THE TOPOLOGY OF $S^2$-FIBER BUNDLES

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Abstract. Let $P \rightarrow M$ be an oriented $S^2$-fiber bundle over a closed manifold $M$ and let $Q$ be its associated $SO(3)$-bundle, then we investigate the ring structure of the cohomology of the total space $P$ by constructing the coupling form $\tau_A$ induced from an $SO(3)$ connection $A$. We show that the cohomology ring of total space splits into those of the base space and the fiber space if and only if the Pontrjagin class $p_1(Q) \in H^4(M;\mathbb{Z})$ vanishes. We apply this result to the twistor spaces of 4-manifolds.

1. Introduction

(1.1) In this article, we are going to investigate the cohomology ring structure of the total space of an $S^2$-fiber bundle $P$ over a closed manifold $M$. Many of such examples can be constructed by the projectivization of rank 2 complex vector bundle $E$ over $M$, i.e., $\pi : P(E) \rightarrow M$. In this case, the cohomology ring $H^*(P(E);\mathbb{R})$ is already known by the Leray-Hirsch theorem as a free $H^*(M;\mathbb{R})$-module generated by 1, $c_1(\xi)$ with a relation such as $c_1^2(\xi) - \pi^* (c_1(E)) \cdot c_1(\xi) + \pi^* (c_2(E)) = 0$ where $\xi$ is the tautological line bundle over $P(E)$. This ring structure of the total space can be recovered by constructing a closed 2-form $\tau$ on $P$ which is called a coupling form. By the result of this paper, we can identify the cohomology class of the coupling 2-form $[\tau] = -c_1(\xi) + \frac{1}{2} c_1(\pi^*(E))$ for the case $P \cong P(E)$. Then we have $[\tau^2] = \frac{1}{4} \pi^*(c_1^2(E) - 4c_2(E)) \in \pi^*(H^4(M;\mathbb{R})) \subset H^4(P(E);\mathbb{R})$ which completely determines the cohomology ring structure of the total space $P(E)$ of the $S^2$-fiber bundle. In turn, we can conclude that $c_1^2(E) = 4c_2(E)$ if and only if the cohomology ring $H^*(P(E))$ splits. This kind of characterization of the
cohomology ring structure of $S^2$-fiber bundle $P$ over $M$ will be studied in terms of coupling form $\tau_A$ which is induced by a symplectic connection $A$ which comes from an $SO(3)$ connection. Let us start with some basic preliminaries about $S^2$-fibration.

(1.2) Suppose $P \xrightarrow{\pi} M$ is an $S^2$-fiber bundle over $M$ and the system of local coefficients on $M$ induced by the fiber is simple. Then there exists an exact sequence, so called Gysin short exact sequence, such as

$$0 \to H^k(M; \mathbb{R}) \xrightarrow{\pi^*} H^k(P; \mathbb{R}) \xrightarrow{\pi_*} H^{k-2}(M; \mathbb{R}) \to 0,$$

where $\pi_* = PD \circ \pi_# \circ PD^{-1}, \pi_# = H_{n-k}(P) \to H_{n-k}(M)$ is the cohomology map induced by $\pi$ and $PD$ is the Poincare dual map [1]. Moreover $\pi_*$ is called the map of integration along the fiber which it will be defined in the Section 3 via a given $SO(3)$-connection on $P$. Let $\tau \in H^2(P; \mathbb{R})$ be an element such that $\pi_*(\tau) = 1 \in H^0(M; \mathbb{R})$. It leads the splitting of the Gysin sequence by defining $s(\alpha) = \tau \cup \pi^*(\alpha) \in H^k(P; \mathbb{R})$ where $\alpha \in H^{k-2}(M; \mathbb{R})$. The splitting induced by the map $s$ is followed by the projection formula [1], i.e., $\pi_*(\tau \cup \pi^*(\alpha)) = \pi_*(\tau) \cup \alpha = \alpha$. Then it completely determine the linear structure of the cohomology of the total space $P$ as the tensor product of those of the base $M$ and the fiber $S^2$. It says that

$$H^*(P; \mathbb{R}) \cong H^*(M; \mathbb{R}) \otimes H^*(S^2; \mathbb{R}),$$

where the isomorphism is induced by the splitting map $s_\tau$ as above. With a given cohomology class $\tau$, the ring structure of $H^*(P; \mathbb{R})$ is determined by the square $\tau^2 \in H^4(P; \mathbb{R})$. Suppose we have $\tau^2 = \pi^*(\alpha) \cup \tau + \pi^*(\beta)$, by changing $\tau$ to $\tau - \frac{1}{2} \pi^*(\alpha)$, we may assume that the square of the cohomology class $\tau$ is the pull-back of some cohomology class $\beta$, i.e., $\tau^2 = \pi^*(\beta)$. We will show that the square of the $\tau$ is equal to the pull-back of $\frac{1}{2}p_1(Q) \in H^4(M; \mathbb{R})$ where $Q$ is the $SO(3)$-bundle over $M$ associated $P$. In the next subsection, it will be discussed the way of getting the principal $SO(3)$-bundle $Q$ from the $S^2$-fiber bundle $P$.

2. Reduction of structure group

(2.1) For a given oriented $S^2$-fiber bundle $\pi : P \to M$, the bundle $P$ admits the structure of symplectic fibration since the $\text{Diff}^+(S^2)/\text{Symp}(S^2, \omega_{S^2})$ can be identified with the space of the symplectic forms on $S^2$, which is the contractible space of positive volume form. Hence the
structure group $\text{Diff}^+(S^2)$ can be reduced to the group of symplectomorphisms, $\text{Symp}(S^2, \omega_{S^2})$. This reduction always holds for the case when the fiber $F$ is a compact Riemann surface [8]. Moreover since the group $\text{Diff}^{+}(S^2)$ deformation retract onto its linear part $\text{SO}(3)$ we can associate the principal $\text{SO}(3)$-bundle $Q$ such that $P \cong Q \times_{\text{SO}(3)} S^2$, where $\text{SO}(3)$ acts $S^2$ as the symplectomorphism of the standard symplectic form $\omega_{S^2}$. Note that linear subgroup $\text{SO}(3)$ is naturally isomorphic to the isometry group of the Kähler metric on $\mathbb{CP}^1 = S^2$. Suppose the dimension of the base space $M$ is less than or equal to 4 then the principal $\text{SO}(3)$-bundles $Q$ over $M$ are completely classified by the pair of characteristic classes $(\omega_2(Q), p_1(Q))$ such that $p_1(Q) \equiv \omega_2(Q)^2 \pmod{2}$ where $\omega_2(Q) \in H^2(M; \mathbb{Z}/2)$ is the 2nd Stiefel-Whitney class and $p_1 \in H^4(M; \mathbb{Z})$ is the first Pontrjagin class. This classification result is due to the theorem of Dold and Whitney[3]. The diffeomorphism class of the principal $\text{SO}(3)$-bundles over $M$ is unique up to homotopy class of maps from $M \to B\text{SO}(3)$ where $B\text{SO}(3)$ is the classifying space of $\text{SO}(3)$. We now discuss the extensions of the linear structure group $\text{SO}(3)$ to $\text{Spin}(3)$ or $\text{Spin}^c(3)$ structure. Recall that $\text{Spin}(3) \cong SU(2)$ and $\text{Spin}^c(3) \cong \text{Spin}(3) \times_{\mathbb{Z}_2} U(1) \cong U(2)$. The following results can be found in [5].

(2.1.1) $\text{Spin}(3) = SU(2)$ case. The obstruction for the extension of the structure group from $\text{SO}(3)$ to $SU(2)$ is completely determined by the second Stiefel-Whitney class $\omega_2(Q) \in H^2(M; \mathbb{Z}_2)$. It implies that the vanishing of $\omega_2(Q) \in H^2(M; \mathbb{Z}_2)$ gives an equivalent condition of the extension to $\text{Spin}(3)$. In this case $p_1(Q) = -4c_2(E) \in H^4(M; \mathbb{Z})$ where $E$ is the complex $SU(2)$-bundle associated the extension.

(2.1.2) $\text{Spin}^c(3) = U(2)$ case. The obstruction is that there is a complex line bundle $L$ whose first Chern class $c_1(L) \in H^2(M; \mathbb{Z})$ is the integral lift of $\omega_2(Q) \in H^2(M; \mathbb{Z}_2)$ i.e., $c_1(L) \equiv \omega_2(Q) \pmod{2}$. And we have $p_1(Q) = c_1^2(E) - 4c_2(E) \in H^4(M; \mathbb{Z})$ where $E$ is the complex vector bundle associated to the extension.

(2.2) Existence of section of $\pi : P \to M$. Since the action of $\text{SO}(3)$ on $S^2$ is transitive, we can view $S^2$ as a homogeneous space as $\text{SO}(3)/\text{SO}(2) = \text{SO}(3)/S^1$. Hence the existence of a section of $\pi : P \cong Q \times_{\text{SO}(3)} S^2 = Q \times_{\text{SO}(3)} \text{SO}(3)/S^1 \to M$ gives an existence condition such that there is an $S^1$ reduction of the principal $\text{SO}(3)$-bundle, i.e., $Q \cong \mathbb{Q}_{S^1} \times_{S^1} \text{SO}(3)$. It also gives an equivalent condition such that there exist a line bundle $L$ whose first Chern class $c_1(L)$ is an integral lift of $\omega_2(Q)$ and $c_1(L)^2 = p_1(Q)$. 

(2.2.1) Note that the condition for the existence of the section of an $S^2$-fiber bundle is exactly the same as that of existence of an almost complex structure on oriented 4-manifold by the Wu's theorem. This is just because an almost complex structure on 4-manifold can be realized as a section of the twistor space $\tau(X)$ which is an $S^2$-fiber bundle over $M[5]$.

(2.2.2) Note that even though we know all the characteristic classes $p_1(Q), \omega_2(Q)$ associated to the $SO(3)$-bundle $Q$, it does not determine all the homotopy classes of maps from $M$ to $BSO(3)$ for $\text{dim}M \geq 5$. However, the cohomology ring structure of the associated $S^2$-fiber bundle $P \cong Q \times_{SO(3)} S^2$ is completely determined by the characteristic classes of $SO(3)$-bundle $Q$, which will be discussed in the Section 3. Before getting into that, we need to discuss the Hamiltonian group action of $SO(3)$ on $S^2$ which induces an invariant positive definite pairing on the Lie algebra of $SO(3)$ in terms of the Hamiltonian functions.

3. Hamiltonian group action and semi-simple Lie algebra

(3.1) In this section, we discuss the Hamiltonian group action of a semi-simple Lie group on a symplectic manifold and the local isometry between its Lie algebra and the Hamiltonian functions induced by a moment map. Let us recall some basic facts from the Hamiltonian group action. Let $G$ be a compact Lie group with its Lie algebra $\mathfrak{g} = \text{Lie}(G)$ which acts covariantly on a symplectic manifold $(X, \omega)$ by symplectomorphisms. This implies that there is a group homomorphism $G \to \text{Symp}(X, \omega) : g \mapsto \psi_g$. The infinitesimal action determines a Lie algebra homomorphism $\mathfrak{g} \to \chi(X, \omega) : \xi \mapsto X_\xi$ defined by

$$X_\xi = \left. \frac{d}{dt} \right|_{t=0} \psi_{\exp(t\xi)}$$

for every $\xi \in \mathfrak{g}$.

Since $\psi_g$ is a symplectomorphism for every $g \in G$ it follows that each $X_\xi$ is a symplectic vector field. This means that the 1-form $\iota(X_\xi)\omega$ is closed for every $\xi$. Suppose the 1-form $\iota(X_\xi)\omega$ is exact, $dH_\xi = \iota(X_\xi)\omega$, for every $\xi \in \mathfrak{g}$, we call the action of $G$ on $X$ weakly Hamiltonian. Moreover the action is called Hamiltonian if the map

$$\mathfrak{g} \to C^\infty(X) : \xi \mapsto H_\xi$$

can be chosen to be a Lie algebra homomorphism with respect to be the Lie algebra structure on $\mathfrak{g}$ and Poisson structure on $C^\infty(X)$. Note that
in general, a weakly Hamiltonian action need not be Hamiltonian. The obstruction takes the form of a Lie algebra cocycle in $H^2(G; \mathbb{R})$. For details, see chapter 5 in [8]. However suppose $(X, \omega)$ is a compact symplectic manifold then there is a way of normalizing the Hamiltonian function so that $\int_X H \omega^n = 0$. Since $\int_X H_{[\xi,\eta]} \omega^n = 0$ and $\int_X \{H_\xi, H_\eta\} \omega^n = \int_X dH_\xi \wedge dH_\eta \wedge \omega^{n-1} = 0$, we have $H_{[\xi,\eta]} = \{H_\xi, H_\eta\}$. Hence with this normalization, we can show that every weakly Hamiltonian action is Hamiltonian. Assume that the action of $G$ on $X$ is Hamiltonian and $G$ is connected. Then it follows by straightforward calculation that

$$H_{g^{-1}g} = H_\xi \circ \psi_g$$

for $g \in G$ and $\xi \in G$.

(3.2) Consider a bilinear symplectic paring on the Lie Algebra $G$ with a Hamiltonian action of $G$ on a compact symplectic manifold $(X, \omega)$:

$$\langle \xi, \eta \rangle := \int_X H_\xi \cdot H_\eta \omega^n,$$

where $\omega^n = \omega \wedge \cdots \wedge \omega$. By the equation(1) and $\psi_g^* \omega = \omega$, we have

$$\langle (Adg)\xi, (Adg)\eta \rangle = \int_X H_{g^{-1}g}(x) \cdot H_{g^{-1}g}(x) \omega^n$$

$$= \int_X H_\xi(\psi_g(x)) \cdot H_\eta(\psi_g(x)) \psi_g^*(\omega^n)$$

$$= \int_X H_\xi \cdot H_\eta \omega^n$$

$$= \langle \xi, \eta \rangle.$$ 

Thus we can prove the following proposition.

**Proposition 3.2.1.** Let $G$ be a connected Lie group. Suppose the action of $G$ on a compact symplectic manifold, $(X, \omega)$ is Hamiltonian. Then the paring

$$\langle \xi, \eta \rangle := \int_X H_\xi \cdot H_\eta \omega^n$$

defines an adjoint invariant semi-positive definite form on the Lie algebra $G$.

Note that we have chosen the canonical orientation induced by the form $\omega^n \in \Omega^{2n}(X)$. To make a the form $\langle \cdot, \cdot \rangle$ being positive definite, it only needs to have $H_\xi \neq 0$ for all $0 \neq \xi \in G$. It leads to the following definition.
DEFINITION 3.2.2. The symplectic group action of $G$ on $(X, \omega)$ is effective if the induced Lie algebra homomorphism $\mathcal{G} \rightarrow \chi(X, \omega)$ is injective.

The definition of effectiveness is equivalent to that it has only discrete stabilizers in $G$. For instance, consider the Hamiltonian action of $G = U(n)$ on $(\mathbb{C}P^{n-1}, \tau_0)$ induced by the obvious action on $\mathbb{C}^n$ where $\tau_0$ the standard symplectic form on $\mathbb{C}P^{n-1}$. Then this action is not effective since the diagonal matrices $(e^{it}E)$ act trivially on $\mathbb{C}P^{n-1}$ where $E$ is the identity matrix. However if the action is restricted to the subgroup $SU(n) \subset U(n)$ then it becomes effective. In that case, one can compare the positive definite form $\langle \cdot, \cdot \rangle$ defined above and canonical inner product $\langle \xi, \eta \rangle = \text{trace}(\xi^* \eta)$, where $\xi^*$ denotes the conjugate transpose of $\xi$. By the uniqueness of the invariant definite form on the semisimple Lie algebra $su(n)$, one can compute the constant $c$ such that $\langle \cdot, \cdot \rangle = c \langle \cdot, \cdot \rangle$. For the sake of this exposition, we are going to compute this constant for the effective Hamiltonian action of $SU(2)$ on $\mathbb{C}P^1 \cong S^2$.

(3.3) The effective Hamiltonian action of $SU(2)$ on $(\mathbb{C}P^1 = S^2, \omega_{S^2})$. Let us define the symplectic form $\omega_{S^2}$ on $S^2$ as follows. Let $pr : \mathbb{C}^2 - 0 \rightarrow \mathbb{C}P^1$ denote the obvious projection and define $pr^* \omega_{S^2} = \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2$ where $\|z\|^2 = z_0 \bar{z}_0 + z_1 \bar{z}_1$. Then it is easily checked that $\omega_{S^2}$ is a well defined, $U(2)$ invariant symplectic 2-form and $\int_{S^2} \omega_{S^2} = 1$.

Now let $\{\omega = z_1/\bar{z}_0\}$ be the coordinates on the open set $U_0 \equiv \{(z_0 \neq 0)\}$ in $\mathbb{C}P^1$ and use the lifting $z = (1, \omega)$ on $U_0$; we have

$$\omega_{S^2} = \frac{i}{2\pi} \frac{d\omega_1 \wedge d\bar{\omega}_1}{(1 + |\omega_1|^2)^2}.$$ 

Let $\omega = \omega$ be the complex coordinate on $U_0 = (z_0 \neq 0)$ and let

$$\xi = \begin{pmatrix} -i \\ 0 \\ 0 \\ i \end{pmatrix} \in su(2).$$

Then in the polar coordinate system, $X_\xi = \frac{d}{dt} \big|_{t=0} e^{2it} \omega(= re^{i\theta}) = 2(\frac{d}{d\theta}) \omega$ where $\frac{d}{d\theta}$ is the angular tangent vector such that $d\theta(\frac{d}{d\theta}) = 1$ so we have $\omega = \frac{1}{\pi(1+r^2)^2} dr d\theta$ and $X_\xi \omega = -\frac{1}{\pi(1+r^2)^2} dr = d(-\frac{1}{\pi(1+r^2)})$. Thus we have $H_\xi = \frac{1}{\pi}(\frac{1}{1+r^2} - \frac{1}{2}) = -\frac{1}{2\pi} \frac{1-r^2}{1+r^2}$, here we take the normalization such that $\int_{S^2} H_\xi \omega_{S^2} = 0$. Then by the direct integration we have

$$\int_{S^2} H_\xi^2 \omega = \frac{1}{12\pi^2}.$$
Also \(\langle \xi, \xi \rangle = \text{trace}_{\xi} \xi = -\text{trace}_{\xi}^2 = 2\). Then we have \(\langle \xi, \xi \rangle = 24\pi^2 \ll \xi, \xi \gg\). By the invariance of the adjoint action of the both inner product, the constant is universal i.e., \(\langle \xi, \eta \rangle = 24\pi^2 \ll \xi, \eta \gg_{su(2)}\) for all \(\xi, \eta \in su(2)\). In particular, we have \(\text{Tr}(\xi^2) = -24\pi^2 \ll \xi, \xi \gg\) where \(\text{Tr}\) is the trace map.

(3.4) **Local Hamiltonian action of** \(SO(3)\) **on** \(\mathbb{C}P^1\). As we already know, there is a local isometry between \(su(2)\) and \(so(3)\) which is induced from the double cover \(SU(2) \to SO(3)\). Note that \(SU(2)\) is naturally identified with \(\text{Spin}(3) \cong S^3\). By using this local isometry, any \(\xi \in so(3)\) can be viewed as an element \(\frac{1}{2} \xi \in su(2)\). Let \(\xi \in so(3) = su(2)\) be a element of the Lie algebra of \(SO(3)\). Let \(\text{exp}_\xi \in SO(3)\) be a local curve in \(SO(3)\). Then \(\text{exp}_\xi\) becomes its local lifting to \(SU(2)\). Since the group action of \(SU(2)\) and \(SO(3)\) on \(S^2\) coincide for the lifting of \(g \in SO(3)\) to \(\tilde{g} \in SU(2)\), the symplectic vector field \(X_\xi\) induced by \(SO(3)\) action is the same as that of \(\frac{1}{2} X_\xi\) by the \(SU(2)\) action. Also the Hamiltonian function \(H_\xi\) from the \(SO(3)\) action is the half of that from \(SU(2)\). It follows that

\[
\langle \xi, \eta \rangle_{so(3)} = \frac{1}{4} \langle \xi, \eta \rangle_{su(2)}
\]

\[
= \frac{1}{24\pi^2} \ll \xi, \eta \gg_{su(2)}
\]

\[
= \frac{1}{24\pi^2} \ll \xi, \eta \gg_{so(3)}
\]

Hence it can be summarized as the following lemma.

**Lemma 3.4.1.** Under the assumption of the Hamiltonian action of \(SO(3)\) on \(S^2\), we have

\[
\text{Tr}(\xi \cdot \eta) = -24\pi^2 \int_{S^2} H_\xi \cdot H_\eta \omega_{S^2},
\]

where \(H_\xi\) is the normalized Hamiltonian function associated to \(\xi, \eta \in su(2)\).

4. The ring structure of \(H^*(P; \mathbb{R})\)

(4.1) **\(SO(3)\) connection and coupling 2-form.** As we discussed in Section 1, for every \(S^2\) fiber bundle \(P\) over \(M\) there is an \(SO(3)\) principal bundle \(Q\) over \(M\) such that \(P \cong Q \times_{SO(3)} S^2\) by the reduction of structure group to the linear subgroup \(SO(3)\) of \(\text{Symp}(S^2, \omega)\). Note that we take the symplectic form \(\omega\) to the canonical one defined as
above. Then each fiber $F_m$ of the symplectic fibration $\pi : P \to M$ carries a natural symplectic structure $\omega_m \in \Omega^2(F_m)$ defined by

$$\omega_m = \phi_\alpha(m)^* \omega$$

for some local trivialization $\phi_\alpha : \pi^{-1}(U_\alpha) \simeq U_\alpha \times S^2$ and $\phi_\alpha(m) = \phi_\alpha|_{F_m} : F_m \cong S^2$. Note that this form is independent of the choice of $\alpha$. We call 2-form $\tau \in \Omega^2(P)$ is compatible with the symplectic fibration $\pi : P \to M$ if the restriction of $\tau$ to each fiber $F_m$ is equal to $\omega_m$ defined as above. Note that the symplectic fibration $P$ is induced from the principal $SO(3)$-bundle $Q$, i.e., $P \cong Q \times_{SO(3)} S^2$ and $SO(3) \to \text{Symp}(S^2, \omega_{S^2})$ is a Hamiltonian action. We can apply the following theorem due to Weinstein [9].

**Theorem 4.1.1.** Let $G \to \text{Sypm}(F, \omega) : g \mapsto \psi_g$ be a Hamiltonian action. Then every connection $A$ on a principal $G$-bundle $Q \to M$ gives rise to a closed 2-form $\tau_A$ on the associated fibration $Q \times_G F \to M$ which restricts to the forms $\omega_m$ on the fibers.

Such a $\tau_A$ is called the coupling 2-form of the symplectic connection induced by the connection $A$. The above theorem is generalized by Guillemin-Lerman-Sternberg by constructing the coupling 2-form induced by the symplectic connection with a compact simply-connected fiber. This construction is extensively discussed in the book [4, 8]. Let us briefly explain how the construction goes. At each point $x \in P$ denote by $\text{Vert}_x = \ker d\pi(x) = T_x F_{\pi(x)}$ the vertical tangent space to the fiber. Let us define $\Gamma$ to be the connection on the fibration $\pi : P \to M$, which defines a field of horizontal subspace $\text{Hor}_x \subset T_x P$ such that $TP_x = \text{Vert}_x \oplus \text{Hor}_x$. This leads to a splitting of the tangent bundle of $P$, i.e., $TP = \text{Vert} \oplus \text{Hor}$. Then every path $\gamma : [0, 1] \to M$ determines a diffeomorphism $\Psi_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$. The diffeomorphism $\Psi_\gamma$ is called the holonomy of the path $\gamma$. The connection $\Gamma$ is called symplectic if the associated diffeomorphism $\Psi_\gamma$ preserves the symplectic structure in the fiber, i.e., $\Psi_\gamma^* \omega_{\gamma(1)} = \omega_{\gamma(0)}$ for every path $\gamma$. Let $\tilde{v} \in \text{Hor}_x$ be a horizontal vector then $\tau_\Delta = 0$ for all vertical vector $\omega \in \text{Vert}_x$. It now remains to define $\tau_T(\tilde{v}_1, \tilde{v}_2)$. It is defined as follows, let $v_1, v_2$ be two vector fields on $M$ then the vertical part of the communicator $[\tilde{v}_1, \tilde{v}_2]$ of the horizontal lifts $\tilde{v}_1, \tilde{v}_2$ respectively is a symplectic vector field on each fiber $F_{\pi(x)}$ and so, by the assumption of Hamiltonian action, is generated by a unique Hamiltonian function $H(\tilde{v}_1, \tilde{v}_2)$ of mean value zero, i.e., $\int_F H(\tilde{v}_1, \tilde{v}_2) \omega = 0$. We therefore define

$$\tau_T(\tilde{v}_1, \tilde{v}_2) = H_{[\tilde{v}_1, \tilde{v}_2]}^{\text{vert}}(x).$$
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The proof that $\tau_T$ is a well-defined closed 2-form reduces to some basic facts about connection, gauge transformation, and curvature on symplectic fibration [8].

Let $A$ be a connection on $\pi : Q \to M$ then it induces a connection $\Gamma_A$ on $\pi : P = Q \times_{SO(3)} S^2 \to M$. This connection becomes a symplectic connection because the holonomy is induced from the $SO(3) \subset \text{Symp}(S^2, \omega_{S^2})$. Let $\tau_A$ be the 2-form on $P = Q \times_{SO(3)} S^2$ associated with an $SO(3)$-connection $A$ on $Q$. In our case, we have

$$\tau_A(\tilde{v}_1(x), \tilde{v}_2(x)) = H_A[\tilde{v}_1, \tilde{v}_2](x),$$

where $\xi(m) = [q, A[\tilde{v}_1, \tilde{v}_2](q)] \in \Gamma(M, \text{ad}Q)$ and $H_\xi$ the normalized Hamiltonian function on each fiber $F_m = S^2_{\pi(x)}$. Here we denote $\xi(m) = [q, \xi_q] = [q \cdot g, g^{-1}\xi_g] \in \text{ad}Q$ and note that $R_g^*((A[\tilde{v}_1, \tilde{v}_2])(q)) = A[\tilde{v}_1 g, \tilde{v}_2 g] (q \cdot g) = g^{-1}A[\tilde{v}_1, \tilde{v}_2](q) g$.

It can be explained as follows. Let $x = [q, y] = [q \cdot g, g^{-1} \cdot y] \in P = Q \times_{SO(3)} S^2$ where we denote that $g^{-1} \cdot y = \psi_{g^{-1}}(y) \in S^2$ is the group action of $g \in SO(3)$ at $y \in S^2$. Let $\xi \in \Gamma(M, \text{ad}Q)$ be a section of the adjoint vector bundle, ad$Q$, associated by $Q$. Then it defines a vector field $X_\xi$ on $P$ which is vertical along the fiber as follows,

$$X_{\xi,x} = [q \cdot \xi, y] = \left[ \frac{d}{dt} \right]_{t=0} q \cdot \exp t \xi, y = \left[ q, \frac{d}{dt} \right]_{t=0} \exp t \xi \cdot y.$$ 

It follows from the equivariance of $\xi$, i.e., $\xi_{q \cdot g} = g^{-1} \xi_g g$, this vector field is well defined and independent of the representative $x = [q, y] = [q \cdot g, g^{-1} \cdot y]$. By definition, it defines a symplectic vector field $\xi$ which induces the unique Hamiltonian funtions $H_\xi$ on each fiber $F_m \cong S^2$ of mean value zero. For any pair of vector fields $v_1, v_2$ on $M$, the vertical part of the commutator of the horizontal lifts $[\tilde{v}_1, \tilde{v}_2]$ is exactly defined by $\xi_{v_1, v_2} = A[\tilde{v}_1, \tilde{v}_2] \in \Gamma(M, \text{ad}Q)$. Hence it defines the Hamiltonian function $H_{A[\tilde{v}_1, \tilde{v}_2]}(x) : P \to \mathbb{R}$. Moreover note that $A((\tilde{v}_1, \tilde{v}_2)) = F_A(\tilde{v}_1, \tilde{v}_2)$ where $F_A \in \Omega^2(M, \text{ad}Q)$ is the curvature tensor induced by $A$.

(4.2) The ring structure of $H^*(P; \mathbb{R})$ and the Pontrjagin class of $Q$. From Section 1.2, we have the Gysin sequence of an $S^2$-fiber bundle $P$ over $M$ as follows,

$$0 \to H^k(M; \mathbb{R}) \xrightarrow{\pi^*} H^k(P; \mathbb{R}) \xrightarrow{\pi_*} H^{k-2}(M; \mathbb{R}) \to 0,$$

where $\pi_*$ is the integration along the fiber.

We want to define this map $\pi_* : \Omega^k(P, \mathbb{R}) \to \Omega^{k-2}(M, \mathbb{R})$ as follows

$$\pi_*(\alpha)(v_1, v_2, \cdots, v_{k-2})(m) = \int_{F_m} \alpha(\tilde{v}_1, \cdots, \tilde{v}_{k-2})(x),$$
where $\tilde{v}_i$ is the horizontal lift of $v_i$ with respect to the connection $A$. Note that this map does not make any difference if $\tilde{v}_i$ has been chosen to be any lifting of $v_i$.

Then we have $\pi_*(\tau_A) = 1$ and by the projection formula we have $\pi_*(\pi^*(\beta) \wedge \tau_A) = \beta$ for all $\beta \in \Omega^k(M, \mathbb{R})$ and the commutativity of $\pi_*$ and $d$ follows from the direct local computation, i.e., we have $\pi_* d\alpha = d\pi_* \alpha$ [1]. We need to verify the following identities to show the main result Theorem 4.2.2 below of this paper.

**Lemma 4.2.1.** $\tau_A^2 = \frac{1}{4} \pi^* p_1(Q)$ where $p_1(Q)$ is the Pontrjagin class of $Q$.

**Proof.** We are going to establish the following identities to prove Lemma 4.2.1.

1. $\tau_A^2 = \pi^*(\beta) + da$.

   From the Gysin sequence, we have
   
   $$\tau_A^2 = \tau_A \wedge \tau_A = \pi^*(\beta) + \pi^*(\alpha) \wedge \tau_A + d\gamma$$

   and we have $\pi_*(\tau_A^2) = \alpha + d(\pi_*(\gamma))$. By the normalization condition, $\pi_*(\tau_A^2)(\tilde{v}_1, \tilde{v}_2) = 2 \int_F \tau_A(v_1, v_2)\tau_A = 0$, it implies that $\alpha = \pi^*(\pi_*(\gamma)) \wedge \tau_A + \gamma$.

2. $\pi_*(\tau_A^3)(v_1, \cdots, v_4) = 3 \int_F \tau_A^2(\tilde{v}_1, \cdots, \tilde{v}_4)\tau_A$.

   For sake of brevity, we denote
   
   $$\tau_A(\tilde{v}_i, \tilde{v}_j) = H_{ij}, \quad \tau_A(\tilde{v}_i, \cdot) = \tau_i,$$
   $$\tau_A^2(\tilde{v}_1, \cdots, \tilde{v}_4) = 2(H_{12}H_{34} - H_{13}H_{24} + H_{14}H_{23}),$$
   $$\tau_A^3(\tilde{v}_i, \tilde{v}_j, \cdot, \cdot) = 2(H_{ij}\tau_A = \tau_i \wedge \tau_j),$$
   $$\iota(\tilde{v}_4) \cdots \iota(\tilde{v}_1)(\tau_A^3) = \iota(\tilde{v}_4) \cdots \iota(\tilde{v}_1)\tau_A^2\tau_A$$

   $$+ \sum_{i > j > k > l} (-1)^{l-1}\iota(\tilde{v}_i)\iota(\tilde{v}_j)\iota(\tilde{v}_k)\tau_A^2 \wedge \iota(\tilde{v}_l)\tau_A$$

   $$+ \sum_{\sigma(1) < \sigma(2), \sigma(3) < \sigma(4)} \text{sign}(\sigma)\iota(\tilde{v}_{\sigma(2)})\iota(\tilde{v}_{\sigma(1)})\tau_A^2\iota(\tilde{v}_{\sigma(4)})\iota(\tilde{v}_{\sigma(3)})\tau_A$$

   $$= \tau_A(\tilde{v}_1, \cdots, \tilde{v}_4)\tau_A + 4(H_{12}H_{34} - H_{13}H_{24} + H_{14}H_{23})\tau_A$$

   $$+ \sum a_{ij}\tau_i \wedge \tau_j$$

   $$= 3\tau_A^2(\tilde{v}_1, \cdots, \tilde{v}_4)\tau_A + \sum a_{ij}\tau_i \wedge \tau_j.$$

   Since each term $\tau_i \wedge \tau_j$ in the last sum vanishes by the integration along the fiber, it completes the equation.

3. $[\beta] = \frac{1}{4} p_1(Q)$. 

From above, we have
\[
\tau_A^3 = \tau_A^2 \wedge \tau_A = \pi^*(\beta) \wedge \tau_A + d(a \wedge \tau_A),
\]
\[
\beta(v_1, \cdots, v_4) = \pi_*(\tau_A^3) - d(\pi_*(a \wedge \tau_A)),
\]
\[
\pi_*(\tau_A^3)(v_1, \cdots, v_4) = \int_F \tau_A^2(\bar{v}_1, \cdots, \bar{v}_4) \tau_A
\]
\[
= 3 \int_F 2(H_{12}H_{34} - H_{13}H_{24} + H_{14}H_{23}) \tau_A
\]
\[
= 3 \int_{S^2} 2(H_{F_{12}}F_{34} - H_{F_{13}}F_{24} + H_{F_{14}}F_{23})\omega_{S^2}
\]
\[
= \frac{1}{4\pi^2}(<F_{12}, F_{34}> - <F_{13}, F_{24}> + <F_{14}, F_{23}>)
\]
\[
= -\frac{1}{4\pi^2} \text{Tr}(F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23})
\]
\[
= -\frac{1}{8\pi^2} \text{Tr}((F_A^2)(v_1, \cdots, v_4)),
\]
where \(F_{i,j} = F_A(v_1, v_2) \in \text{so}(3) \cong \text{su}(2)\), and \(H_{F_{i,j}}\) is the normalized Hamiltonian function induced by \(F_{i,j} \in \text{so}(3)\). Hence we have \([\pi^*(\beta)] = [\tau_A^2] = \frac{1}{4}\pi^*p_1(Q)\).

**Theorem 4.2.2.** Let \(P\) be the \(S^2\)-fiber bundle over \(M\) then there is a closed two form \(\tau \in \Omega^2(P, \mathbb{R})\) such that its cohomology class defines the linear isomorphism \(H^*(P; \mathbb{R}) \cong H^*(M) \otimes H^*(S^2)\) and it also determine the ring structure \(H^*(P; \mathbb{R})\) such that \([\tau]^2 = \frac{1}{2}\pi^*p_1 \in \pi^*(H^4(M; \mathbb{R})) \subset H^4(P; \mathbb{R})\), where \(p_1 = p_1(Q)\) is the first Pontryagin class of the associated principal \(SO(3)\)-bundle \(Q\).

**Corollary 4.2.3.** \(H^*(P; \mathbb{R}) \cong H^*(M; \mathbb{R}) \otimes H^*(S^2; \mathbb{R})\) splits as a ring iff the Pontryagin class of the associated \(SO(3)\)-bundle \(Q\) vanishes i.e., \(p_1(Q) = 0 \in H^4(M; \mathbb{R})\). For the case \(P = P(E)\) is a projectivization of rank 2 vector bundle, \(p_1(Q) = p_1(\text{ad}E) = c_1(E)^2 - 4c_2(E) \in H^4(M; \mathbb{R})\).

**Proof.** Suppose the ring \(H^*(P; \mathbb{R})\) splits as a ring \(H^*(M; \mathbb{R}) \otimes H^*(S^2; \mathbb{R})\) then there is an element \(\tau \in H^2(P; \mathbb{R})\) such that \(\pi_*(\tau) = 1\) and \([\tau]^2 = 0\). Comparing the coupling 2-form \(\tau_A\) with \(\tau\), we know that \([\tau_A] - [\tau] = [\pi^*a]\) by the Gysin sequence. Therefore we have \([\tau_A]^2 = [\tau]^2 + 2[\tau][\pi^*a] + [\pi^*a]^2\) and \(0 = \pi_*[\tau_A^2] = 2[a] \in H^2(M; \mathbb{R})\), i.e., \(p_1(Q) = 4[\tau_A^2] = 0 \in H^4(M; \mathbb{R}) \subset H^4(P; \mathbb{R})\). This completes the proof. \(\square\)
Note that the splitting condition of $P(E)$, the projectivization of rank 2 vector bundle $E$, is achieved if $E$ is projectively flat which implies that $\text{End}(E)$ is flat.

**Lemma 4.2.3.** We can prove that the cohomology class of the coupling two-form $[\tau_A]$ does not depend on any symplectic connection $\Gamma$ on $P$ by the same argument given in the proof of the Corollary 4.2.2, i.e., $[\tau_A] = [\tau_\Gamma] \in H^2(P; \mathbb{R})$. See [4].

In the $S^2$-fiber bundle case, the cohomology class of the coupling form is characterized uniquely as an element $\tau \in H^2(P; \mathbb{R})$ such that $\pi_* \tau = 1$ and $\tau^2 \in \pi^*(H^4(M; \mathbb{R}))$.

## 5. Twistor space of 4-manifold

**5.1** In this section, we are going to study the twistor space $\tau(M)$ of oriented 4-manifold $M$ which is an $S^2$-fiber bundle over $M$. The twistor space $\tau(M)$ is naturally induced by the projectivization of positive pure spinors on $M$ which is isomorphic to the space of orthogonal almost complex structures on $M$. Suppose the dimension of $M$ is 4, nonzero positive spinor defines a pure spinor in turn, the twistor space $\tau(M)$ is canonically isomorphic to the projectivization of positive spinor bundle, i.e., $\tau(M) \cong P(S_C^+)$ where $S_C^+$ is the positive spinor bundle which is rank 2 complex vector bundle. Thus the twistor-space $\tau(M)$ is an $S^2$-fiber bundle over $M$ canonically associated to the Riemannian manifold $M$. Topological characterization of the existence of the positive spinor bundle is whether there exists an integral lifts of the second Stiefel-Whitney class of $M$, $\omega_2(M)$ which indicates the Spin$^c$ structure of given manifold $M$. It is well known that there exists as Spin$^c$-structure on any oriented smooth 4-manifold. For more discussion on the twistor space and Spin$^c$ structures, see [5]. Note that the twistor space is well-defined independent of Spin$^c$ structure, $S_C^+$. As we discussed before, there is an associated principal $SO(3)$-bundle $Q_{\tau(M)}$ which isomorphic to the adjoint bundle of the unitary bundle $S_C^+$. We will show that $p_1(Q) = 3\sigma(M) + 2\chi(M)$ where $\chi(M)$ is the Euler characteristic of $M$ and $\sigma(M)$ is the signature of $M$.

**Lemma 5.1.1.** Let $S^+$ be a positive spinor bundle of almost complex 4-manifold $M$ then we have

$$c_2(S^+) = \frac{1}{4}(c_1(S^+)^2 - 3\sigma - 2\chi),$$
where $\sigma$ is the signature of $M$ and $\chi$ is the Euler characteristic of $M$.

Before we prove the lemma, it might be good place to recall the basic facts about the spinor bundle over 4-manifold. This material can be found in [5] and [6]. Let $\omega_2(M) \in H^2(M; \mathbb{Z}/2)$ be the second Stiefel-Whitney class and then the $\text{Spin}^c$ structures are naturally isomorphic to the cohomology class of the integral lift of $\omega_2(M)$ which is naturally isomorphic to the set of characteristic line bundles $\{L \mid \text{complex line bundle } c_1(L) \equiv \omega_2(M)\}$. It is a principal $H^2(M; \mathbb{Z})$ space since the difference between two characteristic line bundles contained in $2H^2(M; \mathbb{Z})$. We will abuse the notation for complex line bundle $L$ as $l = c_1(L) \in H^2(M; \mathbb{Z})$, vice versa. This sums up to as follows:

$$\{\text{Spin}^c \text{ structures over } M\} \cong \{L_0 + 2l \mid l \in H^2(M; \mathbb{Z})\}$$

where $L_0$ is a characteristic line bundle. Suppose $S^+$ be the positive spinor bundle then the determinantal line bundle $\det S^+ = L$ defines the $\text{Spin}^c$ structure. We denote $S^+ = S^+(L)$ for $L = \det S^+$. For any other spinor bundle, it can be written as tensor product of some line bundle $l$, i.e., $S^+(L) = S_0^+ \otimes l$ where $l = (L \otimes L_0)^{1/2}$. It induces $c_2(S^+(L)) = c_2(S_0^+) + c_1(S_0^+) \cdot c_1(l) + c_1^2(l)$ and $c_1(S^+(L))^2 = c_1^2(S_0^+) + 4c_1(S_0^+) \cdot c_1(l) + 4c_1^2(l)$ which proves the lemma. It suffices to show that there exists a positive spinor bundle $S_0^+$ such that $c_2(S_0^+) = \frac{1}{4}(c_1^2(S_0^+)) - 3\sigma - 2\chi$.

**Proof of Lemma 5.1.1.** Suppose $M$ has an almost complex structure then the $J : TM \to TM$ defines a canonical $\text{Spin}^c$ structure and the induced positive spinor bundle is isomorphic to $S^+_J \cong II \otimes K^{-1}_J$ with $K^{-1}_J = \det TM_J$ for $TM_J$ being the complex tangent bundle induced by $J$ and $II$ being trivial line bundle ([6]). We have $c_2(S^+_J) = 0$ and $c_2^2(K^{-1}_J) = 2\chi + 3\sigma$ by the Hirzebruch signature theorem.

Note that the canonical negative spinor bundle $S^-_J$ induced by an almost complex structure $J$ is canonically isomorphic to complex tangent bundle $TM_J$. Since $c_2(S^-_J) = c_2(M) = \chi(M)$ and $c_1^2(S^-_J) = c_1^2(M) = 2\chi(M) + 3\sigma(M)$, we have $c_2(S^-_J) = \frac{1}{4}(c_1^2(S^-_J)) - 3\sigma(M) - 2\chi(M))$. Let $Q_{S^-_C}$ be the principal $SO(3)$-bundle associated to the the negative spinor bundle $S^-_C$. Then we have $p_1(Q_{S^-_C}) = 3\sigma(M) - 2\chi(M)$.

**Corollary 5.1.2.** The rational cohomology ring of $P(S^\pm_C)$, the projection of the positive (resp. negative) spinor bundle, splits if and only if $3\sigma(M) = \mp 2\chi(M)$ respectively.

**Example 5.1.3.** We know that the complex projective space $\mathbb{C}P^3$ becomes the twistor space over $S^4$. Identity $\mathbb{C}^4$ with quaternionic plane
$\mathbb{H}^2$ then the obvious fibration $\pi \mathbb{C}P^3 \to \mathbb{H}P^1$ induces the $S^2$-fiber bundle structure. Canonically we can identify $\mathbb{H}P^1 \cong S^4$ and $\mathbb{C}P^3 = \tau(S^4)$. For details, see [5], [8]. In this case, the cohomology class of the form induces by the Fubini-Study metric, $\omega \in H^2(\mathbb{C}P^3; \mathbb{R})$ defines the same class of the coupling 2-form since $\int_{S^2} \omega = \deg_{\omega} S^2 = 1$ where $S^2$ is the fiber class.

Note that the above example explains that the square of the cohomology class of coupling form is the generator of $\mathbb{H}^4(\mathbb{C}P^3; \mathbb{Z})$ which equals to $\pi^*(\text{positive generator}) = \frac{1}{4} \pi^* p_1(Q) = \frac{1}{4} (3\sigma(S^4) + 2\chi(S^4)) = 1 \in \mathbb{Z} \cong \mathbb{H}^4(M; \mathbb{Z})$.

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