FRACTIONAL INTEGRAL ALONG HOMOGENEOUS CURVES IN THE HEISENBERG GROUP

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ABSTRACT. We obtain the type set for the fractional integral operator along the curve \((t, t^2, \alpha t^3)\) on the three dimensional Heisenberg group when \(\alpha \neq \pm 1/6\). The proof is based on the Fourier inversion formula and the angular Littlewood-Paley decompositions in the Heisenberg group in [5].

1. Introduction

Let us consider the fractional integral along a curve with \(0 < \sigma \leq 1\):

\[
A^\sigma_d f(x) = \int_0^\infty f(x - (t, \cdots, t^d)) t^{d-1} dt.
\]

The type set of \(A^\sigma_d\) is the set of points \((1/p, 1/q)\) such that \(A^\sigma_d\) maps \(L^p(\mathbb{R}^d)\) to \(L^q(\mathbb{R}^d)\). Let \(ABCD\) be the closed trapezoid with vertices

\(A = (0, 0), B = (1, 1), C = (2/3, 1/2),\) and \(D = (1/2, 1/3)\).

For each \(\sigma > 0\) and \(d \in \mathbb{Z}^+\), let \(l_3^\sigma\) be the line on \(\mathbb{R}^2\)-plane given by

\[
l_3^\sigma = \{(u, v) : u - v = \frac{\sigma}{1 + 2 + \cdots + d}\}.
\]

In [6] and [8], it is known that the type set of \(A^\sigma_3\) is given by

\(ABCD \cap l_3^\sigma\).
The purpose of this paper is to investigate the type set for the operator
\begin{equation}
S_\alpha^\sigma f(x) = \int_0^\infty f(x \cdot (t, t^2, \alpha t^3)^{-1}) t^{\sigma - 1} dt,
\end{equation}
where \( \alpha \in \mathbb{R} \) and \( \cdot \) is the group multiplication on the 3 dimensional Heisenberg group \( \mathbb{H}^1 \) defined by
\[(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)).\]
For a finite interval \( I \) on \( \mathbb{R} \), we define an average operator:
\begin{equation}
T_\alpha f = \int_I f(x \cdot (t, t^2, \alpha t^3)^{-1}) dt.
\end{equation}
In the theorems 5.2 and 5.3 of [7], S. Secco obtains the typeset of \( T_\alpha \) for \( \alpha \in \mathbb{R} \).

**Theorem 1.** Let \( T_\alpha \) be the operator defined in (1.2). If \( \alpha \neq \pm 1/6 \), then the type set of \( T_\alpha \) is the closed trapezoid \( ABCD \).

For the case \( \alpha = \pm 1/6 \), the type set is strictly contained in \( ABCD \).

**Theorem 2.** Let \( T_\alpha \) be the operator defined in (1.2).
(i) If \( \alpha = 1/6 \), then the type set of \( T_\alpha \) is the closed triangle \( ABC \).
(ii) If \( \alpha = -1/6 \), then the type set of \( T_\alpha \) is the closed triangle \( ABD \).

From the result of Theorem 1 and the fact that with some \( C_\sigma > 0 \)
\begin{equation}
T_\alpha f(x) \leq C_\sigma S_\alpha^\sigma f(x),
\end{equation}
we see that the type set of \( S_\alpha^\sigma \) is contained in \( ABCD \) when \( \alpha \neq \pm 1/6 \). From Theorem 2 and (1.3), the type set of \( S_{1/6}^\sigma \) is contained in \( ABC \) and \( S_{-1/6}^\sigma \) is contained in \( ABD \). Let us define a dilation in \( \mathbb{R}^3 \) by \( \delta x = (\delta x_1, \delta^2 x_2, \delta^3 x_3) \) with \( \delta > 0 \). Then we have the dilation invariance under our group multiplication such that \( \delta(x \cdot y) = \delta x \cdot \delta y \). This implies that \( (S_\alpha^\sigma f)_\delta = \delta^\sigma S_\alpha^\sigma(f_\delta) \) where \( g_\delta(x) = g(\delta x) \) for a function \( g \) defined on \( \mathbb{H}^1 \). If the operator norm \( \| S_\alpha^\sigma \|_{L^q \to L^q} \) is finite, then we have
\begin{align}
\delta^{-(1+2+3)/q} \| S_\alpha^\sigma f \|_{L^q(\mathbb{H}^1)} &= \| (S_\alpha^\sigma f)_\delta \|_{L^q(\mathbb{H}^1)} = \delta^\sigma \| S_\alpha^\sigma f_\delta \|_{L^q(\mathbb{H}^1)} \\
&\leq \| S_\alpha^\sigma \|_{L^p \to L^q} \delta^\sigma \| f_\delta \|_{L^p(\mathbb{H}^1)} \\
&= \| S_\alpha^\sigma \|_{L^p \to L^q} \delta^\sigma^{-(1+2+3)/p} \| f \|_{L^p(\mathbb{H}^1)}. \tag{1.4}
\end{align}
Fractional integral along homogeneous curves

In order to satisfy the above inequality with both small and large $\delta$, we need the condition $1/p - 1/q = \sigma/(1 + 2 + 3)$. Therefore from Theorems 1.2 and (1.3)-(1.4), it follows that

(i) the type set for $S_\alpha$ is contained in $ABCD \cap l_3^2$ ($\alpha \neq \pm 1/6$).

(ii) the type set of $S_{1/6}$ is contained in $ABC \cap l_3^2$.

(iii) the type set of $S_{-1/6}$ is contained in $ABD \cap l_3^2$.

In this paper we show that $ABCD \cap l_3^2$ is the type set for the operator $S_\alpha$ when $\alpha \neq \pm 1/6$. The type set of $S_\alpha$ with each $\alpha = \pm 1/6$ is not known. However we show in Section 4 that neither $ABC \cap l_3^2$ nor $ABD \cap l_3^2$ is the type set of each $S_{1/6}$ or $S_{-1/6}$.

**Main Theorem.** Let $S_\alpha$ be the operator defined in (1.1). If $\alpha \neq \pm 1/6$, then the type set of $S_\alpha$ is the line segment $ABCD \cap l_3^2$.

**Remark.**

(1) It is conjectured that the type set of the translation invariant operator $A_d^\alpha$ is $ABCD \cap l_3^2$ where $A = (0,0)$, $B = (1,1)$, $C_d = (\frac{d^2-d+2}{d(d+1)}, \frac{d-1}{d+1})$ and $D_d = (\frac{2}{d+1}, \frac{d-2}{d^2-d+2})$. This conjecture is known for $d \leq 3$. For $d \geq 4$, M. Christ has obtained the $L^p \rightarrow L^q$ bound except the endpoints $C_d$ and $D_d$ for the average operator along that curve in [2]. This proves the conjecture for any $d \geq 4$ except those endpoints $C_d$ and $D_d$, see in [4] how we obtain the bound of $A_d^\alpha$ from the bound of the average operator on the finite interval.

(2) In [7], S. Secco has obtained the typeset for more general curves satisfying some curvature and torsion properties.

The proof of $L^p(\mathbb{R}^3) \rightarrow L^q(\mathbb{R}^3)$ boundedness for $A_d^\alpha$ is based on the decomposition of the frequency variables of convolution kernel into several angular sectors. Some of those sectors are estimated by using the Hardy-Littlewood-Sobolev theorem, and the others are handled by the Littlewood-Paley inequality and the interpolation argument made in [1]. In applying this standard argument on the Heisenberg group, we use the Fourier inversion formula on the Hisenberg group and the angular Littlewood-Paley decomposition developed in [5]. In Section 2 we discuss about these two basic tools. In Section 3, we show how we apply these tools for the proof of Main Theorem when $\alpha \neq \pm 1/6$. In
Section 4 we discuss about the case $\alpha = \pm 1/6$.

Notations. We write $*$ for the convolution on $\mathbb{H}^1$, and $\mathcal{F}$ or $\mathcal{F}^{-1}$ denotes the Fourier transform or its inverse for the given Euclidean space. Given two quantities $a$ and $b$, we write $a \lesssim b$ or $b \gtrsim a$ if there is a constant $C > 0$ such that $a \leq Cb$. If $a \lesssim b$ and $b \gtrsim a$, then we write $a \approx b$.

2. Background on the Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n} \times \mathbb{R}^1$ endowed with the group multiplication:

$$(x,y,t) \cdot (x',y',t') = (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - y \cdot x'))$$

for $x, y, x', y' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}^1$. The left invariant vector fields on $\mathbb{H}^n$ are defined as for each $1 \leq j \leq n$,

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$ 

Then $\{X_j, Y_j, T\}$ forms a basis of the Lie algebra $\mathfrak{h}_n$ corresponding to the Lie group $\mathbb{H}^n$. The canonical commutation relation in $\mathfrak{h}$ is given by $[X_j, Y_k] = \delta_{j,k} T$ with all the other commutators vanished. In [1], it is proved that the operator with a little less singular convolution kernel $|(x,y)|^{\sigma-2n}\delta(t)$ maps $L^p(\mathbb{H}^n)$ to $L^q(\mathbb{H}^n)$ if and only if $1/p - 1/q = \sigma/(2n + 2)$ and $(1/p, 1/q)$ belong to $ABC$ which is the closed triangle with vertices $(0,0), (1,1)$ and $(\frac{2n+1}{2n+2}, \frac{1}{2n+2})$. The essential part for the proof of this theorem is the Littlewood-Paley theory for the Laplacian $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$ and $T$. The commutativity of these two vector fields makes it possible to use spectral calculus for the computation of dyadic decomposition of $\mathcal{L}$, $T$ and the mixed one. However our convolution kernel is not radial, so we need to make the Littlewood-Paley decompositions for noncommutative vector fields $X_j, Y_j, T$ (with $j = 1$ since $n = 1$ for our case) instead of $\mathcal{L}$ and $T$. 
2.1. Group Fourier transform

For each \( \lambda \in \mathbb{R}^1 \), we define one parameter Shrödinger representation by a mapping \( R^\lambda \) from the Heisenberg group \( \mathbb{H}^n \) to the group of unitary operators on \( L^2(\mathbb{R}^n) \) such that for \( h \in L^2(\mathbb{R}^n) \) and \( (p, q, t) \in \mathbb{H}^n \),

\[
[R^\lambda(p, q, t)h](x) = e^{2\pi i \lambda[q \cdot x + p \cdot q/2 + t]}h(x + p).
\]

Let \( \mathcal{B}(L^2(\mathbb{R}^n)) \) be the space of bounded operators on \( L^2(\mathbb{R}^n) \). The group Fourier transform of \( f \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n) \) is defined as an operator-valued function from \( \mathbb{R}^1 \) to \( \mathcal{B}(L^2(\mathbb{R}^n)) \) such that \( \lambda \in \mathbb{R}^1 \mapsto \hat{f}(\lambda) \in \mathcal{B}(L^2(\mathbb{R}^n)) \), given by

\[
\hat{f}(\lambda)h(x) = \int_{\mathbb{H}^n} [R^\lambda(-p, -q, -t)h](x)f(p, q, t)d\mathcal{P}d\mathcal{Q}dt.
\]

From (2.1), \( \hat{f}(\lambda) \) is an integral operator on \( L^2(\mathbb{R}^n) \) given by

\[
[\hat{f}(\lambda)h](x) = \int_{\mathbb{R}^n} \mathfrak{F}^{2,3}f(x - y, \lambda(x + y)/2, \lambda)h(y)dy,
\]

where \( \mathfrak{F}^{2,3} \) is the Euclidean Fourier transform with respect to the second and third component of \( f \). From the definition of (2.1), we can show

\[
\hat{k} \ast \hat{f}(\lambda) = \hat{k}(\lambda) \cdot \hat{f}(\lambda),
\]

where the multiplication on the right is the composition of operators. From (2.1) and (2.3), we can prove the Fourier inversion formula such as

\[
f(p, q, t) = \int_{\mathbb{R}^1} \text{tr}(\hat{f}(\lambda) \cdot R^\lambda(p, q, t)) |\lambda|^n d\lambda.
\]

Here \( \text{tr}(T) \) denotes the trace of the operator \( T \). If \( Tf(x) \) is given by \( \int L(x, y)f(y)dy \), then \( \text{tr}(T) = \int L(x, x)dx \). For further study of the group Fourier transform, see [3] and [9].

2.2. Dyadic decomposition

Choose an even function \( \psi \in C^\infty_0(-1, 1) \) such that \( \psi \equiv 1 \) on \([-1/2, 1/2]\). Put \( \chi(\xi) = \psi(\xi/2) - \psi(\xi) \) and define

\[
L_j^2(y) = 2^{2j}[\mathfrak{F}^{-1}\chi](2^{2j}y_2)\delta(y_1, y_3),
\]

\[
L_j^3(y) = 2^{3j}[\mathfrak{F}^{-1}\chi](2^{3j}y_3)\delta(y_1, y_2).
\]
where $\delta$ is the Dirac measure at 0 in $\mathbb{R}^2$. In [5], it is shown that $L_j^2$ and $L_j^3$ are convolution kernels of the dyadic decompositions for $Y_1$ and $T$. They satisfy the Littlewood-Paley type inequalities:

**Lemma 1.** For $1 < p < \infty$, and $\nu = 2, 3$,

$$\left\| \left( \sum_j |L_j^\nu * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^1)} \lesssim \| f \|_{L^p(\mathbb{R}^1)}.$$

### 2.3. Angular decomposition

Put $\Omega(\xi, \eta) = \chi(\xi/\eta)$ and define three measures for each $j \in \mathbb{Z}$:

$$A_k^1(y_1, y_2, y_3) = 2^{5k}|3^{-1}\Omega|(2^ky_2, 2^ky_3)\delta(y_1),$$

$$A_k^2(y_1, y_2, y_3) = 2^{3k}|3^{-1}\Omega|(2^ky_1, 2^ky_2)\delta(y_3 - y_1y_2/2),$$

$$A_k^3(y_1, y_2, y_3) = 2^{4k}|3^{-1}\Omega|(2^ky_1, 2^ky_3)\delta(y_2).$$

In [5], it is also proved that $A_k^\nu$ is a convolution kernel of the angular decomposition for each pair of $X_1, Y_1, T$. We can see that $\mathcal{F}(A_k^1)(\xi_1, \xi_2, \xi_3)$ is supported on $|\xi_2/\xi_3| \approx 2^{-k}$. They also satisfy the following Littlewood-Paley type inequalities:

**Lemma 2.** For $1 < p < \infty$ and for $\nu = 1, 2, 3$,

$$\left\| \left( \sum_{j=-\infty}^{\infty} |A_j^\nu * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^1)} \lesssim \| f \|_{L^p(\mathbb{R}^1)}.$$

Let $\epsilon_\alpha = 1 - 3(\alpha + 1/2)$ and $\epsilon^*_\alpha = 1 - 3(-\alpha + 1/2)$. Then we choose $n_\alpha, m_\alpha \in \mathbb{Z}$ such that $2^{n_\alpha - 1} < \min\{|\epsilon_\alpha|, |\epsilon^*_\alpha|\} \leq 2^{m_\alpha}$ and $2^{m_\alpha - 1} < |\epsilon_\alpha| + 1 \leq 2^{m_\alpha}$. Note that we can choose $n_\alpha$ when $\alpha \neq \pm 1/6$. Let us define three subsets of $\mathbb{Z}$ so that

$$H(1, j) = \{ k \in \mathbb{Z} : 2^{n_\alpha - m_\alpha - j - 5} \leq 2^{-k} \leq 2^{m_\alpha - j + 5} \}$$

$$H(2, j) = \{ k \in \mathbb{Z} : 2^{-k} > 2^{m_\alpha - j + 5} \}$$

$$H(3, j) = \{ k \in \mathbb{Z} : 2^{-k} < 2^{n_\alpha - m_\alpha - j - 5} \}.$$

For fixed $\alpha \neq \pm 1/6$, let us define three measures by using $A_k^1$'s for each $\nu = 1, 2, 3$:

$$E_j^\nu(y_1, y_2, y_3) = \sum_{k \in H(\nu, j)} A_k^1(y_1, y_2, y_3).$$
We have the following vector valued inequalities for the operator \( \{ E_j^\nu \ast f_j \} \) on \( L^p(l^2) \) in [5]:

**Lemma 3.** We have for each \( \nu = 1, 2, 3 \), with \( 1 < p < \infty \)
\[
\left\| \left( \sum_{j=\pm \infty} |E_j^\nu \ast f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(l^1)} \lesssim \left\| \left( \sum_{j=\pm \infty} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(l^1)}.
\]

All measures defined in this section are dilation invariant in the sense that \( U_j(y_1, y_2, y_3) = 2^{6j} U_0(2^j y_1, 2^{2j} y_2, 2^{3j} y_3) \) for \( U_j = L_j^\nu, A_j^\nu, E_j^\nu \), or the convolution of these measures.

**3. Proof of main theorem with \( \alpha \neq \pm 1/6 \)**

Let \( \omega(t) \) be a smooth function on \( \mathbb{R}^+ \) and such that \( \sum_{j \in \mathbb{Z}} \omega(2^j t) = 1 \) for \( t > 0 \). For each fixed \( \sigma \), put \( \varphi_j(t) = \omega(2^j t)t^{\sigma-1} \). We define the convolution kernel \( K_j^\alpha \) of the operator \( S_j^\alpha \) by
\[
<K_j^\alpha, f> = \int f(t, t^2, \alpha t^3) \varphi_j(t) dt.
\]

Then \( S_j^\alpha f = \sum_j S_j^\alpha f = \sum_j K_j^\alpha \ast f \), and each convolution kernel \( K_j^\alpha \) is scaled so that:
\[
K_j^\alpha(x_1, x_2, x_3) = 2^{(6-\sigma)j} K_0^\alpha(2^j x_1, 2^{2j} x_2, 2^{3j} x_3).
\]

By using (2.2), the integral kernel of the operator \( \widehat{K_j^\alpha} (\lambda) \tilde{3}^{-1} \) can be written as,
\[
\int e^{-2\pi i(\lambda x - y)^2(x+y)/2 + \lambda \alpha(x-y)^3 - \xi y)} \varphi_j(x - y) dy.
\]

(3.1)

The size of the derivative of phase function is determined by \( \xi, \lambda y, \lambda \) which can be considered as the three frequency variables \( \mathbb{H}^1 \). These three frequency variables are controlled by the convolution with \( L_j^\nu \)'s or \( A_j^\nu \)'s. For example we can observe that by using the computation formula (2.2) and (2.4), each of \( \widehat{L_j^\nu} (\lambda) \) determines the range of the frequency variable \( \lambda y \ (\nu = 2) \) or \( \lambda \ (\nu = 3) \) in (3.1). Moreover \( \widehat{A_j^\nu} (\lambda) \) controls the ratio \( \lambda y/\lambda \ (\nu = 1), \xi/\lambda y \ (\nu = 2) \) or \( \xi/\lambda \ (\nu = 3) \) in (3.1).
For each $\nu = 2, 3$, we put $[L_0]_0 = \sum_{l=-\infty}^{10} L_0^{\nu}$. Fix $\sigma$, and split
\[ K_j^\alpha = K_j^\alpha \ast [L_0^2]_0 \ast [L_0^3]_0 + K_j^\alpha \ast (\delta - [L_0^2]_0 \ast [L_0^3]_0) = K_j^{loc} + K_j^{glo}. \]
With $\nu = 0, 1, 2, 3$, we set $R_j^\nu = \sum_j R_j^\nu$ such that
\[
R_j^0 f = K_j^{loc} \ast f, \\
R_j^\nu f = K_j^{glo} \ast E_j^\nu \ast f.
\]
Note that the group Fourier transform of $E_j^\nu$ splits the range of $(\xi, \lambda y, \lambda)$ in (3.1) into three sectors
\[
\frac{2^{-5} \min\{|\epsilon_\alpha|, |\epsilon_\alpha^*|\}}{1 + |\epsilon_\alpha|} |\lambda 2^{-j}| \leq |\lambda y| \leq 2^5(1 + |\epsilon_\alpha|) |\lambda 2^{-j}| \quad \text{when} \quad \nu = 1, \\
|\lambda y| \geq 2^5(1 + |\epsilon_\alpha|) |\lambda 2^{-j}| \quad \text{when} \quad \nu = 2, \\
|\lambda y| \leq \frac{2^{-5} \min\{|\epsilon_\alpha|, |\epsilon_\alpha^*|\}}{1 + |\epsilon_\alpha|} |\lambda 2^{-j}| \quad \text{when} \quad \nu = 3.
\]
For the proof of Main Theorem, we show that there exists a constant $C > 0$ such that for each fixed $\sigma$, $\alpha \neq \pm 1/6$ and $\nu = 0, 1, 2, 3$,
\[
(3.2) \quad \| R^\nu \|_{L^q(\mathbb{R}^3)} \leq C(1 + \| S_\alpha^\sigma \|_{L_p^\infty, L_q}) \| f \|_{L_p^\infty} \quad \text{for} \quad \theta \in (0, 1).
\]

3.1. Proof of (3.2) for $\nu = 0$

Let $\phi_2(\xi) = \psi(\xi/2^{20})$ and $\phi_3(\xi) = \psi(\xi/2^{30})$. Then by the straightforward computation, $K_0^{loc}(x_1, x_2, x_3)$ is written as
\[
(3.3) \quad \varphi_0(x_1)(\mathcal{F}^{-1}\phi_2)(x_2 - x_1^2)(\mathcal{F}^{-1}\phi_3)(x_3 - x_1^2 - x_1 x_2/2).
\]
Let $\rho(x_1, x_2, x_3) = (|x_1|^6 + |x_2|^3 + |x_3|^2)^{1/6}$. Then from (3.3) we have for large $N > 0$,
\[
(3.4) \quad |K_0^{loc}(x_1, x_2, x_3)| \leq C_N(1 + \rho(x_1, x_2, x_3))^{-N}.
\]
Hence,
\[
\sum_j |K_j^{loc}(x_1, x_2, x_3)| = \sum_j 2^{(6-\sigma)j} K_0^{loc}(2^j x_1, 2^j x_2, 2^j x_3) \lesssim \rho(x_1, x_2, x_3)^{-6+\sigma}.
\]
By applying the Hardy-Littlewood-Sobolev theorem with \( \frac{1}{p} - \frac{1}{q} = \frac{\sigma}{6} \), we have

\[
\| \mathcal{R}^0 \ast f \|_{L^q(\mathbb{H}^1)} \lesssim \| f \|_{L^p(\mathbb{H}^1)}.
\]

### 3.2. Proof of (3.2) for \( \nu = 2, 3 \)

Let us decompose the support of the integral in (3.1) into two sectors

\[
\{|\xi| \gg |\lambda y| \text{ or } |\xi| \ll |\lambda y|\} \text{ and } \{|\xi| \approx |\lambda y|\}
\]

for \( \nu = 2 \). For \( \nu = 3 \), the two sectors are \( \{|\xi| \gg (|\epsilon_\alpha| + 1)|\lambda| \text{ or } |\xi| \ll \min\{|\epsilon_\alpha|, |\epsilon_\alpha^*|\}|\lambda|\} \) and \( \{|\xi| \approx |\lambda|\} \). For this decomposition we define two subsets of \( \mathbb{Z} \):

\[
M(2, j) = \{ k \in \mathbb{Z} : 2^{-j-5} \leq 2^{-k} \leq 2^{-j+5} \}
\]

\[
M(3, j) = \{ k \in \mathbb{Z} : 2^{2^n - 2j-5} \leq 2^{-2k} \leq 2^{2^n - 2j+5} \}.
\]

By using \( M(\nu, j) \) with \( \nu = 2, 3 \), let us define measures:

\[
G_j^{\nu} = \sum_{k \in M(\nu, j)} A_k^{\nu},
\]

\[
W_j^{\nu} = \delta - G_j^{\nu},
\]

where \( \delta \) is a dirac mass at 0. Split for each \( \nu = 2, 3 \)

\[
(3.5) \quad K_j^{glo} \ast E_j^{\nu} = K_j^{glo} \ast E_j^{\nu} \ast W_j^{\nu} + K_j^{glo} \ast E_j^{\nu} \ast G_j^{\nu}.
\]

Let us start with the first term above. In order to apply the Hardy-Littlewood-Sobolev theorem for the kernel \( \sum_j K_j^{glo} \ast E_j^{\nu} \ast W_j^{\nu} \) as (3.4), we need to show that

\[
(3.6) \quad |K_0^{glo} \ast E_0^{\nu} \ast W_0^{\nu}(x_1, x_2, x_3)| \leq C_N(1 + \rho(x_1, x_2, x_3))^{-N}.
\]

**Proof of (3.6).** Let \( \Theta_j(u, v, \lambda) = \varphi_j(u)(1 - \phi_2(2^{-2^j} v) \phi_3(2^{-3^j} \lambda)) \). By using (2.2) and (2.3), we can compute the integral kernel \( K_j^{\nu}(x, z) \) of the group Fourier transform \( K_j^{glo} \ast E_j^{\nu} \ast W_j^{\nu}(\lambda) \): For \( \nu = 2 \),

\[
(3.7) \quad K_j^{\nu}(x, z) = \int \int e^{-2\pi \lambda(x-y)^2(x+y)/2 + \alpha(x-y)^3 - \xi(y-z)} \Theta_j(x-y, \lambda y, \lambda) \chi(\frac{\lambda y}{2^{-k} \lambda}) \sum_{k \in M(\nu, j)} \chi(\frac{\xi}{2^{-k} \lambda y}) dy d\xi.
\]
For \( \nu = 3 \), the last sum of the above integrand is replaced by the sum \( \sum_{k \in M(v, j)} \chi(\frac{v}{2 - 2k\lambda}) \). Let \( P_{p,q,t}(z,w) \) be the integral kernel of the operator \( R^\lambda(p,q,t) \). By the Fourier inversion formula in (2.4), \( K^{glo}_0 \ast E^\nu_0 \ast W_0^\nu(p,q,t) \) is written as \( \iint K^\nu_0(x,z)P_{p,q,t}(z,x)dzdx|\lambda|d\lambda \). This integral is computed as

\[
(3.8) \quad \iint e^{2\pi i \Psi(p,q,t,u,\xi,v,\lambda)} M^\nu(u,\xi,v,\lambda) du d\xi dv d\lambda,
\]

where the oscillatory term \( \Psi(p,q,t,u,\xi,v,\lambda) \) given by

\[
\xi(p - u) + v(q - u^2) + \lambda(t - \frac{pq}{2} + qu - (\frac{1}{2} + \alpha)u^3).
\]

The amplitudes \( M^\nu \)'s are

\[
M^2(u,\xi,v,\lambda) = \Theta_0(u) \left( \sum_{k \in H(2,0)} \chi(\frac{v}{2 - 2k\lambda}) \right) \sum_{k \in M(2,0)} \chi(\frac{\xi}{2 - 2k\lambda}),
\]

\[
M^3(u,\xi,v,\lambda) = \Theta_0(u) \left( \sum_{k \in H(3,0)} \chi(\frac{v}{2 - 2k\lambda}) \right) \sum_{k \in M(3,0)} \chi(\frac{\xi}{2 - 2k\lambda}).
\]

Here we note that \( \xi,v,\lambda \) are three frequency variables and \( v \) and \( u \) is given by the change of variables \( \lambda y = v \) and \( x - y = u \) in (3.7). Let us define a differential operator \( D_w \) for each \( w = u,v,\xi,\lambda \) by

\[
D_w g(u,\xi,v,\lambda) = \frac{1}{2\pi i \Psi'_w(p,q,t,u,\xi,v,\lambda)} \frac{\partial g(u,\xi,v,\lambda)}{\partial w}.
\]

Then \( D_w e^{2\pi i \Psi(p,q,t,u,\xi,v,\lambda)} = e^{2\pi i \Psi(p,q,t,u,\xi,v,\lambda)} \). The transpose of \( D_w \) is given by

\[
D^T_w g(u,\xi,v,\lambda) = \frac{1}{\partial w} \left( \frac{g(u,\xi,v,\lambda)}{2\pi i \Psi'_w(p,q,t,u,\xi,v,\lambda)} \right).
\]

Let us choose a cutoff function \( \zeta \) supported on the region \( 1 < 2^{-10} < t < 1 + 2^{-10} \) and \( \zeta(t) = 1 \) on \( 1 - 2^{-11} < t < 1 + 2^{-11} \). We define \( [\theta]^2 = \lambda q/(\xi + 2vu) \) and \( [\theta]^3 = \lambda(q - u^2)/(\xi + \epsilon_0 \lambda u^2) \) as functions of \( u,\xi,v,\lambda \) and \( q \). To apply integration by parts in (3.7), we define a differential operator for each \( \nu = 2,3 \), \( r = p,q,(t - \frac{pq}{2}) + qu \) and \( w = \xi,v,\lambda \),

\[
D^\nu g = [D^T_u]^N(\zeta^c([\theta]^\nu)g) + [D^T_v]^N(\zeta([\theta]^\nu)g),
\]

\[
D^w g = [D^T_w]^N(\psi^c(r/2^{5+m_0})g) + \psi(r/2^{5+m_0})g.
\]
For each \( \nu = 2, 3 \), the integration by parts yields that \( K_{\delta}^{\alpha} * E_{0}^{\nu} \ast W_{0}^{\nu}(p, q, t) \) is majorized by

\[
\int \int \int |D_{\lambda}^{(t-\frac{N}{2})+\nu}D_{\xi}^{\nu}D_{\eta}^{\nu}M^{\nu}(u, \xi, v, \lambda)|dud\xi dv d\lambda \\
\lesssim ((1 + |p|)(1 + |q|)(1 + |t - 2pq|))^{-N}.
\]

(3.10)

This is the desired estimate for the proof of (3.6). Let us sketch the proof of (3.10). As the first step we need to observe that \( D^{\nu} \) makes the decay such as

\[
|D^{\nu}M^{\nu}(u, \xi, v, \lambda)| \lesssim (1 + |\xi| + |v| + |\lambda|)^{-N}.
\]

(3.11)

**Proof of (3.11).** Derivatives of our phase function are computed as follows:

\[
\begin{align*}
\Psi_{u}'(p, q, t, u, \xi, v, \lambda) &= -(\xi + 2uv) + \epsilon_{\alpha}\lambda u^2 + \lambda(q - u^2), \\
\Psi_{v}'(p, q, t, u, \xi, v, \lambda) &= q - u^2.
\end{align*}
\]

(3.12)

Recall that \( \epsilon_{\alpha} = 1 - 3(\alpha + 1/2) \neq 0 \) when \( \alpha \neq -1/6 \). In (3.7) we need to observe that \( M^{\nu} \) is supported on the set \( B(\nu) \) where

\[
\begin{align*}
B(2) &= \{ u \approx 1, |v| \gg 1, |v| \gg (1 + |\epsilon_{\alpha}|)|\lambda| \} \cap \{|\xi| \gg |v| \text{ or } |\xi| \ll |v|\}, \\
B(3) &= \{ u \approx 1, |\lambda| \gg 1, |v| \ll \min(|\epsilon_{\alpha}|, |\epsilon_{\alpha}|)|\lambda| \} \cap \\
&\quad \{|\xi| \gg (|\epsilon_{\alpha}| + 1)|\lambda| \text{ or } |\xi| \ll \min(|\epsilon_{\alpha}|, |\epsilon_{\alpha}|)|\lambda|\}.
\end{align*}
\]

(3.13)

On the intersection of the support of \( \zeta([\theta]^{\nu}) \) and \( B(\nu) \), we have rapid change of the phase function \( \Psi \) with respect to \( u \):

\[
|\Psi_{u}'(p, q, t, u, \xi, v, \lambda)| \gtrsim (|\xi| + |v| + |\lambda| + 1).
\]

(3.14)

On the intersection of the support of \( \zeta([\theta]^{\nu}) \) and \( B(\nu) \), we also have enough change of \( \Psi \) with respect to \( v \):

\[
|\Psi_{v}'(p, q, t, u, \xi, v, \lambda)| \gtrsim (|\xi| + |v| + |\lambda| + 1)/|\lambda|.
\]

(3.15)

We can also easily observe that on \( B(\nu) \),

\[
\begin{align*}
\left| \frac{\partial}{\partial u} \left( \zeta([\theta]^{\nu})M^{\nu}(u, \xi, v, \lambda) \right) \right| &\lesssim \frac{1}{u}, \\
\left| \frac{\partial}{\partial v} \left( \zeta([\theta]^{\nu})M^{\nu}(u, \xi, v, \lambda) \right) \right| &\lesssim \frac{1}{|\lambda|}.
\end{align*}
\]
The derivative of $\partial_u(1/\Psi'_u)$ or $\partial_v(1/\Psi'_v)$ gains the decay of $1/u$ or $1/|\lambda|$ respectively. This combined with (3.8) and (3.13)-(3.15) gives the inequality of (3.11).

As the second step, we show that each of $D^p_\xi$, $D^q_v$, and $D^{(t-pq/2)+qu}_\lambda$ gives the decay of $p$, $q$ and $t - pq/2$ respectively. For these estimates we need to observe that on each region of $|x| \geq 2^{4+m\alpha}$ with each $x = p, q, (t - \frac{pq}{2}) + qu$,

\begin{align}
|\Psi'_u(p, q, t, u, \xi, v, \lambda)| & = |p - u| \gtrsim |p|,
|\Psi'_v(p, q, t, u, \xi, v, \lambda)| & = |q - u^2| \gtrsim |q|,
|\Psi_\lambda(p, q, t, u, \xi, v, \lambda)| & \gtrsim |(\frac{t}{4} - \frac{pq}{2}) + qu|.
\end{align}

By using (3.11) and (3.16) with (3.9) we can apply $D^{(t-pq/2)+qu}_\lambda D^q_v D^p_\xi$ to obtain (3.10). Therefore the decay estimate (3.6) is proved.

Let us turn to the second term of (3.5). Let us consider $\nu = 2$. In view of (3.7), we can observe that $x - y \approx 2^{-j}$ and $|y| \gg 2^{-j}$, which implies that $|x| \gg 2^{-j}$. Therefore from this and the Fourier inversion formula (2.4), we can write $K^{glo}_j \ast E^2_j \ast G^2_j = [E^2_j]_b \ast K^{glo}_j \ast E^2_j \ast G^2_j$. Here $[E^2_j]_b = \sum_{k \in H_b(2,j)} A^1_k$ where $H_b(2,j) = \{k : 2^{-k} > 2^{m\alpha-j+4}\}$. We decompose $K^{glo}_j \ast E^2_j \ast G^2_j$ so that

\begin{align}
K^{glo}_j \ast E^2_j \ast G^2_j = G^2_j \ast [E^2_j]_b \ast K^{glo}_j \ast E^2_j \ast G^2_j
+ W^2_j \ast E^2_j \ast K^{glo}_j \ast [E^2_j]_b \ast G^2_j.
\end{align}

Let us consider the second term in (3.17). Let $1/p' = 1 - 1/p$ and $1/q' = 1 - 1/q$. Then the $L^p \to L^q$ boundedness of the convolution operator $B \ast f$ is equivalent to the $L^{q'} \to L^{p'}$ boundedness of the convolution operator $B \ast f$ with $B(x) = B(-x)$. Let $r_j(t)$'s be the Rademacher
functions. Then we have the estimate
\[ \| \sum_{j} (G_{j}^2)^\ast (E_{j}^2) \ast (K_{j}^{glo})^\ast ([E_{j}]_{b}) \ast (W_{j}^2)^\ast f \|_{L^{p'}(R^{d})} \]
\[ \lesssim \| (\sum_{j} |(K_{j}^{glo})^\ast ([E_{j}]_{b}) \ast (W_{j}^2)^\ast f|^2)^{1/2} \|_{L^{p'}(R^{d})} \]
\[ \approx (\int_{0}^{1} \| \sum_{j} r_{j}(t)(K_{j}^{glo})^\ast ([E_{j}]_{b}) \ast (W_{j}^2)^\ast f \|_{L^{p'}(R^{d})} dt)^{1/p'} \]
\[ \lesssim \| f \|_{L^{p'}(R^{d})}. \]

The first inequality follows from Lemma 3 and the dual estimates of Lemma 2. The last inequality follows from the fact that $(1/q', 1/p') \in ABCD \cap l^{2}$ and
\[ \sum_{j} |r_{j}(t)(K_{j}^{glo})^\ast ([E_{j}]_{b}) \ast (W_{j}^2)(x_{1}, x_{2}, x_{3})| \lesssim \rho(x_{1}, x_{2}, x_{3})^{-6+\sigma}, \]
which is shown in the similar way as (3.6). Now there remains the first term in (3.17). By Lemmas 1,2,3, we have the estimate
\[ \| \sum_{j} G_{j}^2 \ast [E_{j}]_{b} \ast K_{j}^{glo} \ast E_{j}^2 \ast G_{j}^2 \ast f \|_{L^{q}} \]
\[ \lesssim \| \{K_{j}^{glo} \ast E_{j}^2 \ast G_{j}^2 \ast f\} \|_{L^{q}(l^{2})} \]
\[ \lesssim \| \{S_{j}^\alpha\} \|_{L^{P}(l^{2})} \| f \|_{L^{p}} \]
\[ \lesssim (1+ \| S_{j}^\alpha \|_{L^{p}(l^{2})}^{\frac{1-\theta}{\theta}} \| f \|_{L^{p}}, \]
where $0 < \theta < 1$ such that $\frac{1}{2} = \frac{\theta}{p} + \frac{1-\theta}{\infty}$. The last inequality can be shown as follows. Let us assume that $p \leq 2$, since the other range is treated by duality. By using the Hölder inequality,
\[ \| \{S_{j}^\alpha\} \|_{L^{P}(l^{2})} \leq \| \{S_{j}^\alpha\} \|_{L^{P}(l^{2})} \leq \| \{S_{j}^\alpha\} \|_{L^{P}(l^{2})} \leq C. \]

We also have for $(1/p, 1/q) \in ABCD \cap l^{2}$,
\[ (3.18) \| \{S_{j}^\alpha\} \|_{L^{P}(l^{2})} \leq \| S_{j}^\alpha \|_{L^{P}} \leq C. \]

The positivity for the kernel of $S_{j}^\alpha$ shows (3.18). The first inequality of (3.19) follows from the Minkowski inequality and the fact $q/p \geq 1$. 

The positivity for the kernel of $S_{j}^\alpha$ shows (3.18). The first inequality of (3.19) follows from the Minkowski inequality and the fact $q/p \geq 1$. 

The second inequality follows from a scaling estimation. The last follows by the application of Theorem 1. For this, we need to use the fact that \(|S_0^\alpha f(x)| \lesssim |T_0 f(x)|\) and the condition \((1/p, 1/q) \in ABCD \cap l_3^\infty\).

When \(\nu = 3\), in view of (3.7) with the condition \(|x - y| \approx 2^{-j}\) and \(|y| \approx 2^{-j}\), we have the support condition \(|x| \approx 2^{-j}\). So we can write \(K_j^{glo} * E_j^3 * G_j^3 = E_j^1 * K_j^{glo} * E_j^3 * G_j^3\). This case is also handled in the similar way,

\[
\| \sum_j E_j^1 * K_j^{glo} * E_j^3 * G_j^3 * f \|_{L^q} \lesssim (1 + \| S^\alpha_{\frac{1}{2}} \|_{L^p \to L^q}^{-\theta}) \| f \|_{L^p}.
\]

3.3. Proof of (3.2) for \(\nu = 1\)

By using the support condition of the integral kernel of \(\widehat{K_j^{glo}} * E_j^1(\lambda)\) we have \(K_j^{glo} * E_j^1 = (E_j^1 + E_j^3) * K_j^{glo} * E_j^1\). By applying the previous estimate for the case \(\nu = 3\), we can show that for \((1/p, 1/q) \in ABCD \cap l_3^\infty\) with \(0 < \theta < 1\) satisfying \(\frac{1}{2} = \frac{\theta}{p} + \frac{1-\theta}{\infty}\),

\[
\| \sum_j E_j^1 * K_j^{glo} * E_j^3 * f \|_{L^q} \lesssim (1 + \| S^\alpha_{\frac{1}{2}} \|_{L^p \to L^q}^{-\theta}) \| f \|_{L^p},
\]

\[
\| \sum_j G_j^3 * E_j^3 * K_j^{glo} * E_j^1 * f \|_{L^q} \lesssim (1 + \| S^\alpha_{\frac{1}{2}} \|_{L^p \to L^q}^{-\theta}) \| f \|_{L^p},
\]

\[
\| \sum_j W_j^3 * E_j^3 * K_j^{glo} * E_j^1 * f \|_{L^q} \lesssim \| f \|_{L^p}.
\]

For the last inequality we need to show the dual part estimate

\[
\| \sum_j E_j^1 * [K_j^{glo}] * E_j^3 * W_j^3 * f \|_{L^{p'}} \lesssim \| f \|_{L^{p'}}.
\]

The above estimate follows from the decay condition

\[
|\langle K_0^{glo} \rangle * E_0^3 * W_0^3(x_1, x_2, x_3)| \leq C_N (1 + \rho(x_1, x_2, x_3))^{-N}.
\]

We show (3.20) by the same way as (3.6). For this case the corresponding phase function \(\Psi(p, q, t, u, \xi, v, \lambda)\) for the integral (3.8) is

\[
\xi(p - u) + v(q - u^2) + \lambda(t - \frac{pq}{2} + qu - (\frac{1}{2} - \alpha)u^3).
\]
Its derivatives are
\begin{align}
\Psi'_u(p, q, t, u, \xi, v, \lambda) &= -(\xi + 2uv) + \epsilon^*_\alpha \lambda u^2 + \lambda(q - u^2), \\
\Psi'_v(p, q, t, u, \xi, v, \lambda) &= q - u^2.
\end{align}

(3.21)

If \( \alpha \neq 1/6 \), then \( \epsilon^*_\alpha \neq 0 \). So we can apply the same estimation as (3.10). Now the proof of (3.2) is finished. This implies that for some \( 0 < \theta < 1 \),
\begin{equation}
\| S^\sigma \|_{L^p \to L^q} \leq C \{ 1 + \| S^{1-\theta} \|_{L^p \to L^q} \}. \tag{3.22}
\end{equation}

Let us replace \( S^\sigma \) by \( S^\sigma_{\alpha, N} = \sum_{j=-N}^{N} S^\sigma_{j} \) with a finite \( N \). Then from (3.22) we obtain that

\[ \| S^\sigma_{\alpha, N} \|_{L^p \to L^q} \leq C, \]

where \( C \) is independent of \( N \).

4. The case \( \alpha = \pm 1/6 \)

The decay properties (3.6) and (3.20) are crucially used for our \((L^p, L^q)\) estimate. The decay estimates (3.6) and (3.20) are made by using the fact that at least one of \( \Psi'_u \) and \( \Psi'_v \) in each (3.12) and (3.21) does not vanish. However for the case \( \alpha = 1/6 \) we have \( \epsilon^*_\alpha = 0 \) in (3.21), which makes all the partial derivatives of \( \Psi \) with respect to \( u, \xi, v, \) and \( \lambda \) vanish where \( |\xi + 2uv|, |p - u|, |q - u^2| \) and \( |t - \alpha u^3| \) are very small. Similarly when \( \alpha = -1/6 \), we have \( \epsilon_\alpha = 0 \) in (3.12), and all the derivatives of \( \Psi \) vanish where \( |\xi + 2uv| \approx |p - u| \approx |q - u^2| \approx |t - \alpha u^3| \approx 0 \).

We do not know what is the type set of \( S^\sigma_{\pm1/6} \), but we can show that \( ABD \cap l_3^\sigma \) is not the type set of \( S^\sigma_{-1/6} \), and the set \( ABC \cap l_3^\sigma \) is not the type set of \( S^\sigma_{1/6} \). Let us define two sets in \( \mathbb{R}^2 \):

\[ U_\sigma = \{(1/p, 1/q) : 1/p \geq 2\sigma/3\}, \]
\[ V_\sigma = \{(1/p, 1/q) : 1/q \leq 1 - 2\sigma/3\}. \]

We show the following nontrivial results about the case \( \alpha = \pm 1/6 \):

(a) The type set of \( S^\sigma_{1/6} \) is contained in \( ABC \cap l_3^\sigma \cap U_\sigma \).

(b) The type set of \( S^\sigma_{-1/6} \) is contained in \( ABD \cap l_3^\sigma \cap V_\sigma \).
In showing (a) and (b), it suffices to show (b) since the other case $\alpha = 1/6$ follows from the dual estimate of $(L^p, L^q)$ and the fact

$$[S^*_\alpha] f(x_1, x_2, x_3) = S^-\alpha \tilde{f}(-x_1, -x_2, x_3),$$

where $\tilde{f}(x_1, x_2, x_3) = f(-x_1, -x_2, x_3)$.

**Proof of (b).** Recall that

$$S^{-1/6}_\alpha f(x) = \int f(x_1 - t, x_2 - t^2, x_3 + \frac{1}{6} t^3 - \frac{1}{2} (x_1 t^2 - x_2 t)) t^{\sigma} dt.$$

Let $R \gg r$ and define two sets depending on $r$ and $R$ and an integer $1 \leq k \leq R/r$:

$$D(r, R) = \{(x_1, x_2, x_3) : |x_1| < 10r, |x_2| < 10rR, |x_3| < 10r^2 R\},$$

$$Q_k(r, R) = \{(x_1, x_2, x_3) : (k - 1)r < |x_1| < kr, |x_2 - x_1^2| < rR,$n

$$|x_3 + \frac{1}{2} x_1 (x_2 - x_1^2)| < r^2 R\}.$$

Let us define $f_{D(r, R)}$ be a characteristic function supported on $D(r, R)$. Then for any $(x_1, x_2, x_3) \in Q_k(r, R)$ we can show that if $(k - 1)r < t < kr$, then

$$(4.1) \quad f_{D(r, R)}(x_1 - t, x_2 - t^2, x_3 + \frac{1}{6} t^3 - \frac{1}{2} (x_1 t^2 - x_2 t)) = 1.$$ 

(4.1) follows from the fact that

$$|x_1 - t| < 10r, |x_2 - t^2| \leq |x_2 - x_1^2| + |(x_1 - t)(x_1 + t)| < 10rR,$$

and the third component $|x_3 + \frac{1}{6} t^3 - \frac{1}{2} (x_1 t^2 - x_2 t)|$ is written as

$$|x_3 + \frac{1}{6} (t - x_1)^3 + \frac{1}{2} (x_2 - x_1^2)(t - x_1) + \frac{1}{2} x_1 (x_2 - x_1^2)| < 10r^2 R.$$ 

Therefore for each $x \in Q_k(r, R)$,

$$S^\sigma \alpha f_{D(r, R)}(x) \geq (kr)^{\sigma - 1} r.$$
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So from this and the measure estimate \( m(Q_k(r, R)) = r(rR)(r^2R) \),

\[
\left\| \mathcal{S}_\alpha f_D(r, R) \right\|_{L^q(\mathbb{H}^1)} \geq \left( \sum_{1 < k < R/r} \frac{1}{r} \int_{Q_k(r, R)} |\mathcal{S}_\alpha f_D(r, R)(x)|^q \, dx \right)^{1/q}
\]

\[
\geq \left( \sum_{1 < k < R/r} (k)^{(\sigma-1)q} (r)^{\sigma} (r(rR)(r^2R))^{1/q} \right)^{1/q}
\]

\[
\geq C_{\sigma,q}(R/r)^{(\sigma-1)+1/q} (r)^{\sigma} (r(rR)(r^2R))^{1/q},
\]

and

\[
\left\| f_D(r, R) \right\|_{L^p(\mathbb{H}^1)} = m(D(r, R))^{1/p} = (r(rR)(r^2R))^{1/p}.
\]

Suppose that for \((1/p, 1/q) \in ABD \cap l_3^1\), \( \left\| S_{-1/6} \right\|_{L^p \rightarrow L^q} < \infty \). Then we can write

\[ (R/r)^{(\sigma-1)+1/q} (r)^{\sigma} \leq (r(rR)(r^2R))^{1/p-1/q}. \]

Then by taking \( r = R^{1-\epsilon} \) with some positive number \( \epsilon \), then we need the following relation for sufficiently large \( R \),

\[ 1/q \leq 1 - 2\sigma/3. \]

\[ \square \]

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References


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