WEYL'S THEOREMS FOR POSINORMAL OPERATORS

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Abstract. An operator $T$ belonging to the algebra $B(H)$ of bounded linear transformations on a Hilbert $H$ into itself is said to be posinormal if there exists a positive operator $P \in B(H)$ such that $TT^* = T^*PT$. A posinormal operator $T$ is said to be conditionally totally posinormal (resp., totally posinormal), shortened to $T \in CTP$ (resp., $T \in TP$), if to each complex number $\lambda$ there corresponds a positive operator $P_\lambda$ such that $|(T-\lambda I)^*|^2 = |P_\lambda^{1/2}(T-\lambda I)|^2$ (resp., if there exists a positive operator $P$ such that $|(T-\lambda I)^*|^2 = |P^{1/2}(T-\lambda I)|^2$ for all $\lambda$). This paper proves Weyl's theorem type results for $TP$ and $CTP$ operators. If $A \in TP$, if $B^* \in CTP$ is isoloid and if $d_{AB} \in B(B(H))$ denotes either of the elementary operators $\delta_{AB}(X) = AX - XB$ and $\Delta_{AB}(X) = AXB - X$, then it is proved that $d_{AB}$ satisfies Weyl's theorem and $d_{AB}^*$ satisfies a-Weyl's theorem.

1. Introduction

Denoting the algebra of operators (equivalently, bounded linear transformations) on an infinite dimensional complex Hilbert space $H$ into itself by $B(H)$, an operator $T \in B(H)$ is said to be posinormal (short for positive-normal) if there exists a $P \geq 0$ in $B(H)$ such that $TT^* = T^*PT$. Equivalently, $T \in B(H)$ is posinormal if there exists a co-isometry $V^* \in B(H)$ and a positive operator $P \in B(H)$ such that $T = T^*PV^*$. The class of posinormal operators is large: it contains in particular the classes consisting of hyponormal ($T \in B(H) : TT^* \leq T^*T$), $M$-hyponormal ($T \in B(H) : ||(T-\lambda I)^*||^2 \leq M||(T-\lambda I)||^2$ for some real number $M > 0$ and all complex numbers $\lambda$) and dominant operators ($T \in B(H) : ||(T-\lambda I)^*||^2 \leq M_\lambda||T-\lambda I||^2$ for some real number $M_\lambda > 0$.

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and all complex numbers \( \lambda \). The class of posinormal operators was introduced by Rhaly [22], and has since by considered by Jeon et al [14]. It is elementary to see that the restriction of a posinormal operator to an invariant subspace is again posinormal.

It is immediate from the definition of posinormality that a posinormal operator \( T \) satisfies \( T^{-1}(0) \subseteq T^{*-1}(0) \), which implies that a posinormal operator has ascent \( \leq 1 \). A posinormal operator \( T \) is said to be conditionally totally posinormal (resp., totally posinormal), shortened to \( T \in CTP \) (resp., \( T \in TP \)), if to each complex number \( \lambda \) there corresponds a positive operator \( P_\lambda \) such that \( |(T - \lambda I)^*|^2 = |P_\lambda^{1/2}(T - \lambda)|^2 \) (resp., if there exists a positive operator \( P \) such that \( |(T - \lambda I)^*|^2 = |P^{1/2}(T - \lambda)|^2 \) for all \( \lambda \)). CTP operators have been considered by Jeon et al [14](where they have been called totally posinormal). Obviously, if \( T \in CTP \), then \( (T - \lambda I) \) has ascent \( \leq 1 \). Furthermore, \( T \in CTP \) if and only if \( T \) is dominant [22, Proposition 3.5]. Restricting themselves to only those \( T \in CTP \) for which the spectrum \( \sigma((T - \lambda I)|_{\mathcal{M}}) = \{0\} \implies (T - \lambda I)|_{\mathcal{M}} = 0 \) for every \( \mathcal{M} \in \text{Lat}(T) \), Jeon et al [14, Theorem 13] have shown that \( T \) satisfies Weyl’s theorem. In this note we prove that posinormal operators satisfy Weyl’s theorems under conditions which are visibly weaker than those considered in [14]. The plan of this note is as follows: we explain our notation and terminology in Section 2, with Section 3 devoted to proving our main results. In addition to proving Weyl’s theorem type results for \( TP \) and \( CTP \) operators, we prove that if \( A \in TP \), if \( B^* \in CTP \) is isoloid and if \( d_{AB} \in B(B(H)) \) denotes either of the elementary operators \( \delta_{AB}(X) = AX - XB \) and \( \Delta_{AB}(X) = AXB - X \), then \( d_{AB} \) satisfies Weyl’s theorem and \( d_{AB}^* \) satisfies \( \alpha \)-Weyl’s theorem.

2. Notation and terminology.

A Banach space operator \( T, T \in B(\mathcal{X}) \), is said to be Fredholm, \( T \in \Phi(\mathcal{X}) \), if \( T(\mathcal{X}) \) is closed and both the deficiency indices \( \alpha(T) = \dim(T^{-1}(0)) \) and \( \beta(T) = \dim(\mathcal{X}/T(\mathcal{X})) \) are finite, and then the index of \( T, \text{ind}(T) \), is defined to be \( \text{ind}(T) = \alpha(T) - \beta(T) \). The ascent of \( T, \text{asc}(T) \), is the least non-negative integer \( n \) such that \( T^{-n}(0) = T^{-(n+1)}(0) \) and the descent of \( T, \text{dsc}(T) \), is the least non-negative integer \( n \) such that \( T^n((\mathcal{X})) = T^{n+1}((\mathcal{X})) \). We shall, henceforth, shorten \( (T - \lambda I) \) to \( (T - \lambda) \). The operator \( T \) is Weyl if it is Fredholm of index zero, and \( T \) is said to be Browder if it is Fredholm “of finite ascent and descent”. Let \( \mathbf{C} \) denote the set of complex numbers. The (Fredholm)
essential spectrum \( \sigma_e(T) \), the Browder spectrum \( \sigma_b(T) \) and the Weyl spectrum \( \sigma_w(T) \) of \( T \) are the sets

\[
\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm} \};
\]

\[
\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Browder} \}
\]

and

\[
\sigma_w(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl} \}.
\]

If we let \( \rho(T) \) denote the resolvent set of the operator \( T \), \( \sigma(T) \) denote the usual spectrum of \( T \) and \( \text{acc} \sigma(T) \) denote the set of accumulation points of \( \sigma(T) \), then:

\[
\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup \text{acc} \sigma(T).
\]

Let \( \pi_0(T) \) denote the set of Riesz points of \( T \) (i.e., the set of \( \lambda \in \mathbb{C} \) such that \( T - \lambda \) is Fredholm of finite ascent and descent), and let \( \pi_{a0}(T) \) denote the set of eigenvalues of \( T \) of finite geometric multiplicity. Also, let \( \pi_{a0}(T) \) be the set of \( \lambda \in \mathbb{C} \) such that \( \lambda \) is an isolated point of \( \sigma_a(T) \) and \( 0 < \dim \ker(T - \lambda) < \infty \), where \( \sigma_a(T) \) denotes the approximate point spectrum of the operator \( T \in \mathcal{B}(\mathcal{X}) \). Clearly, \( \pi_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{a0}(T) \).

We say that \textit{Browder's theorem holds for} \( T \in \mathcal{B}(\mathcal{X}) \) if

\[
\sigma(T) \setminus \sigma_w(T) = \pi_0(T),
\]

\textit{Weyl's theorem holds for} \( T \) if

\[
\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),
\]

and \textit{a-Weyl's theorem holds for} \( T \) if

\[
\sigma_{wa}(T) = \sigma_a(T) \setminus \pi_{a0}(T),
\]

where \( \sigma_{wa}(T) \) denote the \textit{essential approximate point spectrum} (i.e., \( \sigma_{wa}(T) = \cap \{ \sigma_a(T + K) : K \in \mathcal{K}(\mathcal{X}) \} \) with \( \mathcal{K}(\mathcal{X}) \) denoting the ideal of compact operators on \( \mathcal{X} \)). If we let \( \Phi_+(\mathcal{X}) = \{ T \in \mathcal{B}(\mathcal{X}) : \alpha(T) < \infty \text{ and } T(\mathcal{X}) \text{ is closed} \} \) denote the semi-group of \textit{upper semi-Fredholm operators} in \( \mathcal{B}(\mathcal{X}) \), then \( \sigma_{wa}(T) \) is the complement in \( \mathbb{C} \) of all those \( \lambda \) for which \( (T - \lambda) \in \Phi_+(\mathcal{X}) \) and \( \text{ind}(T - \lambda) \leq 0 \). The concept of \( a \)-Weyl's theorem was introduced by Rakočević: \( a \)-Weyl's theorem for \( T \implies \) Weyl's theorem for \( T \), but the converse is generally false [21].

An operator \( T \in \mathcal{B}(\mathcal{X}) \) has the \textit{single-valued extension property} at \( \lambda_0 \in \mathbb{C} \), SVEP at \( \lambda_0 \in \mathbb{C} \) for short, if for every open disc \( \mathcal{D}_{\lambda_0} \) centered at \( \lambda_0 \) the only analytic function \( f : \mathcal{D}_{\lambda_0} \to \mathcal{X} \) which satisfies

\[
(T - \lambda)f(\lambda) = 0 \text{ for all } \lambda \in \mathcal{D}_{\lambda_0}
\]
is the function $f \equiv 0$. Trivially, every operator $T$ has SVEP at points of the resolvent $\mathbb{C} \setminus \sigma(T)$; also $T$ has SVEP at $\lambda \in isoc(T)$. We say that $T$ has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. It is known that a Banach space operator $T$ with SVEP satisfies Browder's theorem [1, Corollary 2.12] and that Browder's theorem holds for $T \iff$ Browder's theorem holds for $T^*$ [12].

The analytic core $K(T - \lambda)$ of $(T - \lambda)$ is defined by

$$K(T - \lambda) = \{ x \in \mathcal{X} : \text{there exists a sequence } \{ x_n \} \subset \mathcal{X} \text{ and }$$
$$\delta > 0 \text{ for which } x = x_0, (T - \lambda)(x_{n+1}) = x_n \text{ and }$$
$$\| x_n \| \leq \delta^n \| x \| \text{ for all } n = 1, 2, ... \}.$$  

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T - \lambda)$ such that $(T - \lambda)^{-p}(0) \subseteq H_0(T - \lambda)$ for all $p = 0, 1, 2, \ldots$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [18].

The operator $T \in B(\mathcal{X})$ is said to be semi-regular if $T(\mathcal{X})$ is closed and $T^{-1}(0) \subset T^\infty(\mathcal{X}) = \bigcap_{m \in \mathbb{N}} T^m(\mathcal{X})$; $T$ admits a generalized Kato decomposition, GKD for short, if there exists a pair of $T$-invariant closed subspaces $(M, N)$ such that $X = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(\mathcal{X})$ has a GKD at every $\lambda \in isoc(T)$, namely $\mathcal{X} = H_0(T - \lambda) \oplus K(T - \lambda)$. We say that $T$ is of Kato type at a point $\lambda$ if $(T - \lambda)|_M$ is nilpotent in the GKD for $(T - \lambda)$. Fredholm operators are Kato type [15, Theorem 4], and Operators $T \in B(\mathcal{X})$ satisfying property $H(p)$, $H(p)$

$$H_0(T - \lambda) = (T - \lambda)^{-p}(0)$$

for some integer $p \geq 1$, are Kato type at isolated points of $\sigma(T)$ (but not every Kato type operator $T$ satisfies property $H(p)$). Let $\sigma_{kt}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Kato type} \}$. The set $\sigma_{kt}(T)$ is known to be a closed subset of $\mathbb{C}$ such that $\sigma_{kt}(T) \subseteq \sigma_a(T)$ [3].

3. Main results

We say in the following that an operator $T \in B(H)$ is conditionally totally posinormal, CTP for short, if to each $\lambda \in \mathbb{C}$ there corresponds an operator $P_\lambda \geq 0$ such that $|(T - \lambda)^*|^2 \leq |P_{1/2}^{1/2} (T - \lambda)|^2$; $T$ will be said to be totally posinormal, TP for short, if $T$ is CTP and the positive operator $P_{1/2}$ can be chosen independent of $\lambda$. It is easy to see that $T$ is CTP if and only if it is dominant, and that $T$ is TP if and only if it is $M$-hyponormal. Clearly, TP operator satisfy Bishop's condition $(\beta)$ [11] and hence are subscalar. If we let $\hat{T}$ denote the generalized
scalar extension of $T$ to some Hilbert space $K \supset H$, then there exists
an integer $p \geq 1$ such that $H_0(\hat{T} - \lambda) = (\hat{T} - \lambda)^{-p}(0)$ for each $\lambda \in \mathbb{C}$ [5, Theorem 4.5, Chapter 4]. The Hilbert space $H$ being invariant for $\hat{T}$,
$H_0(\hat{T}|_H - \lambda) = H_0(\hat{T} - \lambda) \cap H = (\hat{T} - \lambda)^{-p}(0) \cap H = (\hat{T}|_H - \lambda)^{-p}(0)$, i.e. $T$ satisfies the property $\mathcal{H}(p)$ for all $\lambda \in \mathbb{C}$. Finally, since asc($T - \lambda$) $\leq 1$, $T$ satisfies property $\mathcal{H}(1)$.

Let $\mathcal{H}(\sigma(T))$ (resp., $\mathcal{H}_1(\sigma(T))$) denote the set of analytic functions which are defined on an open neighborhood $U$ of $\sigma(T)$ (resp., the set of $f \in \mathcal{H}(\sigma(T))$ which are non-constant on each of the connected components of the open neighborhood $U$ of $\sigma(T)$ on which $f$ is defined). Recall that an operator $T$ is said to be isoloid if each $\lambda \in \text{iso}\sigma(T)$ is an eigen-value of $T$.

**Theorem 3.1.** Let $T \in TP$. Then:

(i) $f(T)$ and $f(T^*)$ satisfy Weyl’s theorem for every $f \in \mathcal{H}(\sigma(T))$.

(ii) $T^*$ satisfies a-Weyl’s theorem.

(iii) If $T^*$ has SVEP, then $T$ satisfies a-Weyl’s theorem.

(iv) If $T$ is the quasi-affine transform of an operator $S \in B(H)$, then $f(S)$ and $f(S^*)$ satisfy Weyl’s theorem for every $f \in \mathcal{H}(\sigma(T))$.

**Proof.** The proof of (i) follows from [19, Corollary 3.6] (see also [2]). We note that if $T$ is the quasi-affine transform of $S$ and $T$ satisfies property $\mathcal{H}(p)$, then $S$ satisfies property $\mathcal{H}(p)$ [19, Lemma 3.2], and this implies (iv) [19, Corollary 3.7]. To prove (ii), we notice that $T$ satisfies property $\mathcal{H}(1) \implies T$ has finite ascent (and hence SVEP), which implies that $\sigma(T^*) = \sigma_a(T^*)$ [17, p.35] and $\pi_{a_0}(T^*) = \pi_{00}(T^*)$. We prove that $\sigma_{wa}(T^*) = \sigma_w(T^*)$: since $T^*$ satisfies Weyl’s theorem by (i), this would then imply that $\sigma_a(T^*) \setminus \sigma_{wa}(T^*) = \pi_{a_0}(T^*)$. It being clear that $\sigma_{wa}(T^*) \subseteq \sigma_w(T^*)$, we prove the reverse inclusion. Since

$$\lambda \notin \sigma_{wa}(T^*) \iff (T - \lambda)^* \notin \Phi_+(H) \text{ and } ind(T - \lambda)^* \leq 0,$$

the hypothesis $T$ has SVEP implies that

$$dsc(T - \lambda)^* < \infty \text{ ([1, Theorem 2.9]),}$$

$$\alpha(T - \lambda)^* < \infty \text{ and } ind(T - \lambda)^* \leq 0.$$

Since $dsc(T - \lambda)^* < \infty \implies ind(T - \lambda)^* \geq 0$ [13, Proposition 38.5], it follows that

$$dsc(T - \lambda)^* < \infty, \alpha(T - \lambda)^* = \beta(T - \lambda)^* < \infty \implies \lambda \notin \sigma_w(T^*).$$

This leaves us with the proof of (iii).
The hypothesis $T^*$ has SVEP implies $\sigma(T) = \sigma_a(T)$ [17, p.35], and hence $\pi_{a0}(T) = \pi_{00}(T)$. We prove that $\sigma_{wa}(T) = \sigma_w(T)$: since $T$ satisfies Weyl's theorem (by part (i)), this would then imply that $\sigma_a(T) \setminus \sigma_{wa}(T) = \pi_{a0}(T)$. It being clear that $\lambda \notin \sigma_w(T) \implies \lambda \notin \sigma_{wa}(T)$, we prove that $\lambda \notin \sigma_{wa}(T) \iff \lambda \notin \sigma_w(T)$. Since $\lambda \notin \sigma_{wa}(T) \iff (T - \lambda) \in \Phi_+(X)$ and $\text{ind}(T - \lambda) \leq 0$, the hypothesis $T^*$ has SVEP implies that $dsc(T - \lambda) < \infty [1, \text{Theorem 2.9}], \alpha(T - \lambda) < \infty$ and $\text{ind}(T - \lambda) \leq 0$. Again, since $dsc(T - \lambda) < \infty$ implies $\text{ind}(T - \lambda) \geq 0$ [13, Proposition 38.5], we have:

$$\lambda \notin \sigma_{wa}(T) \iff dsc(T - \lambda) < \infty \text{ and } \alpha(T - \lambda) = \beta(T - \lambda) < \infty,$$

which implies that $\lambda \notin \sigma_w(T)$. \hfill \Box

The example of a quasi-nilpotent CTP operator shows that CPT operators do not satisfy property $\mathbf{H}(p)$. (Such operators exist: see [14, Example 8].) Since $(T - \lambda)^{-1}(0) \subseteq (T - \lambda)^{-1}(0)$ for CTP operators $T$, CTP operators have ascent $\leq 1$. In particular, CTP operators have SVEP.

**Lemma 3.2.** A necessary and sufficient condition for the isolated points of the spectrum of a Banach space operator $T$, $T \in B(\mathcal{X})$, to be poles of the resolvent of $T$ is that $\text{iso}\sigma(T) \cap \sigma_{kt}(T) = \emptyset$.

**Proof.** If $\text{iso}\sigma(T) \cap \sigma_{kt}(T) = \emptyset$, then $\lambda \in \text{iso}\sigma(T) \implies T - \lambda$ is Kato type. Since both $(T - \lambda)$ and $(T - \lambda)^*$ have SVEP at 0, it follows (from [1, Theorems 2.6 and 2.9] and [17, Proposition 4.10.6]) that $\text{asc}(T - \lambda) = dsc(T - \lambda) < \infty \implies \lambda$ is a pole of the resolvent of $T$ [13, Proposition 50.2]. Conversely, if each $\lambda \in \text{iso}\sigma(T)$ is a pole (of some finite order $p$) of the resolvent of $T$, then $\mathcal{X} = (T - \lambda)^{-p}(0) \oplus (T - \lambda)^p(\mathcal{X}) \implies \lambda \notin \sigma_{kt}(\mathcal{X}) \implies \text{iso}\sigma(T) \cap \sigma_{kt}(T) = \emptyset$. \hfill \Box

**Theorem 3.3.** If $T \in CTP$ is such that $\text{iso}\sigma(T) \cap \sigma_{kt}(T) = \emptyset$, then:

(i) $f(T)$ satisfies Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$ and $T^*$ satisfies a-Weyl's theorem.

If also $T^*$ has SVEP, then:

(ii) $f(T)$ satisfies a-Weyl's theorem for every $f \in \mathcal{H}_1(\sigma(T))$.

**Proof.** (i) We start by proving that $T$ satisfies Weyl's theorem. Since $T \in CTP \implies (T - \lambda)^{-1}(0) \subseteq (T - \lambda)^{-1}(0)$, $\text{asc}(T - \lambda) \leq 1$ for all $\lambda \in \mathbb{C} \implies T$ has SVEP $\implies T$ satisfies Browder's theorem $\implies \sigma(T) \setminus \sigma_w(T) = \pi_0(T) \subseteq \pi_{00}(T)$. For the reverse inclusion, let $\lambda \in \pi_{00}(T)$. Then, since $\text{iso}\sigma(T) \cap \sigma_{kt}(T) = \emptyset$, $\lambda \in \pi_0(T)$ (by Lemma
3.2) \[ \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T), \] and \( T \) satisfies Weyl's theorem. It is clear from Lemma 3.2 that \( T \) is isoloid \[ \implies \sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(f(T))) \] [16, Lemma]. Since \( T \) has SVEP, \( \sigma_w(f(T)) = f(\sigma_w(T)) \) for every \( f \in \mathcal{H}(\sigma(T)) \) [6, Corollary 2.6]. We already know that \( T \) satisfies Weyl's theorem. Hence
\[ \sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma_w(T)) = \sigma_w(f(T)), \]
i.e., \( f(T) \) satisfies Weyl's theorem.

The proof that \( T^* \) satisfies \( a \)-Weyl's theorem is similar to that of Theorem 3.1(ii), and is left to the reader.

(ii) Let \( f \in \mathcal{H}_1(\sigma(T)) \). Then \( T^* \) has SVEP \( \implies f(T^*) = f(T)^* \) has SVEP [17, Theorem 3.3.9] \( \implies \sigma(f(T)) = \sigma_a(f(T)) \) [17, pp. 35]. Arguing as before it is seen that \( \sigma_w(f(T)) = \sigma_{wa}(f(T)) \). Since \( f(T) \) satisfies Weyl's theorem by part (i),
\[ \sigma_a(f(T)) \setminus \sigma_{wa}(f(T)) = \sigma(f(T)) \setminus \sigma_w(f(T)) = \pi_{00}(f(T)) = \pi_{0a}(f(T)), \]
i.e. \( f(T) \) satisfies \( a \)-Weyl's theorem. \( \square \)

We remark here that Theorem 3.3 has a more general Banach space version; see [9]. A CTP operator \( T \) such that \( \sigma(T - \lambda) = \{0\} \implies T = \lambda I \) satisfies \( isos(\sigma(T)) \cap \sigma_{KL}(T) = \emptyset \): this is seen as follows. If \( \lambda \in isos(\sigma(T)) \), then \( H = H_0(T - \lambda) \oplus K(T - \lambda) \). The operator \( (T - \lambda)|_{H_0(T - \lambda)} \) being CTP with \( \sigma((T - \lambda)|_{H_0(T - \lambda)}) = \{0\} \), \( (T - \lambda)|_{H_0(T - \lambda)} = 0 \implies T \) is Kato type. Obviously, Theorem 3.3 contains [14, Theorems 13 and 16].

An elementary operator. Let \( A \in TP \) and let \( B^* \in CTP \). Define the elementary operator \( \Delta_{AB} \in B(B(H)) \) and the generalized derivation \( \delta_{AB} \in B(B(H)) \) by \( \Delta_{AB}(X) = AXY - X \) and \( \delta_{AB}(X) = AX - XB \). Let \( d_{AB} \in B(B(H)) \) denote either of the operators \( \Delta_{AB} \) and \( \delta_{AB} \). We prove in the following that if \( B^* \) has the isoloid property (i.e., if the isolated points of \( \sigma(B^*) \) are eigenvalues of \( B^* \)), then \( f(d_{AB}) \) satisfies Weyl's theorem for each \( f \in \mathcal{H}(\sigma(d_{AB})) \), thereby generalizing [8, Theorem 3.1]. We start with the following lemma on the ascent of \( d_{AB} \).

Lemma 3.4. \( asc(d_{AB} - \lambda) \leq 1 \) for all \( \lambda \in \mathbb{C}. \)

Proof. Recall that \( A \in TP \) if and only if \( A \) is \( M \)-hyponormal and \( B^* \in TCP \) if and only if \( B^* \) is dominant. Recall also that if \( A \in B(H) \) is \( M \)-hyponormal and \( B^* \in B(H) \) is dominant, then \( \delta_{AB}(X) = 0 \implies \delta_{A^*B^*}(X) = 0 \) [20]. Thus \( (d_{AB})^{-1}(0) \subset (d_{A^*B^*})^{-1}(0) \) [7, Theorem 2]. Evidently, if \( B^* \in CTP \), then the operators \( \alpha B^* \) and \( \frac{1}{\beta} B^* \) are \( CTP \) for all \( \alpha \) and non-zero \( \beta \) in \( \mathbb{C} \). Hence \( (d_{AB} - \lambda)^{-1}(0) \subset (d_{A^*B^*} - \lambda)^{-1}(0) \).
for all $\lambda \in \mathbb{C}$. This, by [8, Proposition 2.3], implies that $asc(d_{AB} - \lambda) \leq 1$.

**Lemma 3.5.** If $B^*$ is isoloid, then $isos(d_{AB}) \cap \sigma_{kl}(d_{AB}) = \emptyset$.

**Proof.** We prove that $H_0(d_{AB} - \lambda) = (d_{AB} - \lambda)^{-1}(0)$ for all $\lambda \in isos(d_{AB})$. Since $\lambda \in isos(d_{AB}) \implies B(H) = H_0(d_{AB} - \lambda) \oplus K(d_{AB} - \lambda)$, this would then imply that $B(H) = (d_{AB} - \lambda)^{-1}(0) \oplus (d_{AB} - \lambda)(B(H))$, i.e. $d_{AB} - \lambda$ is Kato type and hence $\lambda \notin \sigma_{kl}(d_{AB})$. Recall that an $M$-hyponormal (equivalently, $TP$) operator is isoloid (indeed, the isolated points of the spectrum of such an operator are simple poles of the resolvent of the operator), and that the eigenvalues of a dominant (equivalently, $CTP$) operator are normal eigenvalues of the operator.

**The case $d_{AB} = \Delta_{AB}$.** Let $\lambda \in isos(\Delta_{AB})$. We divide the proof into the cases $\lambda = -1$ and $\lambda \neq -1$. Let $\Phi_{AB} = L_AR_B$, where $L_A$ and $R_B \in B(B(H))$ are operators of “left multiplication by $A$” and “right multiplication by $B$” (respectively). If $\lambda = -1$, then $0 \in isos(\Phi_{AB})$. Since $\sigma(\Phi_{AB}) = \sigma(B) \cup \sigma(zA)$, we must have either $0 \in isos(B)$ or $0 \in isos(A)$. Suppose that $0 \in isos(B)$. (The other case is similarly dealt with.) Then 0 cannot be a limit point of $\sigma(A)$. For if $0$ is a limit point of $\sigma(A)$, then there exists a sequence $\{\alpha_n\} \in \sigma(A)$ such that $\alpha_n \to 0 \in \sigma(A)$. Choosing a non-zero $z \in \sigma(B)$ we then have a sequence $\{z\alpha_n\} \in \sigma(\Phi_{AB})$ such that $z\alpha_n \to 0$, which contradicts the fact that $0 \in isos(\Phi_{AB})$. (We remark here that such a choice of $z$ is always possible, for if not then $\sigma(B) = \{0\}$ and $B$ is the zero operator.) The conclusion that 0 cannot be a limit point of $\sigma(A)$ implies that either $0 \notin \sigma(A)$ or $0 \notin isos(A)$. If $0 \notin \sigma(A)$, then $A$ is invertible and $H_0(\Phi_{AB}) = H_0(\Phi_{IB})$. Notice that $0 \in isos(B) \implies 0 \in isos(B^*)$. Since $B^* \in CTP$ is isoloid, $ker(B^*)$ reduces $B$ and $B = 0 \oplus B_0$ with respect to the decomposition $H = ker(B^*) \oplus ker(B^*) = H_1 \oplus H_2$, say, of $H$. Clearly, the operator $B_2 = B|_{H_2}$ is invertible. Let $X : H_1 \oplus H_2 \to H_1 \oplus H_2$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then

$$\lim_{n \to \infty} \|\Phi^n_{IB}(X)\|^{\frac{1}{n}} = \lim_{n \to \infty} \left\| \begin{bmatrix} 0 & X_{12}B^n_2 \\ 0 & X_{22}B^n_2 \end{bmatrix} \right\|^{\frac{1}{n}} = 0$$

if and only if both $\|X_{12}B^n_2\|^{\frac{1}{n}}$ and $\|X_{22}B^n_2\|^{\frac{1}{n}} \to 0$ as $n \to \infty$. The operator $B_2$ being invertible, $\Phi_{IB_2}$ is invertible $\implies X_{12} = X_{22} = 0$ and $H_0(\Phi_{AB}) = (\Phi_{AB})^{-1}(0)$. Now let $0 \in isos(A)$. Then $A = 0 \oplus A_2$ with respect to the decomposition $H = ker(A) \oplus ker(A) = H_1 \oplus H'_2$, say, of $H$, where the operator $A_2 = A|_{H'_2}$ is invertible. Let $X : H_1 \oplus H_2 \to$
$H_1' \oplus H_2'$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then
\[
\lim_{n \to \infty} \left\| \Phi_{AB}^n(X) \right\|_n^{\frac{1}{n}} = \lim_{n \to \infty} \left\| \begin{bmatrix} 0 & 0 \\ 0 & A_2^n X_{22} B_2^n \\ \end{bmatrix} \right\|_n^{\frac{1}{n}} = 0
\]
\[\iff \lim_{n \to \infty} \left\| A_2^n X_{22} B_2^n \right\|_n^{\frac{1}{n}} = 0 \iff X_{22} = 0,
\]
which implies $H_0(\Phi_{AB}) = (\Phi_{AB}^{-1})(0)$. This leaves us with the case $\lambda \neq -1$, which we consider next.

If $\lambda \neq -1$, then $X(\Delta_{AB} - \lambda)(X) = AXB - (1 + \lambda)X$, and it follows from [10, Theorem 3.2] that
\[
\sigma(\Delta_{AB} - \lambda) = \bigcup \{ \sigma(-(1 + \lambda) + zA) : z \in \sigma(B) \}.
\]
If $\lambda \in \text{iso}\sigma(\Delta_{AB})$, then $0 \in \text{iso}\sigma(\Delta_{AB} - \lambda)$. There exists a finite set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ of distinct non-zero values of $z \in \text{iso}\sigma(B)$, and corresponding to these values of $z$ a finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of distinct non-zero values $\alpha_i \in \text{iso}\sigma(A)$ such that $\alpha_i \beta_i = 1 + \lambda$ for all $1 \leq i \leq n$. Let
\[
H_1 = \bigwedge_{i=1}^n \ker(B - \beta_i)^*, \quad H_1' = \bigwedge_{i=1}^n \ker(A - \alpha_i),
\]
\[
H_2 = H \ominus H_1 \text{ and } H_2' = H \ominus H_1'.
\]
Then $A$ and $B$ have the direct sum decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where $A_1 = A|_{H_1}$ and $B_1 = B|_{H_1}$ are normal operators with finite spectrum, $B_1$ is invertible, $A_2 = A|_{H_2}$, $B_2 = B|_{H_2}$, and $\sigma(A_1) \cap \sigma(A_2) = \emptyset = \sigma(B_1) \cap \sigma(B_2)$. Let $X : H_1 \oplus H_2 \to H_1' \oplus H_2'$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$. Then
\[
\lim_{n \to \infty} \left\| (\Delta_{AB} - \lambda)^n(X) \right\|_n^{\frac{1}{n}} = \lim_{n \to \infty} \left\| \begin{bmatrix} (\Delta_{A_1} B_1 - \lambda)^n(X_{11}) & (\Delta_{A_1} B_2 - \lambda)^n(X_{12}) \\ (\Delta_{A_2} B_1 - \lambda)^n(X_{21}) & (\Delta_{A_2} B_2 - \lambda)^n(X_{22}) \end{bmatrix} \right\|_n^{\frac{1}{n}} = 0
\]
\[\iff \lim_{n \to \infty} \left\| (\Delta_{A_1} B_j - \lambda)^n(X_{ij}) \right\|_n^{\frac{1}{n}} = 0 \]
for all $1 \leq i, j \leq 2$. Clearly, $0 \notin \sigma(\Delta_{A_i} B_j - \lambda)$ for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$; hence
\[
\lim_{n \to \infty} \left\| (\Delta_{A_1} B_j - \lambda)^n(X_{ij}) \right\|_n^{\frac{1}{n}} = 0 \implies X_{ij} = 0.
\]
for all $1 \leq i, j \leq 2$ such that $i, j \neq 1$. The operators $A_1$ and $B_1$ are normal. Since $B_1$ is invertible implies $(\Delta_{A_1} B_1 - \lambda)(X_{11}) = (A_1 X_{11} - \lambda X_{11})$,

\[X_{11}((1 + \lambda)B_1^{-1}))B_1 = \delta_{A_1((1+\lambda)B_1^{-1})}(X_{11}B_1),\]
and since
\[
\lim_{n \to \infty} \|\delta_{C_D}^n(Y)\|^\frac{1}{n} = 0 \iff \delta_{C_D}(Y) = 0
\]
for normal C and D [20, Lemma 2] (see also [5]),
\[H_0(\triangle_{A_1B_1} - \lambda) = (\triangle_{A_1B_1} - \lambda)^{-1}(0) \implies H_0(\triangle_{AB} - \lambda) = (\triangle_{AB} - \lambda)^{-1}(0).\]

**The case \(d_{AB} = \delta_{AB}\).** Let \(\lambda \in i\sigma(\delta_{AB})\). Then \(0 \in i\sigma(\delta_{AB} - \lambda)\), where \(\sigma(\delta_{AB} - \lambda) = \sigma(A) - \sigma(B + \lambda)\) [10]. Hence \(\sigma(A) \cap \sigma(B + \lambda)\) consists of points which are isolated in both \(\sigma(A)\) and \(\sigma(B + \lambda)\). In particular, \(\sigma(A) \cap \sigma(B + \lambda)\) does not contain any limit points of \(\sigma(A) \cup \sigma(B + \lambda)\). There exist finite sets \(S_1 = \{\alpha_1, \alpha_2, ..., \alpha_n\}\) and \(S_2 = \{\beta_1, \beta_2, ..., \beta_n\}\) of distinct values \(\alpha_i\) and \(\beta_i\) such that each \(\alpha_i\) is an isolated point of \(\sigma(A)\), each \(\beta_i\) is an isolated point of \(\sigma(B)\), and \(\alpha_i - \beta_i = \lambda\) for all \(1 \leq i \leq n\). Let
\[H_1 = \bigvee_{i=1}^{n} \ker(B - \alpha_i)^* , \ H'_1 = \bigvee_{i=1}^{n} \ker(A - \alpha_i),\]
\[H_2 = H \ominus H_1 \text{ and } H'_2 = H \ominus H'_1.\]
Define the normal operators \(A_1\) and \(B_1\), and the operators \(A_2\) and \(B_2\), as before. Letting \(X : H_1 \oplus H_2 \to H'_1 \oplus H'_2\) have the matrix representation \(X = [X_{ij}]_{i,j=1}^{2}\), it is then seen that
\[
\lim_{n \to \infty} \|((\delta_{AB} - \lambda)^n(X))\|^\frac{1}{n} = 0
\]
\[
\lim_{n \to \infty} \left\| \begin{bmatrix}
(\delta_{A_1B_1} - \lambda)^n(X_{11}) & (\delta_{A_1B_2} - \lambda)^n(X_{12}) \\
(\delta_{A_2B_1} - \lambda)^n(X_{21}) & (\delta_{A_2B_2} - \lambda)^n(X_{22})
\end{bmatrix} \right\|^\frac{1}{n} = 0
\]
\[
\iff \lim_{n \to \infty} \|((\delta_{A_1B_j} - \lambda)^n(X_{ij}))\|^\frac{1}{n} = 0
\]
for all \(1 \leq i, j \leq 2\). Since \(\sigma(A_i) \cap \sigma(B_j + \lambda) = \emptyset\) for all \(1 \leq i, j \leq 2\) such that \(i, j \neq 1\) (so that \(0 \notin \sigma(\delta_{ab} - \lambda)\) for all \(1 \leq i, j \leq 2\) such that \(i, j \neq 1\)), \(X_{ij} = 0\) for all \(1 \leq i, j \leq 2\) such that \(i, j \neq 1\). The operators \(A_1\) and \(B_1\) being normal
\[
\lim_{n \to \infty} \|((\delta_{A_1B_1} - \lambda)^n(X_{11}))\|^\frac{1}{n} = 0 \iff (\delta_{A_1B_1} - \lambda)(X_{11}) = 0
\]
[20, Lemma 2]. Hence \(H_0(\delta_{A_1B_1} - \lambda) = (\delta_{A_1B_1} - \lambda)^{-1}(0)\), which implies that \(H_0(d_{AB} - \lambda) = (d_{AB} - \lambda)^{-1}(0)\) \(\square\)

**Theorem 3.6.** Let \(A \in TP\) and \(B^* \in CTP\). If \(B^*\) is isoloid, then we have the following:

(i) \(f(d_{AB})\) satisfies Weyl's theorem for each \(f \in \mathcal{H}(\sigma(d_{AB}))\).

(ii) \(d_{AB}^*\) satisfies a-Weyl's theorem.

If also \(d_{AB}^*\) has SVEP, then:
(iii) \( f(d_{AB}) \) satisfies a-Weyl’s theorem for each \( f \in \mathcal{H}_1(\sigma(d_{AB})) \).

Proof. (i) We start by proving that \( d_{AB} \) and \( d_{AB}^* \) satisfy Weyl’s theorem. Since \( \operatorname{asc}(d_{AB} - \lambda) \leq 1 \) (by Lemma 3.4), \( d_{AB} \) has SVEP \( \implies \) \( d_{AB} \) and \( d_{AB}^* \) satisfy Browder’s theorem (see [1, Corollary 2.12] and [12]). We prove that \( \pi_{00}(d_{AB}) \subseteq \pi_0(d_{AB}) \), which would then imply \( \pi_{00}(d_{AB}) = \pi_0(d_{AB}) \) and hence that \( d_{AB} \) satisfies Weyl’s theorem. Let \( \lambda \in \pi_{00}(d_{AB}) \); then \( \lambda \in \operatorname{iso}(d_{AB}) \) and \( 0 < \alpha(d_{AB} - \lambda) < \infty \). Since \( \operatorname{iso}(d_{AB}) \cap \sigma_{k1}(d_{AB}) = \emptyset \) (by Lemma 3.5), \( d_{AB} - \lambda \) is Kato type and \( B(H) = (d_{AB} - \lambda)^{-1}(0) \oplus (d_{AB} - \lambda)(B(H)) \implies \lambda \) is a simple pole of the resolvent of \( d_{AB} \implies \pi_{00}(d_{AB}) \subseteq \pi_0(d_{AB}) \). The conclusion \( d_{AB} \) satisfies Weyl’s theorem implies that \( \sigma(d_{AB}) \setminus \sigma_w(d_{AB}) = \pi_{00}(d_{AB}) = \pi_0(d_{AB}) \).

Since

\[
\lambda \notin \sigma_w(d_{AB}) \iff (d_{AB} - \lambda) \in \Phi(B(H)) \text{ and ind}(d_{AB} - \lambda) = 0
\]
\[
\iff (d_{AB}^* - \lambda I^*) \in \Phi(B(H)) \text{ and ind}(d_{AB}^* - \lambda I^*) = 0
\]
\[
\iff \lambda \notin \sigma_w(d_{AB}^*),
\]

\( \sigma_w(d_{AB}) = \sigma_w(d_{AB}^*) \). Hence, since \( \sigma(d_{AB}) = \sigma(d_{AB}^*) \),

\[
\sigma(d_{AB}^*) \setminus \sigma_w(d_{AB}^*) = \sigma(d_{AB}) \setminus \sigma_w(d_{AB})
\]
\[
= \pi_{00}(d_{AB}) = \pi_0(d_{AB}) = \pi_0(d_{AB}^*) \subseteq \pi_{00}(d_{AB}^*). 
\]

For the reverse inclusion, let \( \lambda \in \pi_{00}(d_{AB}^*) \). Then \( \alpha(d_{AB}^* - \lambda I^*) < \infty \implies \beta(d_{AB} - \lambda) < \infty \). Since \( \lambda \in \operatorname{iso}(d_{AB}^*) \implies \lambda \in \operatorname{iso}(d_{AB}) \), both \( d_{AB} \) and \( d_{AB}^* \) have SVEP at \( \lambda \). Thus, since \( (T - \lambda) \) is Kato type, \( \operatorname{asc}(d_{AB} - \lambda) = \operatorname{asc}(d_{AB} - \lambda) < \infty \) ([1, Theorems 2.6 and 2.9]) and \( 0 < \alpha(d_{AB} - \lambda) = \beta(d_{AB} - \lambda) < \infty \) ([13, Proposition 38.6]). Hence \( \lambda \in \pi_0(d_{AB}) = \pi_{00}(d_{AB}) \), which implies that \( \sigma(d_{AB}^*) \setminus \sigma_w(d_{AB}^*) = \pi_{00}(d_{AB}^*) \).

The isoloid property of \( d_{AB} \) (see the proof of Lemma 3.5) implies that \( \sigma(f(d_{AB}) \setminus \pi_{00}(f(d_{AB}))) = f(\sigma(d_{AB}) \setminus \pi_{00}(d_{AB})) \) [16, Lemma]. Since \( d_{AB} \) has SVEP, \( \sigma_w(f(d_{AB})) = f(\sigma_w(d_{AB})) \) for every \( f \in \mathcal{H}(\sigma(d_{AB})) \) [6, Corollary 2.6].

Hence, since \( \sigma(d_{AB}) \setminus \sigma_w(d_{AB}) = \pi_{00}(d_{AB}) \),

\[
\sigma(f(d_{AB}) \setminus \pi_{00}(f(d_{AB}))) = f(\sigma_w(d_{AB})) = \sigma_w(fd_{AB})),
\]
i.e., \( f(d_{AB}) \) satisfies Weyl’s theorem.

To prove parts (ii) and (iii) one argues as in the proof of Theorem 3.3. \( \square \)
References

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