IDEALS AND SUBMODULES OF MULTIPLICATION MODULES

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ABSTRACT. Let $R$ be a commutative ring with identity and let $M$ be an $R$-module. Then $M$ is called a multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. Let $M$ be a non-zero multiplication $R$-module. Then we prove the following:

1. there exists a bijection: $N(M) \cap V(\text{ann}_R(M)) \to \text{Spec}_R(M)$
and in particular, there exists a bijection:

$$N(M) \cap \text{Max}(R) \to \text{Max}_R(M),$$

2. $N(M) \cap V(\text{ann}_R(M)) = \text{Supp}(M) \cap V(\text{ann}_R(M))$,
3. for every ideal $I$ of $R$,

$$((\sqrt{I + \text{ann}_R(M)})M : R M) = \cap_{P \subseteq N(M) \cap V(\text{ann}_R(M))} P.$$

The ideal $\theta(M) = \sum_{m \in M} (Rm : R M)$ of $R$ has proved useful in studying multiplication modules. We generalize this ideal to prove the following result: Let $R$ be a commutative ring with identity, $P \in \text{Spec}(R)$, and $M$ a non-zero $R$-module satisfying

1. $M$ is a finitely generated multiplication module,
2. $PM$ is a multiplication module, and
3. $P^n M \neq P^{n+1} M$ for every positive integer $n$,

then $\cap_{n=1}^{\infty} (P^n + \text{ann}_R(M)) \in V(\text{ann}_R(M)) = \text{Supp}(M) \subseteq N(M)$.

1. Introduction

Throughout this paper, we consider only commutative rings with identity and modules which are unitary. Let $R$ be a commutative ring

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and $M$ an $R$-module. Then $\text{Spec}(R)$ denotes the set of all prime ideals of $R$ and $\text{Spec}_R(M)$ denotes the set of all prime submodules of $M$. Obviously, $\text{Spec}_R(R) = \text{Spec}(R)$. If $N$ is a submodule of $M$, then $(N :_R M)$ is defined by \{ $r \in R \mid rM \subseteq N$ \}. In particular, $(0 :_R M)$ is called the annihilator of $M$ and is denoted by $\text{ann}_R(M)$. There are three subsets of $\text{Spec}(R)$ which depend on $M$:

1. $N(M) = \{ P \in \text{Spec}(R) \mid PM \neq M \}$,
2. $V(\text{ann}_R(M)) = \{ P \in \text{Spec}(R) \mid \text{ann}_R(M) \subseteq P \}$,
3. $\text{Supp}(M) = \{ P \in \text{Spec}(R) \mid M_P \neq 0 \}$.

$\text{Max}(R)$ denotes the set of all maximal ideals of $R$ and $\text{Max}_R(M)$ denotes the set of all maximal submodules of $M$. Clearly, $\text{Max}_R(R) = \text{Max}(R)$.

By a quasi-local ring, we mean a commutative ring with a unique maximal ideal.

Let $R$ be a commutative ring and let $M$ be an $R$-module. Then a submodule $N$ of $M$ is said to be extended if $N = IM$ for some ideal $I$ of $R$. $M$ is called a multiplication module if every submodule of $M$ is extended. For example, every proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is a multiplication module but the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ itself is not. We generalize [8, Theorem 6] as follows. If $M$ is a non-zero multiplication module then there exists a bijection $N(M) \cap V(\text{ann}_R(M)) \rightarrow \text{Spec}_R(M)$ and in particular, there exists a bijection $N(M) \cap \text{Max}(R) \rightarrow \text{Max}_R(M)$.

In commutative ring theory, it is well-known that, for every non-zero finitely generated module over a commutative ring $R$,

$$\emptyset \neq V(\text{ann}_R(M)) = \text{Supp}(M).$$

In Section 2, we prove that if $M$ is a non-zero multiplication module over a commutative ring $R$, then $N(M) \cap V(\text{ann}_R(M)) = \text{Supp}(M) \cap V(\text{ann}_R(M))$.

In Section 3, we are concerned with relationships between the ideals of a commutative ring and the submodules of a multiplication module over the ring. A well-known result of commutative algebra saying that the radical of an ideal $I$ of a commutative ring is the intersection of all prime ideals containing $I$ is generalized to non-zero multiplication modules. Let $R$ be a commutative ring and $M$ an $R$-module. For an ideal $I$ of $R$, we define the ideal $\theta(IM) = \sum_{x \in I,M}(Rx :_R M)$ of $R$. This is a generalization of the ideal $\theta(M)$ of $R$ which was introduced in [1] and recently, the ideal $\theta(M)$ was studied in [3]. Let $R$ be a commutative ring with identity and let $P \in \text{Spec}(R)$. If $M$ is a non-zero $R$-module satisfying

1. $M$ is a finitely generated multiplication module,
(2) $PM$ is a multiplication module, and
(3) $P^n M \neq P^{n+1} M$ for every positive integer $n$,
then we prove by making use of the notion of the ideal \( \theta(M) \) of \( R \) that

\[
\bigcap_{n=1}^{\infty} (P^n + \text{ann}_R(M)) \in V(\text{ann}_R(M)) = \text{Supp}(M) \subseteq N(M).
\]

Let \( R \) be a quasi-local ring with unique maximal ideal \( P \). Let \( M \) be a non-zero \( R \)-module satisfying

(1) \( M \) is a finitely generated multiplication module,
(2) \( PM \) is a multiplication module, and
(3) \( P^n M \neq P^{n+1} M \) for every positive integer \( n \).

Then we prove that \( R/\text{ann}_R(M) \) is a discrete valuation domain. Finally, in particular, it is found under what conditions a Noetherian local ring is a discrete valuation domain.

Our first lemma gives three well-known results that will be used throughout this paper.

**Lemma 1.1.** Let \( R \) be a commutative ring and \( M \) an \( R \)-module.

(1) If \( M \) is a multiplication \( R \)-module, then it is locally cyclic.
(2) If \( M \) is a multiplication \( R \)-module, then

\[
\bigcap_{I \in \mathcal{I}} (IM) = \left( \bigcap_{I \in \mathcal{I}} (I + \text{ann}_R(M)) \right) M
\]

for any non-empty collection \( \mathcal{I} \) of ideals of \( R \).
(3) Let \( M \) be a non-zero multiplication \( R \)-module. Then
(i) for every proper submodule \( N \) of \( M \), there exists \( K \in \text{Max}_R(M) \) of \( M \) such that \( N \subseteq K \), and
(ii) \( K \in \text{Max}_R(M) \) if and only if there exists \( P \in \text{Max}(R) \) such that \( K = PM \neq M \).

**Proof.** (1) Let \( M \) be a multiplication \( R \)-module. Let \( P \) be any element of \( \text{Spec}(R) \). Then \( M_P \) is a multiplication \( R_P \)-module by [2, Corollary 3.5]. Since over a quasi-local ring every multiplication module is cyclic, \( M_P \) is cyclic. (2) follows from [5, Corollary 1.7]. (3) follows from [5, Theorem 2.5]. \( \square \)
2. Prime spectra of multiplication modules

If \( M \) is a module over a commutative ring \( R \), then for every submodule \( N \) of \( M \), \( (N :_R M) = \text{ann}_R(M / N) \). The following lemma was motivated by definitions in [5, p.765] and [6, p.791].

**Lemma 2.1.** Let \( M \) be a non-zero \( R \)-module and let \( N \) be a submodule of \( M \) with \( N \neq M \). Then the following statements are equivalent:

1. \( (N :_R K) = (N :_R M) \) for every submodule \( K \) of \( M \) such that \( K \supseteq N \).
2. If \( ax \in N \), where \( a \in R \) and \( x \in M \), then \( a \in (N :_R M) \) or \( x \in N \).

**Proof.** Assume (1). Assume \( ax \in N \), where \( a \in R \) and \( x \in M \). Assume \( x \notin N \). Then \( N \not\subseteq N + Rx \subseteq M \). By (1), \( (N :_R (N + Rx)) = (N :_R M) \). Since \( ax \in N \), we have \( a(N + Rx) = aN + Rax \subseteq N \). This shows that \( a \in (N :_R (N + Rx)) \). Hence, \( a \in (N :_R M) \).

Conversely, assume (2). Let \( K \) be any submodule of \( M \) such that \( K \supseteq N \). Then \( K / N \subseteq M / N \) and so,

\[
(N :_R K) = \text{ann}_R(K / N) \supseteq \text{ann}_R(M / N) = (N :_R M)
\]

Let \( a \) be any element of \( (N :_R K) \). Since \( N \not\subseteq K \), we can find an element \( x \) of \( K \setminus N \). Then \( ax \in N \). Hence, by (2), \( a \in (N :_R M) \).

Let \( R \) be a commutative ring and let \( M \) be a non-zero \( R \)-module. Let \( N \) be a submodule of \( M \). Then \( N \) is called a prime submodule of \( M \) if

1. \( N \neq M \) and
2. \( N \) satisfies either (hence both) of the statements in Lemma 2.1.

Let \( R \) be a commutative ring and \( M \) an \( R \)-module. Then a submodule \( N \) of \( M \) is called an extended submodule if there exists an ideal \( I \) of \( R \) such that \( N = IM \). \( M \) is called a multiplication module if every submodule of \( M \) is extended.

**Example 2.2.** Consider the ring \( \mathbb{Z} \) of integers. Let \( p \) be a fixed prime number. If we adapt the proof of the well-known fact that \( \mathbb{Z}(p^{\infty}) \) is divisible, then we can get the following:

1. the only proper, extended submodule of the \( \mathbb{Z} \)-module \( \mathbb{Z}(p^{\infty}) \) is 0, and
2. every proper submodule of the \( \mathbb{Z} \)-module \( \mathbb{Z}(p^{\infty}) \) is a multiplication module but the \( \mathbb{Z} \)-module \( \mathbb{Z}(p^{\infty}) \) itself is not.
Every finite-dimensional vector space with dimension greater than 1 cannot be a multiplication module. \(\square\)

Compare the next result with [5, Corollary 1.7].

**Proposition 2.3.** Let \(R\) be a commutative ring and \(M\) an \(R\)-module. Then \(M\) is a multiplication module if and only if \(\bigcap_{A \in \mathcal{A}} A = (\bigcap_{A \in \mathcal{A}} (A :_R M))M\) for any non-empty collection \(\mathcal{A}\) of submodules of \(M\).

**Proof.** Assume that \(M\) is a multiplication module. Let \(\mathcal{A}\) be any non-empty collection of submodules of \(M\). Then

\[
\bigcap_{A \in \mathcal{A}} A = (\bigcap_{A \in \mathcal{A}} A :_R M)M = (\bigcap_{A \in \mathcal{A}} (A :_R M))M
\]

with the first equality following since \(M\) is a multiplication module and the second since residuation distributes over intersection.

Conversely, assume that \(\bigcap_{A \in \mathcal{A}} A = (\bigcap_{A \in \mathcal{A}} (A :_R M))M\) for any non-empty collection \(\mathcal{A}\) of submodules of \(M\). Let \(N\) be any submodule of \(M\). Then \(\{N\}\) is a non-empty collection of a submodule of \(M\). By our assumption, \(N = (N :_R M)M\). Hence, \(M\) is a multiplication module. \(\square\)

Let \(R\) be a ring. If \(M\) is a non-zero \(R\)-module, then \(\text{ann}_R(M) \neq R\). By Zorn’s Lemma, \(V(\text{ann}_R(M)) \neq \emptyset\).

**Lemma 2.4.** Let \(R\) be a commutative ring. Let \(M\) be a non-zero multiplication module. Then

1. \((PM :_R M) = \begin{cases} \text{P + ann}_R(M) & \text{if } P \in N(M) \\ R & \text{if } P \notin N(M) \end{cases}\)
2. \(PM\) is an element of \(\text{Spec}_R(M)\) if \(P \in N(M)\).

**Proof.** (1) Clearly, \(P + \text{ann}_R(M) \subseteq (PM :_R M)\). Conversely, let \(a\) be any element of \((PM :_R M)\). Then \(aM \subseteq PM\). Assume that \(P \in N(M)\). Then we can take an element \(x \in M \setminus PM\). Hence, \(ax \in PM\).

\(M\) can be given \(R/\text{ann}_R(M)\)-module structure as follows: for any \(r \in R\) and \(m \in M\), define \((r + \text{ann}_R(M))m = rm\). Then the module structure is well-defined. \(M\) becomes an \(R/\text{ann}_R(M)\)-module. Moreover, as an \(R/\text{ann}_R(M)\)-module, \(M\) is a multiplication module. Since \(ax \in PM\), we have \((a + \text{ann}_R(M))x = (P/\text{ann}_R(M))M\). Further, since \(x \notin PM\), we have \(x \notin (P/\text{ann}_R(M))M\). By [5, Lemma 2.10], we have \(a + \text{ann}_R(M) \in P/\text{ann}_R(M)\). This implies \(a \in P + \text{ann}_R(M)\). Thus, \((PM :_R M) \subseteq P + \text{ann}_R(M)\). Therefore, \((PM :_R M) = P + \text{ann}_R(M)\).
Assume now that \( PM = M \). Then \((PM :_RM) = (M :_RM) = R\).

(2) Let \( ax \in PM \), where \( a \in R \) and \( x \in M \). Then as in the proof of (1), we can show that either \( a \in P + \text{ann}_R(M) \) or \( x \in PM \). If \( a \in P + \text{ann}_R(M) \), then \( a \in (PM :_RM) \). Thus, either \( a \in (PM :_RM) \) or \( x \in PM \). Hence, \( PM \) is a prime submodule of \( M \) if \( PM \neq M \). \( \square \)

The following result generalizes [8, Theorem 6. (c) \( \Rightarrow \) (d)] and [7, p.216, Property 1].

**Theorem 2.5.** Let \( R \) be a commutative ring. Let \( M \) be a non-zero multiplication module. Then there is a one-to-one order-preserving correspondence: \( N(M) \cap V(\text{ann}_R(M)) \rightarrow \text{Spec}_R(M) \)

**Proof.** Let \( \mathcal{X} = N(M) \cap V(\text{ann}_R(M)) \) and let \( \mathcal{Y} = \text{Spec}_R(M) \). Define a map \( \varphi : \mathcal{X} \rightarrow \mathcal{Y} \) by \( \varphi(P) = PM \), where \( P \in \mathcal{X} \). Then by Lemma 2.4(2), \( \varphi \) is well-defined. Now, define a map \( \psi : \mathcal{Y} \rightarrow \mathcal{X} \) by \( \psi(N) = (N :_RM) \), where \( N \in \mathcal{Y} \). Let \( N \) be any prime submodule of \( M \). Then \( \text{ann}_R(M/N) \) is a prime ideal of \( R \) and \( \text{ann}_R(M) \subseteq \text{ann}_R(M/N) \) by definitions and hence \((N :_RM)\) is a prime ideal of \( R \) containing \( \text{ann}_R(M) \). Further, since \( M \) is a multiplication module, we have \((N :_RM)M = N \neq M \). Hence, \( \psi \) is well-defined.

Let \( P \) be any element of \( \mathcal{X} \). Then by Lemma 2.4(1),

\[
(\psi \circ \varphi)(P) = \psi(\varphi(P)) = \psi(PM) = (PM :_RM) = P.
\]

Hence, \( \psi \circ \varphi = 1_\mathcal{X} \). Thus, \( \varphi \) is one-to-one.

Let \( N \) be any element of \( \mathcal{Y} \). Then since \( M \) is a multiplication module,

\[
(\varphi \circ \psi)(N) = \varphi(\psi(N)) = \varphi(N :_RM) = (N :_RM)M = N
\]

Hence, \( \varphi \circ \psi = 1_\mathcal{Y} \). Thus, \( \varphi \) is onto. Therefore, \( \varphi \) is a one-to-one correspondence between \( \mathcal{X} \) and \( \mathcal{Y} \). Moreover, it is clear that \( \varphi \) is order-preserving. \( \square \)

If \( M \) is a non-zero multiplication module over a commutative ring \( R \), then it follows from Theorem 2.5 that every prime submodule of \( M \) is of the form \( PM \), where \( P \in N(M) \cap V(\text{ann}_R(M)) \).

**Lemma 2.6.** Let \( R \) be a commutative ring and \( M \) a non-zero module. Then \( N(M) \cap \text{Max}(R) \subseteq V(\text{ann}_R(M)) \).
Proof. Assume that \( P \) is a maximal ideal of \( R \) such that \( PM \neq M \). Suppose \( \text{ann}_R(M) \not\subset P \). Then \( P + \text{ann}_R(M) = R \). Hence,
\[
M = RM = (P + \text{ann}_R(M))M \subseteq PM + (\text{ann}_R(M))M = PM,
\]
and so \( M = PM \). This contradiction shows that \( \text{ann}_R(M) \subseteq P \). \( \square \)

Let \( R \) be a commutative ring and let \( M \) be a non-zero multiplication module. Then by Lemma 1.1 or [8, Theorem 2 (4)], \( \text{Max}_R(M) \neq \emptyset \). Compare the following result with [8, Theorem 2 (1)].

**Corollary 2.7.** Let \( R \) be a commutative ring and \( M \) a non-zero multiplication module. Then there is a one-to-one order-preserving correspondence:
\[
N(M) \cap \text{Max}(R) \to \text{Max}_R(M).
\]

**Proof.** Let \( \mathcal{X} = N(M) \cap V(\text{ann}_R(M)) \) and let \( \mathcal{Y} = \text{Spec}_R(M) \). Define a map \( \varphi : \mathcal{X} \to \mathcal{Y} \) by \( \varphi(P) = PM \), where \( P \in \mathcal{X} \). Then by the proof of Theorem 2.5, \( \varphi \) is a one-to-one correspondence. Let \( \mathcal{X}' = N(M) \cap \text{Max}(R) \) and let \( \mathcal{Y}' = \text{Max}_R(M) \). Since every maximal ideal of \( R \) is prime, it follows from Lemma 2.6 that \( \mathcal{X}' \subseteq \mathcal{X} \). We can now consider the restriction of \( \varphi \) to \( \mathcal{X}' \) \( \varphi|_{\mathcal{X}'} : \mathcal{X}' \to \mathcal{Y} \). Then since \( \varphi \) is one-to-one, so is \( \varphi|_{\mathcal{X}'} \).

Let \( P \) be a maximal ideal of \( R \) such that \( M \neq PM \). Then by Lemma 1.1, there is a maximal submodule \( K \) of \( M \) such that \( PM \subseteq K \). Hence, \( P \subseteq PM :_R M \subseteq K :_R M \neq R \) and so \( P = K :_R M \). Thus, \( K = (K :_R M)M = PM \). This shows that \( PM \) is a maximal submodule of \( M \). Therefore, in particular, \( \text{Im}(\varphi|_{\mathcal{X}'}) \subseteq \mathcal{Y}' \). Further, it follows from Lemma 1.1 that \( \mathcal{Y}' \subseteq \text{Im}(\varphi|_{\mathcal{X}'}) \). Hence, \( \text{Im}(\varphi|_{\mathcal{X}'}) = \mathcal{Y}' \). Thus, \( \varphi|_{\mathcal{X}'} : \mathcal{X}' \to \mathcal{Y}' \) is a one-to-one correspondence. Moreover, it is clear that \( \varphi|_{\mathcal{X}'} \) is order-preserving. \( \square \)

If \( M \) is a non-zero multiplication module over a commutative ring \( R \), then it follows from Corollary 2.7 that every maximal submodule of \( M \) is of the form \( PM \) where \( P \in N(M) \cap \text{Max}(R) \).

### 3. Multiplication modules

Let \( I \) be an ideal of a commutative ring \( R \). Recall from [6, p.792] that an \( R \)-module \( M \) is said to be \( I \)-torsion if for each \( m \in M \) there exists an element \( i \in I \) such that \( (1 - i)m = 0 \). Let \( I \) be an ideal of \( R \) and \( M \) a finitely generated \( R \)-module. Then it follows from standard determinant argument that \( M \) is \( I \)-torsion if and only if \( M = IM \).
LEMMA 3.1. Let $I$ be an ideal of $R$ and $M$ a multiplication $R$-module. Then $M$ is $I$-torsion if and only if $M = IM$.

Proof. Adapt the proof of [10, p.229, Lemma 6] to show this. \qed

Let $P$ be a maximal ideal of a commutative ring $R$. Recall [10, p.223] that an $R$-module $M$ is said to be $P$-cyclic if there exists an element $x \in M$ and an element $p \in P$ such that $(1 - p)M \subseteq Rx$.

DEFINITION 3.2. Let $I$ be an ideal of a commutative ring $R$. An $R$-module $M$ is said to be $I$-cyclic if there exists a maximal ideal $P$ of $R$ containing $I$ such that $M$ is $P$-cyclic.

Every $R$-module is $R$-torsion but no $R$-module is $R$-cyclic.

Let $P$ be a maximal ideal of a commutative ring $R$. Let $M$ be an $R$-module. Then we remark that $M$ is $P$-cyclic when we regard $P$ as an ideal if and only if it is $P$-cyclic when we regard $P$ as a maximal ideal.

PROPOSITION 3.3. Let $R$ be a commutative ring and $M$ an $R$-module. Then the following statements are equivalent.

1. For every proper ideal $I$ of $R$, $M$ is $I$-cyclic.
2. For every maximal ideal $P$ of $R$, $M$ is $P$-cyclic.

Proof. Assume (1). Let $P$ be any maximal ideal of $R$. Then $P$ is a proper ideal of $R$. By (1), there exists a maximal ideal $Q$ of $R$ with $Q \supseteq P$ such that $M$ is $Q$-cyclic. Since $P$ is maximal, we must have $Q = P$. Hence, $M$ is $P$-cyclic.

Assume (2). Let $I$ be any proper ideal of $R$. There exists a maximal ideal $P$ of $R$ such that $P \supseteq I$. By (2), $M$ is $P$-cyclic. Thus, $M$ is $I$-cyclic. \qed

THEOREM 3.4. Let $R$ be a commutative ring and let $M$ be a non-zero $R$-module. Then the following statements are equivalent.

1. $M$ is a multiplication module.
2. For every ideal $I$ of $R$ either $M$ is $I$-torsion or $M$ is $I$-cyclic.
3. For every maximal ideal $P$ of $R$ either $M$ is $P$-torsion or $M$ is $P$-cyclic.

Proof. Assume (1). Let $I$ be any ideal of $R$. Then $M = IM$ or $M \neq IM$.

Assume that $M = IM$. Then by Lemma 3.1, $M$ is $I$-torsion.

Assume now that $M \neq IM$. Then by Lemma 1.1, there is a maximal submodule $K$ of $M$ such that $IM \subseteq K$. Further, by Lemma 1.1, there is
a maximal ideal $P$ of $R$ such that $K = PM$. Since $PM \neq M$, it follows from Lemma 2.6 that $ann_R(M) \subseteq P$. Hence, by Lemma 2.4, $(PM :_R M) = P$. Thus, $I \subseteq (IM :_R M) \subseteq (K :_R M) = (PM :_R M) = P$. Since $PM \not\subseteq M$, we can take an element $x \in M \setminus PM$. By (1), there exists an ideal $J$ of $R$ such that $Rx = JM$. If $J$ were a subset of $P$, then $x$ would be an element of $PM$ since $x \in Rx = JM \subseteq PM$. Hence, $J \not\subseteq P$. Since $P$ is maximal, we have $P + J = R$. There exists an element $p \in P$ such that $1 - p \in J$. Further, $(1 - p)M \subseteq JM = Rx$. Hence, $M$ is $P$-cyclic. This shows that $M$ is $I$-cyclic. Therefore, (2) follows.

It follows from the remark just prior to Proposition 3.3 that (2) implies (3).

Finally, it follows from [5, Theorem 1.2] that (3) implies (1). \qed

**Theorem 3.5.** Let $R$ be a commutative ring and $M$ a non-zero multiplication $R$-module. Then

1. $Supp(M) \subseteq N(M)$.
2. $N(M) \cap V(ann_R(M)) = Supp(M) \cap V(ann_R(M))$.

**Proof.** (1) There are two ways to prove this.

Method I. Use Lemma 3.1 to show this.

Method II. Assume that $P$ is a prime ideal of $R$ and $M$ is a non-zero multiplication module with $M = PM$. By Lemma 1.1, $M_P$ is cyclic. Further, $M_P = PR_PM_P$. By Nakayama's Lemma, $M_P = 0$.

(2) By (1), it suffices to prove $N(M) \cap V(ann_R(M)) \subseteq Supp(M) \cap V(ann_R(M))$.

Assume that $P \in N(M) \cap V(ann_R(M))$. By Lemma 3.1, $M$ is not $P$-torsion. By Theorem 3.4, $M$ is $P$-cyclic. Hence, there exists an element $x \in M$ and an element $p \in P$ such that $(1 - p)M \subseteq Rx$. Then $x/1$ is a non-zero element of $M_P$. For, otherwise there exists an element $s \in R \setminus P$ such that $sx = 0$; hence

$$s(1 - p)M \subseteq s(Rx) = (sR)x = (Rs)x = R(sx) = 0$$

and so $s(1 - p) \in ann_R(M) \subseteq P$, a contradiction. Therefore, $M_P \neq 0$. \qed

### 4. Ideals and submodules of multiplication modules.

In this section we will be concerned with relationships between the ideals of a commutative ring and the submodules of a non-zero multiplication module over the commutative ring.
PROPOSITION 4.1. Let $R$ be a commutative ring and $M$ a non-zero multiplication module. Then the following statements hold.

(1) For every ideal $I$ of $R$ with $M \neq IM$, there exists a maximal ideal $P$ of $R$ containing $I + \text{ann}_R(M)$ such that $PM$ is a maximal submodule of $M$.

(2) If $P$ is a prime ideal of $R$ containing $\text{ann}_R(M)$ such that $M \neq PM$, then $P + J = R$ for every ideal $J$ of $R$ with $M = JM$.

(3) For every ideal $I$ of $R$ with $M \neq IM$ and for every ideal $J$ of $R$ with $M = JM$, there exists a maximal ideal $P$ of $R$ containing $I + \text{ann}_R(M)$ such that $P + J = R$ and $PM$ is a maximal submodule of $M$.

Proof. (1) Let $I$ be any ideal of $R$ with $M \neq IM$. Then by Lemma 1.1, there is a maximal submodule $K$ of $M$ such that $IM \subseteq K$. Further, by Lemma 1.1, there is a maximal ideal $P$ of $R$ such that $K = PM$. Since $PM \neq M$, it follows Lemma 2.6 that $\text{ann}_R(M) \subseteq P$. Suppose that $I \nsubseteq P$. Then $I + P = R$. Since $IM \subseteq K = PM$, it then follows that

$$M = RM = (I + P)M \subseteq IM + PM = PM.$$ 

Hence, $M = PM$. This contradiction shows that $I \subseteq P$. Thus, $I + \text{ann}_R(M) \subseteq P$.

(2) Let $P$ be any prime ideal of $R$ containing $\text{ann}_R(M)$ such that $M \neq PM$. Let $J$ be any ideal of $R$ with $M = JM$. Then there exists an element $x \in M \setminus PM$. Further, since $M$ is a multiplication module and $M = JM$, it follows from Lemma 3.1 that $M$ is $J$-torsion. Hence, there exists an element $j \in J$ such that $(1 - j)x = 0$. Further, $(1 - j)x = 0 \in PM$. By Lemma 2.4(2), $PM$ is a prime submodule of $M$. Hence, $1 - j \in P$. Therefore, $P + J = R$.

(3) follows from (1) and (2). \qed

Given an ideal $I$ of a commutative ring $R$, the radical of $I$, denoted by $\sqrt{I}$, is defined by $\{r \in R \mid r^n \in I$ for some positive integer $n\}$. It is well-known that if $I$ is an ideal of a commutative ring $R$, then $\sqrt{I} = \bigcap_{P \in \mathcal{V}(I)} P$. We will generalize this.

THEOREM 4.2. Let $R$ be a commutative ring. Let $M$ be a non-zero multiplication module. Then for every ideal $I$ of $R$,

$$\left(\left(\sqrt{I + \text{ann}_R(M)}\right)M\right):_RM = \bigcap_{P \in \mathcal{V}(I + \text{ann}_R(M)) \cap N(M)} P.$$
Proof. Let $I$ be any ideal of $R$. Assume that $IM = M$. Then

$$R = (M :_R M) = (IM :_R M) \subseteq ((\sqrt{I + \text{ann}_R(M)})M :_R M).$$

Hence, $$((\sqrt{I + \text{ann}_R(M)})M :_R M) = R.$$ Let $A = V(I + \text{ann}_R(M)) \cap N(M)$. Then $A = \emptyset$. For, otherwise there exists a prime ideal $P$ of $R$ containing $I + \text{ann}_R(M) \subseteq P$ and $PM \neq M$. Then

$$M = IM = (I + \text{ann}_R(M))M \subseteq PM \not\subseteq M,$$

a contradiction. Hence, $\bigcap_{p \in A} P = R$. Therefore,

$$\left(\left(\sqrt{I + \text{ann}_R(M)}\right)M :_R M\right) = \bigcap_{p \in A} P.$$

Now, assume $IM \neq M$. Then $I + \text{ann}_R(M) \neq R$. There exists a prime ideal $Q$ of $R$ such that $I + \text{ann}_R(M) \subseteq Q$. Let $P = V(I + \text{ann}_R(M))$. Then $Q \in P$. In particular, $P \neq \emptyset$. Then it is easy to show that

$$\left(\left(\bigcap_{p \in P} (PM) :_R M\right) = \bigcap_{p \in P} (PM :_R M)\right).$$

By Proposition 4.1(1), $A \neq \emptyset$. Let $B = V(I + \text{ann}_R(M)) \cap (\text{Spec}(R) \setminus N(M))$. Then $P = A \cup B$. Hence, by Lemma 1.1 and Lemma 2.4(1), we have

$$\left(\left(\sqrt{I + \text{ann}_R(M)}\right)M :_R M\right)$$

$$= \left(\left(\bigcap_{p \in P} P\right)M :_R M\right)$$

$$= \left(\left(\bigcap_{p \in P} (PM) :_R M\right)$$

$$= \bigcap_{p \in P} (PM :_R M)$$

$$= \bigcap_{p \in A} (PM :_R M) \cap \bigcap_{p \in B} (PM :_R M)$$

$$= \bigcap_{p \in A} P. \quad \square$$
COROLLARY 4.3. If $M$ is a non-zero faithfully flat multiplication module over a commutative ring $R$, then for every ideal $I$ of $R$,

$\left( (\sqrt{I + \text{ann}_R(M)}M) :_RM \right) = \sqrt{I + \text{ann}_R(M)}$.

Proof. Let $I$ be any ideal of $R$. Then with the same notations as in the proof of Theorem 4.2,

$\sqrt{I + \text{ann}_R(M)} = \bigcap_{P \in \mathcal{P}} P = \left( \bigcap_{P \in \mathcal{A}} P \right) \bigcap \left( \bigcap_{P \in \mathcal{B}} P \right)$.

If $M$ is faithfully flat, it follows from [9, Theorem 7.2] that $\mathcal{B} = \emptyset$. Hence, by Theorem 4.2,

$\sqrt{I + \text{ann}_R(M)} = \bigcap_{P \in \mathcal{A}} P = \left( (\sqrt{I + \text{ann}_R(M)}M :_RM \right)$.

For any ideal $I$ of $R$, let $I^0M = M$ and $I^\infty M = \bigcap_{n=1}^\infty (I^nM)$. [6, p.791, Lemma 3.1 (ii)] can be recast as follows.

LEMMA 4.4. Let $R$ be a commutative ring and $P$ an ideal of $R$. Let $M$ be an $R$-module such that $PM$ is a multiplication module. Then for any submodule $N$ of $PM$, either $N \subseteq P^\infty M$ or there exists a positive integer $k$ and $k$ ideals $I_0, I_1, \cdots, I_{k-1}$ of $R$ with $I_0 \not\subseteq P, I_1 \not\subseteq P^2, \cdots, I_{k-1} \not\subseteq P^k$ such that

$N = I_0P^kM = I_1P^{k-1}M = \cdots = I_{k-1}PM$.

Proof. Assume that $N$ is a submodule of $PM$ such that $N \not\subseteq P^\infty M$. Then there exists a positive integer $k$ such that $N \subseteq P^kM$ but $N \not\subseteq P^{k+1}M$. Since for each $i \in \{0, 1, \cdots, k-1\}$, $N \subseteq P^kM \subseteq P^{k-i}M$ and by [6, Lemma 3.1(i)] $P^{k-i}M$ is a multiplication module, we have, for each $i \in \{0, 1, \cdots, k-1\}$, $N = (N :_R P^{k-i}M)P^{k-i}M$. Further, $(N :_R P^{k-i}M) \supseteq \text{ann}_R(P^{k-i}M)$ implies $(N :_R P^{k-i}M) + \text{ann}_R(P^{k-i}M) = N :_R P^{k-i}M$. Hence, it follows from Lemma 1.1 and the modular law.
that for each \( i \in \{0, 1, \cdots, k - 1\}, \)

\[
N = N \cap P^k M \\
= N \cap (P^i P^{k-i} M) \\
= ((N :_R P^{k-i} M) P^{k-i} M) \cap (P^i P^{k-i} M) \\
= (((N :_R P^{k-i} M) + \text{ann}_R(P^{k-i} M)) \\
\cap (P^i + \text{ann}_R(P^{k-i} M))) P^{k-i} M \\
= ((N :_R P^{k-i} M) \cap (P^i + \text{ann}_R(P^{k-i} M))) P^{k-i} M \\
= ((N :_R P^{k-i} M) \cap P^i) P^{k-i} M \\
= ((N :_R P^{k-i} M) \cap P^i) P^{k-i} M
\]

Now, for each \( i \in \{0, 1, \cdots, k - 1\}, \) let \( I_i = (N :_R P^{k-i} M) \cap P^i. \) Then

\[
N = I_0 P^k M = I_1 P^{k-1} M = \cdots = I_{k-1} P M.
\]

Further, since \( N \not\subseteq P^{k+1} M, \) we get \( I_0 \not\subseteq P, I_1 \not\subseteq P^2, \cdots, I_{k-1} \not\subseteq P^k, \) as required.

Let \( R \) be a commutative ring and \( M \) an \( R \)-module. The ideal \( \theta(M) = \sum_{m \in M}(Rm :_R M) \) of \( R \) has proved useful in studying multiplication modules. We generalize this ideal as follows: \( \theta(IM) = \sum_{x \in IM}(Rx :_R M) \) for an ideal of a commutative ring \( R \) and an \( R \)-module \( M. \) It is always true that \( I\theta(M) \subseteq \theta(IM) \) for every ideal \( I \) of a commutative ring \( R \) and for every module \( M \) over the ring \( R. \) If \( M \) is a multiplication module over a commutative ring \( R, \) then for every ideal \( I \) of \( R, \)

\[
IM = \sum_{x \in IM} Rx \\
= \sum_{x \in IM} ((Rx :_R M) M) \\
= \left( \sum_{x \in IM} (Rx :_R M) \right) M \\
= \theta(IM) M
\]

and \( IM = (IM :_R M) M. \) Hence, we have the following result.
Lemma 4.5. Let $R$ be a commutative ring and $M$ a multiplication $R$-module. Then the following conditions are equivalent:

1. $M$ is finitely generated, and
2. for every ideal $I$ of $R$, $\theta(IM) = (IM :_RM) = I + \text{ann}_R(M)$.

Proof. $(1) \Rightarrow (2)$ follows from [10, Theorem 9 Corollary].

$(2) \Rightarrow (1)$. $(2)$ gives $\theta(M) = R$. Hence, it follows from [3, Corollary 2.2] that $M$ is finitely generated. \hfill \Box

Theorem 4.6. Let $R$ be a commutative ring and let $P$ be a maximal ideal of $R$. Let $M$ be a non-zero $R$-module satisfying

1. $M$ is a finitely generated multiplication module,
2. $PM$ is a multiplication module, and
3. $P^nM \neq P^{n+1}M$ for every positive integer $n$.

Then $\bigcap_{n=1}^{\infty} (P^n + \text{ann}_R(M)) \in V(\text{ann}_R(M)) = \text{Supp}(M) \subseteq \text{N}(M)$.

Proof. By [6, Corollary 3.2], $P^\infty M$ is a prime submodule of $M$. By the statement just prior to Lemma 2.6, there exists a prime ideal $Q$ of $R$ containing $\text{ann}_R(M)$ with $QM \neq M$ such that $P^\infty M = QM$. It suffices to prove that $Q = \bigcap_{n=1}^{\infty} (P^n + \text{ann}_R(M))$.

By Lemma 1.1, we have

$$QM = P^\infty M = \bigcap_{n=1}^{\infty} (P^nM) = \left(\bigcap_{n=1}^{\infty} (P^n + \text{ann}_R(M))\right)M.$$

Hence, by Lemma 4.5, we have

$$Q = \theta(QM) = \theta \left(\left(\bigcap_{n=1}^{\infty} (P^n + \text{ann}_R(M))\right)M\right)$$

$$= \bigcap_{n=1}^{\infty} (P^n + \text{ann}_R(M)),$$

as required. \hfill \Box

Note that intersection of powers of multiplication ideals are considered in [4, Theorem 2.2]. [4, Theorem 4.1] says: Let $(R, P)$ be a quasi-local ring whose maximal ideal $P$ is finitely generated. Then $R$ is Noetherian if and only if for every finitely generated ideal $I$ of $R$, $\bigcap_{n=1}^{\infty} (P^n + I) = I$. Therefore, by Theorem 4.6, we have the following result.
COROLLARY 4.7. Let $R$ be a Noetherian local ring with unique maximal ideal $P$. Let $M$ be a non-zero $R$-module satisfying

1. $M$ is a multiplication module,
2. $PM$ is a multiplication module, and
3. $P^nM \neq P^{n+1}M$ for every positive integer $n$.

Then $R/\text{ann}_R(M)$ is a discrete valuation domain.

Proof. Over a quasi-local ring a multiplication module is cyclic. So $M = R/\text{ann}_R(M)$. Now $PM = P/\text{ann}_R(M)$ is principal so $R/\text{ann}_R(M)$ is a PIR. Then (3) gives that $R/\text{ann}_R(M)$ is a DVR. Further, by Theorem 4.6, $R/\text{ann}_R(M)$ is an integral domain. □

Notice that if a module over a commutative ring satisfies the assumptions of Corollary 4.7, then it is Noetherian module but not Artinian.

COROLLARY 4.8. Let $R$ be a Noetherian local ring with unique maximal ideal $P$ satisfying

1. $P$ is a multiplication ideal of $R$ and
2. $P^n \neq P^{n+1}$ for every positive integer $n$.

Then $R$ is a discrete valuation domain.

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