has shown that the characters of $D_f(z)$ are completely determined by the values at $e^{\otimes h} \otimes \gamma_\mu$ where $2h + |\mu| = f$, and computed the irreducible characters of $D_f(z)$ in terms of the characters of the symmetric groups:

**Theorem 5.1** ([28]). Let $\lambda \in \mathcal{P}(f-2k)$ ($0 \leq k \leq [f/2]$), and $\chi^\lambda_{D_f(z)}$ the irreducible character of $D_f(z)$ corresponding to $\lambda$. For $e^{\otimes h} \otimes \gamma_\mu$ ($2h + |\mu| = f$), we have

$$
\chi^\lambda_{D_f(z)}(e^{\otimes h} \otimes \gamma_\mu) = z^h \sum_{\mu+f-2h \eta+f-2k-2h \eta \text{ even}} (\sum_{\lambda \eta}) N^\nu_{\lambda \eta} \chi^\nu_{S_f-2h} (\sigma_\mu),
$$

where $N^\nu_{\lambda \eta}$ are the Littlewood-Richardson coefficients given in (3.2).

**Remark 5.2.** The formula (5.1) was obtained from the duality relation of the orthogonal groups and the Brauer algebras. Following the same arguments in [28] by using the duality relation between the symplectic groups and the Brauer algebras, it is not difficult to see that

$$
\chi^\lambda_{D_f(z)}(e^{\otimes h} \otimes \gamma_\mu) = (-1)^{|\mu|-\ell(\mu)} z^h \sum_{\mu+f-2h \eta+f-2k-2h \eta \text{ even}} (\sum_{\lambda \eta}) N^\nu_{\lambda \eta} \chi^\nu_{S_f-2h} (\sigma_\mu).
$$
Both (5.1) and (5.2) will be used for our computations.

5.2. Decomposition as $\mathfrak{sp}(n)$ and $\mathfrak{so}(n)$-modules

In this subsection, we assume that $V_1 = 0$, and hence $\mathcal{L}$ is the free Lie algebra generated by $V = V_0 = \mathbb{C}^n$. Suppose that $\mathfrak{g} = \mathfrak{sp}(n)$ (n : even) or $\mathfrak{so}(n) \subset \mathfrak{gl}(n)$.

By restriction, $\mathcal{L}_k$ is a representation of $\mathfrak{g} \subset \mathfrak{gl}(n)$. Note that $V^{\otimes k}$ decomposes into polynomial representations parameterized by the partitions $\mu \in \mathcal{P}(k - 2i)$ ($i = 0, \cdots, [k/2]$) satisfying $\ell(\mu) \leq [n/2]$ (see [32, 31, 36]). We denote by $W^\mu$ the corresponding representation. If $\mathfrak{g} = \mathfrak{so}(n)$ and $\ell(\mu) = n/2$ (n : even), then $W^\mu$ is a sum of two irreducible representations of the same dimension. Otherwise, $W^\mu$ is irreducible. For each partition $\lambda$ with $\ell(\lambda) \leq n$, let $V^{\lambda}$ be the irreducible polynomial representation of $\mathfrak{gl}(n)$. When restricted to a representation of $\mathfrak{g}$, it decomposes into $W^\mu$'s and the multiplicity of $W^\mu$ in $V^{\lambda}$ is given by

$$
\begin{cases}
\sum_{\mu \text{ even}} N^\lambda_{\mu} & \text{if } \mathfrak{g} = \mathfrak{so}(n), \\
\sum_{\nu \text{ even}} N^\lambda_{\nu} & \text{if } \mathfrak{g} = \mathfrak{sp}(n),
\end{cases}
$$

(see [23]). Now, combining (5.1), (5.2) and (5.3), we can describe the multiplicities of irreducible $\mathfrak{g}$-modules in $\mathcal{L}_k (k \leq n)$:

**Proposition 5.3.** For $k \geq 1$, let $\mathcal{L}_k$ be the $k$th homogeneous component of the free Lie algebra generated by $V = \mathbb{C}^n$. Then $\mathcal{L}_k$ is completely reducible as a $\mathfrak{g}$-module. And

(a) if $\mathfrak{g} = \mathfrak{so}(n)$ and $k \leq n$, then for each $\mu \in \mathcal{P}(k - 2i)$ ($i = 0, \cdots, [k/2]$) with $\ell(\mu) \leq [n/2]$, the multiplicity of $W^\mu$ in $\mathcal{L}_k$ is

$$
\frac{1}{k} \sum_{d|k} \mu(d) \chi^\mu_{D_k(z)}(\gamma_{(d^k/d^l)}).
$$

(b) if $\mathfrak{g} = \mathfrak{sp}(n)$ (n : even) and $k \leq n$, then for each $\mu \in \mathcal{P}(k - 2i)$ ($i = 0, \cdots, [k/2]$) with $\ell(\mu) \leq n/2$, the multiplicity of $W^\mu$ in $\mathcal{L}_k$ is

$$
\frac{1}{k} \sum_{d|k} \mu(d) \chi^\mu_{D_k(z)}(\gamma_{(d^k/d^l)})(-1)^{k-(k/d)}.
$$

**Proof.** (a) As a $\mathfrak{gl}(n)$-module, we have $\mathcal{L}_k = \bigoplus_{\ell(\lambda) \leq n} \mathcal{L}^{\lambda} \otimes m_\lambda$ where

$$
m_\lambda = \frac{1}{k} \sum_{d|k} \mu(d) \chi^\lambda_{D_k(z)}(\gamma_{(d^k/d^l)}).
$$
Since \( k \leq n \), the condition \( \ell(\lambda) \leq n \) in the decomposition of \( \mathcal{L}_k \) is sufficient. Hence, for each \( \mu \in \mathcal{P}(k-2i) \) \((i = 0, \cdots, [k/2])\) with \( \ell(\mu) \leq [n/2] \), the multiplicity of \( W^\mu \) in \( \mathcal{L}_k \) is

\[
\sum_{\lambda \vdash k} \left( \sum_{\nu: \text{even}} N^\lambda_{\mu\nu} \right) \frac{1}{k} \sum_{d|k} \mu(d) \chi^\lambda_{\mathcal{S}_k}(\sigma_{(d^k/d)})
\]

(5.7)

\[
= \frac{1}{k} \sum_{d|k} \mu(d) \left( \sum_{\lambda \vdash k} \left( \sum_{\nu: \text{even}} N^\lambda_{\mu\nu} \right) \chi^\lambda_{\mathcal{S}_k}(\sigma_{(d^k/d)}) \right)
\]

\[
= \frac{1}{k} \sum_{d|k} \mu(d) \chi^\mu_{D_k(z)}(\gamma_{(d^k/d)}) \quad \text{by (5.1)}.
\]

(b) The proof for \( \mathfrak{g} = \mathfrak{sp}(n) \) is almost the same as in (a) except using (5.2). \( \square \)

**Remark 5.4.** (1) Proposition 5.3 can be obtained directly by the analogues of Frobenius formula (see Corollary 4.5 and Theorem 4.6 in [28]).

(2) By (5.1) and (5.2), it follows that if \( \mu \vdash k \), then \( \chi^\mu_{D_k(z)}(\gamma_{(d^k/d)}) \) in (5.4) and (5.5) is equal to \( \chi^\mu_{\mathcal{S}_k}(\sigma_{(d^k/d)}) \).

(3) We would like to remark one more application of the Brauer algebras to the decomposition of free Lie algebras. If \( \mathcal{L} \) is the free Lie algebra generated by \( V \otimes \mathfrak{p} \otimes (V^*) \otimes V \), then as a \( \mathfrak{g}(n) \)-module \( \dim V = n \), \( \mathcal{L}_k \) decomposes into rational irreducible representations, whose characters are given by rational Schur functions. In this case, a Frobenius formula is given in [12], and by using (1.1), the multiplicities of irreducible representations in \( \mathcal{L}_k \) can be expressed in terms of the character values of a subalgebra \( D_{p,q}(z) \) of \( D_k(z) \) consisting of \( (p, q) \)-diagrams (cf. [2]).

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