ON THE INCREMENTS OF A
$\textit{d}$-DIMENSIONAL GAUSSIAN PROCESS

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Abstract. In this paper we establish some results on the increments of a $d$-dimensional Gaussian process with the usual Euclidean norm. In particular we obtain the law of iterated logarithm and the Book-Shore type theorem for the increments of a $d$-dimensional Gaussian process, via estimating upper bounds and lower bounds of large deviation probabilities on the suprema of the $d$-dimensional Gaussian process.

1. Introduction and results

Limit theory on the increments of some kinds of stochastic processes, such as the Wiener process, fractional Brownian motions, multifractional Brownian motions, Ornstein-Ulhenbeck processes, $l^2$-valued Ornstein-Ulhenbeck processes, $l^p$-valued Gaussian processes, $l^{\infty}$-valued Gaussian processes and related processes, has been investigated in various directions by many authors, for instance, Choi[5], Choi and Kôno[6], Csáki et al.[7, 8], Csörgő and Révész[10], Kôno[14], Csörgő, Lin and Shao[9], Csörgő and Shao[11], Lin[15], Lin and Lu[17], Lin and Qin[18], Lu[19], Shao[23], Zhang[24, 25].

Furthermore, the law of iterated logarithm and the Book-Shore type theorem [4] for the increments of Gaussian processes have been studied by Arcones[1], He and Chen[13], Monrad and Rootzen[20], Révész[22] and Zhang[24, 25].

In the paper we obtain some results on the increments of a $d$-dimensional Gaussian process with the usual Euclidean norm $\| \cdot \|$. First of
all we establish the law of iterated logarithm and the Book-Shore type theorem for the increments of a $d$-dimensional Gaussian process, via estimating upper bounds and lower bounds of large deviation probabilities on the suprema of the Gaussian process.

Let $\{X_i(t), 0 \leq t < \infty\}, i = 1, \ldots, d$, be real-valued continuous and centered independent Gaussian processes with $X_i(0) = 0$ and $E\{X_i(t) - X_i(s)\} = \sigma_i^2(|t - s|)$, where $\sigma_i(t)$ are positive nondecreasing continuous and regularly varying functions of $t > 0$ with exponents $\alpha_i(0 < \alpha_i < 1)$ at 0 and $\infty$. Hence $\sigma_i(x)/x$ is non-increasing for large $x$. Let $\{X^d(t) = (X_1(t), \ldots, X_d(t)), 0 \leq t < \infty\}$ be a $d$-dimensional Gaussian process with the norm $\| \cdot \|$. For $0 < T < \infty$, let $a_T$ be a positive continuous function of $T$ with $0 \leq a_T \leq T$. Denote

$$\beta(a_T, T) = \left\{ 2(\log(T/a_T) + \log \log T) \right\}^{1/2},$$
$$\sigma(d, t) = \max_{1 \leq i \leq d} \sigma_i(t),$$

where $\log x = \ln(\max\{x, 1\})$.

The following Theorem 1.1 was proved by Lin et. al.[16].

**THEOREM 1.1.** We have

$$\limsup_{T \to \infty} \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a_T} \frac{\|X^d(t + s) - X^d(t)\|}{\sigma(d, a_T) \beta(a_T, T)} \leq 1 \text{ a.s.}$$

Our main results are as follows:

**THEOREM 1.2.** Assume that $\sigma_i^2(x)$ are twice differentiable for $x > 0$ which satisfies

(i) both $a_T$ and $T/a_T$ are nondecreasing;

(ii) $\left| \frac{d\sigma_i^2(x)}{dx} \right| \leq c_1 \frac{\sigma_i^2(x)}{x}$ and $\left| \frac{d^2\sigma_i^2(x)}{dx^2} \right| \leq c_2 \frac{\sigma_i^2(x)}{x^2}$, $i = 1, \ldots, d$

for positive constants $c_1$ and $c_2$. Then we have

$$\limsup_{T \to \infty} \frac{\|X^d(T) - X^d(T - a_T)\|}{\sigma(d, a_T) \beta(a_T, T)} \geq 1 \text{ a.s.}$$

**THEOREM 1.3.** Suppose that

(iii) $\lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} = r, \quad 0 \leq r \leq \infty.$

Then

$$\liminf_{T \to \infty} \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a_T} \frac{\|X^d(t + s) - X^d(t)\|}{\sigma(d, a_T) \beta(a_T, T)} \leq \left( \frac{r}{1 + r} \right)^{1/2} \text{ a.s.}$$
THEOREM 1.4. Assume that \( a_T \) satisfies condition (iii) and there exists a positive constant \( c_2 \) such that
\[
\left| \frac{d^2 \sigma_i^2(x)}{dx^2} \right| \leq c_2 \frac{\sigma_i^2(x)}{x^2}, \quad i = 1, \ldots, d.
\]
Then we have
\[
\liminf_{T \to \infty} \sup_{0 \leq t \leq T} \frac{\|X^d(t + a_T) - X^d(t)\|}{\sigma(d, a_T) \beta(a_T, T)} \geq \left( \frac{r}{1 + r} \right)^{1/2} \text{ a.s.}
\]

Combining Theorems 1.1 and 1.2, we obtain the following limsup value:

COROLLARY 1.1. Under the assumptions of Theorem 1.2, we have
\[
\limsup_{T \to \infty} \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a_T} \frac{\|X^d(t + s) - X^d(t)\|}{\sigma(d, a_T) \beta(a_T, T)}
= \limsup_{T \to \infty} \frac{\|X^d(T) - X^d(T - a_T)\|}{\sigma(d, a_T) \beta(a_T, T)} = 1 \text{ a.s.}
\]

If, furthermore, \( a_T = T \), then we have the law of iterated logarithm for a \( d \)-dimensional Gaussian process:
\[
\limsup_{T \to \infty} \frac{\|X^d(T)\|}{\sigma(d, T) \sqrt{2 \log \log T}} = 1 \text{ a.s.}
\]

From Theorems 1.3 and 1.4, we obtain the following liminf value:

COROLLARY 1.2. Under the assumptions of Theorem 1.4, we have
\[
\liminf_{T \to \infty} \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a_T} \frac{\|X^d(t + s) - X^d(t)\|}{\sigma(d, a_T) \beta(a_T, T)}
= \liminf_{T \to \infty} \frac{\|X^d(t + a_T) - X^d(t)\|}{\sigma(d, a_T) \beta(a_T, T)} = \left( \frac{r}{1 + r} \right)^{1/2} \text{ a.s.}
\]

EXAMPLE. Let \( \sigma_i^2(t) = t^{2\alpha_i} \), \( 0 < \alpha_i < 1 \) for \( i = 1, \ldots, d \), then \( X_i(t) \) is a fractional Brownian motion. Hence \( X^d(t) = (X_1(t), \ldots, X_d(t)) \) is \( d \)-dimensional fractional Brownian motion, obviously, which satisfies condition (ii).
2. Proofs

We shall accomplish the proofs of our theorems through several lemmas. The following lemmas 2.1~2.4 are essential for the proof of Theorem 1.2. Lemma 2.1 is a well-known extension of the second Borel-Cantelli lemma:

**Lemma 2.1.** Let \( \{A_k, k \geq 1\} \) be a sequence of events. If

(a) \( \sum_{k=1}^{\infty} P(A_k) = \infty \),

(b) \( \liminf_{n \to \infty} \sum_{1 \leq j < k \leq n} \frac{P(A_j \cap A_k) - P(A_j)P(A_k)}{\left( \sum_{j=1}^{n} P(A_j) \right)^2} \leq 0 \),

then \( P(A_n, \ i.o.) = 1 \).

The proof of Lemma 2.1 can be found, for example, in Theorem 6.4 in Billingsley[3]. The following Lemma 2.2 is a generalization of Slepian's lemma (cf. e.g. Berman[2]).

**Lemma 2.2.** Let \( \{X_j, j = 1, 2, \cdots, n\} \) be centered stationary Gaussian random variables with \( EX_iX_j = r_{ij} \) and \( r_{ii} = 1 \). Let \( I^+_c = [c, \infty) \), and \( I^-_c = (-\infty, c) \). Denote by \( F_j \) the event \( \{X_j \in I^+_c\} \) for \( c_j \in (-\infty, \infty), j = 1, 2, \cdots, n \), where \( c_j \) is either +1 or -1. Let \( K \subset \{1, 2, \cdots, n\} \) and \( \{K_l, l = 1, 2, \cdots, s\} \) is a partition of \( K \), then

\[
\left| P\left( \bigcap_{j \in K} F_j \right) - \prod_{l=1}^{s} P\left( \bigcap_{j \in K_l} F_j \right) \right| \leq \sum_{1 \leq l < m \leq s} \sum_{i \in K_l} \sum_{j \in K_m} |r_{ij}| \phi(c_i, c_j; r_{ij}^*)
\]

where \( \phi(x, y; r) \) is the standard bivariate normal density with correlation \( r \) and \( r_{ij}^* \) is a number between 0 and \( r_{ij} \).

**Lemma 2.3.** Assume that a function \( \sigma^2(x) \) satisfies that \( |d^2\sigma^2(x)/dx^2| \leq c_2\sigma^2(x)/x^2 \) for some \( c_2 > 0 \) and \( \sigma^2(x)/x^2 \) is non-increasing. Let \( P, Q \) and \( R \) be positive real numbers. Then

\[
\left| \int_{R}^{R+P} d(\sigma^2(x)) - \int_{Q-P}^{Q} d(\sigma^2(x)) \right| \leq c_2 \frac{\sigma^2(Q-P)}{(Q-P)^2} P(R+P-Q).
\]
Proof. We have
\[
\left| \int_{R}^{R+P} d(\sigma^2(x)) - \int_{Q}^{Q+P} d(\sigma^2(x)) \right|
\leq \int_{Q-P}^{Q} \left| \frac{d}{dx} (\sigma^2(x - Q + R + P) - \sigma^2(x)) \right| dx
\leq \int_{Q-P}^{Q} \int_{x}^{x-Q+R+P} \left| \frac{d^2}{dy^2} \sigma^2(y) \right| dy dx
\leq \int_{Q-P}^{Q} \int_{x}^{x-Q+R+P} c_2 \frac{\sigma^2(y)}{y^2} dy dx
\leq c_2 \frac{\sigma^2(Q-P)}{(Q-P)^2} P(R+R-P).
\]
\]
\[\square\]

Lemma 2.4. Assume that for \(i = 1, \ldots, d\), \(\sigma^2_i(x)\) is differentiable for \(x > 0\) and satisfies \(\left| \frac{d\sigma^2_i(x)}{dx} \right| \leq c_1 \frac{\sigma^2_i(x)}{x}\) for a positive constant \(c_1\). Then we have
\[
\limsup_{T \to \infty} \frac{\|X^d(T)\|}{\sigma(d,T)\sqrt{2\log\log T}} \geq 1 \text{ a.s.}
\]

Proof. Take \(i_0 = i_0(T)\) such that \(\sigma_{i_0}(T) = \sigma(d,T)\). Clearly,
\[
\|X^d(T)\|/\sigma(d,T) \geq |X_{i_0}(T)|/\sigma_{i_0}(T).
\]

Using the inequality
\[
\frac{1}{\sqrt{2\pi}} \left( \frac{1}{x} - \frac{1}{x^3} \right) e^{-x^2/2} < 1 - \Phi(x) < \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2},
\]
where \(\Phi(x) = P\{N(0,1) \leq x\}\), we have
\[
P \left\{ \frac{\|X^d(T)\|}{\sigma(d,T)} > \sqrt{2(1-\varepsilon)\log\log T} \right\} \geq \frac{1}{2\sqrt{2\pi}} \frac{\exp \left( -(1-\varepsilon)\log\log T \right)}{\sqrt{2(1-\varepsilon)\log\log T}}
\)
\[
\geq c_3 \left( \log T \right)^{-(1-\varepsilon/2)}
\]

for large \(T\), where \(c_3 > 0\) is a constant. Let \(T_k = \theta^k\) with \(\theta > 1\). Then
\[
\sum_{k=1}^{\infty} P \left\{ \frac{\|X^d(T)\|}{\sigma(d,T)} > \sqrt{2(1-\varepsilon)\log\log T} \right\} \geq c_3 \sum_{k=1}^{\infty} \left( \log T_k \right)^{-(1-\varepsilon/2)}
\]
\[
= \infty.
\]
Hence, in order to prove Lemma 2.4, we need to show that (b) of Lemma 2.1 holds. For \( j < k \), if \( i_0(T_j) \neq i_0(T_k) \), \( E\{X_{i_0(T_j)}(T_j)X_{i_0(T_k)}(T_k)\} = 0 \). If \( i_0(T_j) = i_0(T_k) =: i_0 \), noting that \( \sigma_{i_0}^2(x)/x^2 \) is non-increasing for large \( x \), we have

\[
E\{X_{i_0}(T_j)X_{i_0}(T_k)\} = \frac{1}{2} \left( \sigma_{i_0}^2(T_j) + \sigma_{i_0}^2(T_k) - \sigma_{i_0}^2(T_k - T_j) \right)
\leq \frac{1}{2} \left( \sigma_{i_0}^2(T_j) + \left( 1 - \frac{(T_k - T_j)^2}{T_k^2} \right) \sigma_{i_0}^2(T_k) \right)
\leq \frac{1}{2} \left( \sigma_{i_0}^2(T_j) + \frac{2T_j}{T_k} \sigma_{i_0}^2(T_k) \right)
\]

and

\[
\left| E\left\{ \frac{X_{i_0}(T_j)X_{i_0}(T_k)}{\sigma_{i_0}(T_j)\sigma_{i_0}(T_k)} \right\} \right|
\leq \frac{1}{2} \frac{\sigma_{i_0}(T_j)}{\sigma_{i_0}(T_k)} \frac{T_j}{T_k} \frac{\sigma_{i_0}(T_k)}{\sigma_{i_0}(T_j)}
\leq \frac{1}{2} \left( \frac{T_j}{T_k} \right)^{\alpha_0} \frac{L_{i_0}(T_j)}{L_{i_0}(T_k)} + \left( \frac{T_j}{T_k} \right)^{1-\alpha_0} \frac{L_{i_0}(T_k)}{L_{i_0}(T_j)}
\leq \frac{1}{2} \theta^{-\alpha_0(k-j)} + \theta^{-(1-\alpha_0')(k-j)} =: \eta_{jk}
\]

for large \( j \), where \( L_{i_0}(\cdot) \) is a slowly varying function and \( 0 < \alpha_0 < \min\{\alpha_i, i = 1, \ldots, d\} \), \( \max\{\alpha_i, i = 1, \ldots, d\} < \alpha_0' < 1 \), and the following fact on a slowly varying function \( L(x) \) at the infinite has been used: for any \( \varepsilon > 0 \)

\[
\left( \frac{x}{y} \right)^\varepsilon \frac{L(y)}{L(x)} \to 0 \quad \text{as} \quad \frac{x}{y} \to 0 \quad \text{and} \quad x \to \infty.
\]

We can now prove that (b) of Lemma 2.1 holds. Set

\[
Y_0(T_k) = X_{i_0}(T_k)/\sigma_{i_0}(T_k), \quad x_k = \sqrt{2(1-\varepsilon) \log \log T_k},
\]

\[
r_{jk} = EY_0(T_j)Y_0(T_k) \quad \text{and} \quad A_k = \{Y_0(T_k) > \sqrt{2(1-\varepsilon) \log \log T_k} \}.
\]

By Lemma 2.2 we have, for some fixed \( m \),

\[
\sum_{m \leq j < k \leq n} (P(A_jA_k) - P(A_j)P(A_k))
\leq \sum_{j=m}^{n-1} \sum_{k=j+1}^{n} |r_{jk}| \phi(x_j, x_k; r_{jk}^*)
\]

(3)
\[ = \sum_{j=m}^{n-1} \left( \sum_{k=j+1}^{j+\xi_j} \xi_j' + \sum_{k=j+\xi_j+1}^{n} \xi_j' \right) \frac{|r_{jk}|}{2\pi \sqrt{1 - r_{jk}^*}} \]
\[ \times \exp \left\{ \frac{-x_j^2 + x_k^2 - 2x_jx_kr_{jk}^*}{2(1 - r_{jk}^*)} \right\} \]
\[ =: I_1 + I_2 + I_3, \]

where \( \xi_j = \left[ \frac{2}{\alpha''} \log_\theta j \right] \) with \( \alpha'' = \min(\alpha_0, 1 - \alpha_0') \) and \( \xi_j' = (2j) \land n. \)

Note that for \( j < k, \)
\[ \frac{x_j^2 + x_k^2 - 2x_jx_kr}{2(1 - r^2)} = -\frac{x_j^2}{2} - \frac{(x_k - x_jr)^2}{2(1 - r^2)} \leq -\frac{x_j^2}{2} - \frac{(1 - r)x_j^2}{2(1 + r)} . \]

For any \( 0 < \varepsilon < 1, \) the first sum
\[ I_1 \leq \sum_{j=m}^{n-1} \sum_{k=j+1}^{j+\xi_j} \frac{r}{2\pi \sqrt{1 - r^2}} e^{-x_j^2/2} \exp \left\{ -\frac{1 - r x_j^2}{1 + r} \right\} \]
\[ \leq \sum_{j=m}^{n-1} \frac{r}{2\pi \sqrt{1 - r^2}} \xi_j e^{-x_j^2/2} \exp \left\{ -M \frac{(1 - \varepsilon) \log \log \theta j}{2} \right\} \]
\[ \leq \sum_{j=m}^{n-1} \frac{r}{2\pi \sqrt{1 - r^2}} e^{-x_j^2/2} \frac{2}{\alpha''} (j \log \theta)^{-M(1 - \varepsilon) \log \theta j} \]
\[ \leq \varepsilon \sum_{j=1}^{n} P(A_j) \]

for large \( n, \) where \( M = \frac{1-r}{1+r} \) and \( r \) is the maximum of the covariances \( |r_{jk}| \) for \( j, k = 1, \ldots, n. \) We have \( r < 1 \) by noting (1) and taking \( \theta \) to be large enough. Note that
\[ \frac{x_j^2 + x_k^2 - 2x_jx_kr_{jk}^*}{2(1 - r_{jk}^*)^2} \leq -\frac{x_j^2 + x_k^2 - 2x_jx_kr_{jk}^*}{2} . \]

Choosing \( m \) to be large enough, we have
\[ I_2 \leq \sum_{j=m}^{n-1} \sum_{k=j+\xi_j+1}^{n} \frac{|r_{jk}|}{2\pi \sqrt{1 - r_{jk}^*}} e^{-(x_j^2 + x_k^2)/2} \exp \{ r_{jk}^* x_j x_k \} \]
\[ \leq \sum_{j=m}^{n-1} \sum_{k=j+\xi_j+1}^{n} \frac{1}{2\pi} 2j^{-2} e^{-(x_j^2 + x_k^2)/2} \]
\[ \leq \sum_{j=m}^{n-1} \sum_{k=j+\xi_j+1}^{n} \frac{1}{2\pi} 2j^{-2} e^{-(x_j^2 + x_k^2)/2} \]

(5)
\[ \leq \frac{\varepsilon}{2} \left( \sum_{j=1}^{n} P(A_j) \right)^2, \]

where we have used the fact that for \( k - j > \xi_j \) and \( k \leq 2j \)

\[ |r_{jk}| \leq \eta_{jk} = \frac{1}{2} \frac{\theta^{-\alpha_0(k-j)}}{\theta^{-1-\alpha'_0}(k-j)} + \frac{\theta^{-2\alpha_0/\alpha''}}{j^{-2(1-\alpha'_0)/\alpha''}} \leq \frac{\alpha_0}{2} j^{-2} \]

and

\[ r_{jk}^* x_j x_k \leq \eta_{jk} x_j x_k \]

\[ \leq 3j^{-2} \sqrt{(\log \log \theta_j)(\log \log \theta_k)} \leq 3j^{-2} \log(2j \log \theta) \to 0 \text{ as } j \to \infty. \]

For \( I_3 \) we have the same bound by noting that for \( k - j > \xi_j \) and \( k > 2j \)

\[ r_{jk}^* x_j x_k \leq (\theta^{-\alpha_0 k/2} + 2\theta^{-(1-\alpha'_0)k/2}) \log(k \log \theta) \to 0 \text{ as } k \to \infty. \]

Thus, by combining these results, for any \( \varepsilon > 0 \) there is an \( m \) such that

\[ \sum_{m \leq j < k \leq n} (P(A_j A_k) - P(A_j) P(A_k)) \leq \varepsilon \left( \sum_{j=1}^{n} P(A_j) + \left( \sum_{j=1}^{n} P(A_j) \right)^2 \right), \]

which implies (b) of Lemma 2.1. Therefore Lemma 2.4 is proved. \( \square \)

**Proof of Theorem 1.2.** Let \( \hat{i}_0 = i_0(a_T) \) and

\[ Z_0(T, a_T) = \frac{X_{\hat{i}_0}(T) - X_{\hat{i}_0}(T - a_T)}{\sigma_{\hat{i}_0}(a_T)}. \]

Then, for any \( 0 < \varepsilon < 1 \), we have

\[ P \left\{ \frac{\| X^d(T) - X^d(T - a_T) \|}{\sigma(d, a_T)} > (1 - \varepsilon) \left( 2 \left( \log(T/a_T) + \log \log T \right) \right)^{1/2} \right\} \]

\[ \geq P \left\{ |Z_0(T, a_T)| > (1 - \varepsilon) \left( 2 \left( \log(T/a_T) + \log \log T \right) \right)^{1/2} \right\} \]

\[ \geq \frac{1}{2\sqrt{2\pi}} \exp \left\{ - \left( 1 - \varepsilon \right)^2 \left( \log(T/a_T) + \log \log T \right) \right\} 1^{1/2} \]

\[ \geq \left( \frac{a_T}{T \log T} \right)^{1-\varepsilon}. \]
for $T$ large. Let $T_1 = 1$ and define $T_{k+1}$ by $T_{k+1} - aT_{k+1} = T_k$ if $\rho < 1$, $T_k = \theta^k$ if $\rho = 1$, where $\lim_{T \to \infty} aT/T = \rho$ and $\theta > 1$. In the case of $\rho = 1$, then necessarily $aT = T$ and $\|X^d(T) - X^d(T - aT)\| = \|X^d(T)\|$. By Lemma 2.4, the conclusion of the theorem holds. Consider the case of $\rho < 1$. Set

$$B_k = \{Z_0(T_k, aT_k) > \left(2(1 - \varepsilon)(\log(T_k/aT_k) + \log \log T_k)\right)^{1/2}\}, \ k \geq 2,$$

then

$$\sum_{k=2}^{\infty} P(B_k) \geq \sum_{k=2}^{\infty} \left(\frac{aT_k}{T_k \log T_k}\right)^{1-\varepsilon} = \infty.$$ 

Hence, in order to prove our result, we need to show that (b) of Lemma 2.1 holds. Without loss of generality, assume that $a_1 < 1$. By condition (i) and the definition of $T_k$, we have $T_k(1 - a_1) \leq T_{k-1}, aT_k \leq (1 - a_1)^{-1}aT_{k-1}$ and $\sum_{m=j+1}^{k-1} aT_m \geq (k - j - 1)aT_{j+1}$. For $k \geq j + 2$, if $i_0(aT_j) \neq i_0(aT_k), E\{Z_0(T_j, aT_j)Z_0(T_k, aT_k)\} = 0$. If $i_0(aT_j) = i_0(aT_k) (:= i_0)$, then by Lemma 2.3, we have

$$r_{jk}^0 := E\{Z_0(T_j, aT_j)Z_0(T_k, aT_k)\}$$

$$= \frac{1}{2\sigma_{i_0}(aT_j)\sigma_{i_0}(aT_k)} E\left\{\sigma_{i_0}^2(T_k - T_{j-1}) - \sigma_{i_0}^2(T_k - T_j) - \left(\sigma_{i_0}^2(T_{k-1} - T_{j-1}) - \sigma_{i_0}^2(T_{k-1} - T_j)\right)\right\}$$

$$= \frac{1}{2\sigma_{i_0}(aT_j)\sigma_{i_0}(aT_k)} \left[\int_{T_{k-1} - T_j}^{T_k - T_{j-1}} d\sigma_{i_0}^2(x) - \int_{T_k - T_j}^{T_{k-1} - T_{j-1}} d\sigma_{i_0}^2(x)\right]$$

$$= \frac{1}{2\sigma_{i_0}(aT_j)\sigma_{i_0}(aT_k)} \left[\sum_{m=j+1}^{k-1} aT_m d\sigma_{i_0}^2(x) - \sum_{m=j+1}^{k-1} aT_m d\sigma_{i_0}^2(x)\right]$$

$$\leq c_2 \frac{aT_j aT_k}{\sigma_{i_0}(aT_j)\sigma_{i_0}(aT_k)} \frac{\sigma_{i_0}^2\left(\sum_{m=j+1}^{k-1} aT_m\right)}{\left(\sum_{m=j+1}^{k-1} aT_m\right)^2}. $$

Note that for large $k$,

$$\frac{aT_k}{\sigma_{i_0}(aT_k)} \frac{\sigma_{i_0}\left(\sum_{m=j+1}^{k-1} aT_m\right)}{\sum_{m=j+1}^{k-1} aT_m} = \left(\frac{aT_k}{\sum_{m=j+1}^{k-1} aT_m}\right)^{1-\alpha_0} \frac{L_{i_0}\left(\sum_{m=j+1}^{k-1} aT_m\right)}{L_{i_0}(aT_k)} \leq (1 - a_1)^{1-(1-\alpha_0)}.$$
We have
\[
\begin{align*}
    r_{jk}^0 &\leq c_2 \frac{(1 - a_1)^{-1}(1 - a_0) a_{T_j} \sigma_{i_0}((k - j - 1)a_{T_{j+1}})}{\sigma_{i_0}(a_{T_j}) (k - j - 1)a_{T_{j+1}}} \\
    &\leq c_3 \frac{a_{T_j}}{(k - j - 1)a_{T_{j+1}}} \frac{\sigma_{i_0}((1 - a_1)^{-1}(k - j - 1)a_{T_j})}{\sigma_{i_0}(a_{T_j})} \\
    &\leq c_4 (k - j - 2)^{a_0 - 1}
\end{align*}
\]
for \( k \) large, where \( c_3 \) and \( c_4 > 0 \) are constants. Then along the lines of proof corresponding to that of Theorem 1 in Ortega[21], we obtain that (b) of Lemma 2.1 holds. Theorem 1.2 is proved. \( \square \)

Using another version of Fernique’s lemma [12] on the \( d \)-dimensional Gaussian process, the following lemma estimates an upper bound of the large deviation probability (cf. [16]).

**Lemma 2.5.** For any given \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) depending only on \( \varepsilon \) such that for all \( x > 1 \)

\[
P \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a_T} \frac{\|X^d(t + s) - X^d(t)\|}{\sigma(d, a_T)} \geq x \right\} \leq C_\varepsilon \left( \frac{T}{a_T} \right) \Phi_d \left( \frac{2}{2 + \varepsilon} x \right),
\]

where \( \Phi_d(x) = P\{\|N^d(0, 1)\| \geq x\} \) and \( N^d(0, 1) \) denotes a \( d \)-dimensional standard normal random vector.

**Proof of Theorem 1.3.** First suppose that \( 0 < r \leq \infty \). By condition (iii), for any \( \varepsilon \), \( 0 < \varepsilon < 1 \)

\[
\frac{T}{a_T} \geq (\log T)^{r - \varepsilon}
\]

for sufficiently large \( T \). Note that for sufficiently large \( x > 0 \) we have

\[
\Phi_d(x) \leq C x^{d - 2} e^{-x^2/2} \leq C \exp \left( - \frac{x^2}{2 + \varepsilon} \right)
\]

for some \( C > 0 \), then by Lemma 2.5, we have

\[
P \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a_T} \frac{\|X^d(t + s) - X^d(t)\|}{\sigma(d, a_T) \{2 \log (T/a_T)\}^{1/2}} \geq 1 + 2\varepsilon \right\} \leq C_\varepsilon \left( \frac{T}{a_T} \right) \Phi_d \left( \frac{2(1 + 2\varepsilon)}{2 + \varepsilon} \sqrt{2 \log (T/a_T)} \right)
\]
\[
\leq C_\varepsilon \frac{T}{a_T} \exp \left( -(1 + \varepsilon) \log \left( \frac{T}{a_T} \right) \right)
\leq C_\varepsilon \left( \log T \right)^{-\varepsilon/4} \longrightarrow 0 \quad \text{as} \quad T \rightarrow \infty.
\]

Hence there exists a sequence \( \{T_n\} \) such that

\[
\liminf_{n \to \infty} \sup_{0 \leq t \leq T_n} \sup_{0 \leq s \leq a_{T_n}} \frac{||X^d(t + s) - X^d(t)||}{\sigma(d, a_{T_n}) \{2 \log \left( \frac{T_n}{a_{T_n}} \right) \}^{1/2}} \leq 1 + 2\varepsilon \quad \text{a.s.}
\]

Hence, by condition (iii), we obtain

\[
(7) \quad \liminf_{n \to \infty} \sup_{0 \leq t \leq T_n} \sup_{0 \leq s \leq a_{T_n}} \frac{||X^d(t + s) - X^d(t)||}{\sigma(d, a_{T_n}) \beta(T_n, a_{T_n})} \leq \sqrt{\frac{r}{1 + r}} \quad \text{a.s.}
\]

Next consider the case of \( r = 0 \). It follows from (iii) that for small \( \varepsilon > 0 \), \( T/a_T < \left( \log T \right)^{\varepsilon/(2(2 - \varepsilon))} \) for large \( T \). Applying Lemma 2.5, we get

\[
P \left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a_T} \frac{||X^d(t + s) - X^d(t)||}{\sigma(d, a_T) \beta(T, a_T)} \geq \varepsilon \right\}
\leq C_\varepsilon \left( \frac{T}{a_T} \right) \exp \left( -\frac{\varepsilon}{2} \log \left( \frac{T}{a_T} \log T \right) \right)
\leq C_\varepsilon \left( \log T \right)^{-\varepsilon/4} \longrightarrow 0 \quad \text{as} \quad T \rightarrow \infty
\]

and hence there exists a sequence \( \{T_n\} \) such that

\[
(8) \quad \liminf_{n \to \infty} \sup_{0 \leq t \leq T_n} \sup_{0 \leq s \leq a_{T_n}} \frac{||X^d(t + s) - X^d(t)||}{\sigma(d, a_{T_n}) \beta(T_n, a_{T_n})} \leq 0 \quad \text{a.s.}
\]

Combining (7) and (8) completes the proof of Theorem 1.3.

Proof of Theorem 1.4. When \( r = 0 \), our result is trivial and the result for the case of \( r = \infty \) was proved by Lin et al.[16]. We here consider the case of \( 0 < r < \infty \). For \( \theta > 1 \) and integers \( k \) and \( j \), let

\[
A_{kj} = \{ T : \theta^{k-1} < T \leq \theta^k, \ \theta^{j-1} < a_T \leq \theta^j \}.
\]
For any $0 < \tau < 1$, by condition (iii), $I_k := k - \left[ \frac{r(1+\tau)}{\log \theta} \log k \right] \leq j \leq k - \left[ \frac{r(1-\tau)}{\log \theta} \log k \right] =: I'_k$ provided $k$ is large enough. Clearly

$$\inf_{T \in A_{kj}} \beta(T, a_T) \geq \left( 2 \log \left( \frac{\theta^{k-1}}{\theta^j} \log \theta^{k-1} \right) \right)^{1/2}$$

$$\geq \theta^{-1} \left( 2 \log \left( \frac{\theta^k}{\theta^j} \log \theta^k \right) \right)^{1/2}$$

$$=: \theta^{-1} \beta_{kj}.$$

For some $M > 0$ set $N_{k,j} = \left[ \theta^{k-1}/(M\theta^j) \right]$. By (iii) for large $k$, we have

$$\sup_{T \in A_{kj}} \beta(T, a_T) \leq \left\{ 2 \log \left( \frac{\theta^k}{\theta^{j-1}} \log \theta^k \right) \right\}^{1/2}$$

$$\leq \left\{ 2\theta^2 \left( \frac{1 + r}{r} \right) \log N_{k,j} \right\}^{1/2}.$$

By the regular variation of $\sigma_i(\cdot)$, $i = 1, \ldots, d$, we have

$$\sigma_i(\theta^{j-1}) \geq (\theta - 1)^{-\alpha_0} \sigma_i(\theta^j - \theta^{j-1}).$$

Thus

$$\liminf_{T \to \infty} \sup_{0 \leq t \leq T} \frac{\|X^d(t + a_T) - X^d(t)\|}{\sigma(d, a_T) \beta(T, a_T)}$$

$$\geq \liminf_{k \to \infty} \inf_{I_k \leq j \leq I'_k} \sup_{T \in A_{kj}} \sup_{0 \leq t \leq T} \frac{\|X^d(t + a_T) - X^d(t)\|}{\sigma(d, a_T) \beta(T, a_T)}$$

$$\geq \liminf_{k \to \infty} \inf_{I_k \leq j \leq I'_k} \sup_{0 \leq t \leq \theta^{j-1}} \frac{\|X^d(t + \theta^j) - X^d(t)\|}{\sigma(d, \theta^j) \theta \{2(1 + 1/r) \log N_{k,j}\}^{1/2}}$$

$$- \limsup_{k \to \infty} \sup_{I_k \leq j \leq I'_k} \sup_{0 \leq t \leq \theta^k} \sup_{\theta^{j-1} \leq \theta \leq \theta^j} \frac{(\theta - 1)^{\alpha_0}}{\sigma(d, \theta^j - \theta^{j-1}) \theta^{-1} \beta_{kj}}$$

$$\times \|X^d(t + \theta^j) - X^d(t + s)\|$$

$$=: H_1/(\theta(1 + 1/r)^{1/2}) - (\theta - 1)^{\alpha_0} H_2.$$

At first, we will show that for any $R > 2$,

(11) $H_2 \leq R$ a.s.
Take $\theta$ being close to 1 such that $\theta^{-1} R \geq 2$. By the same way as the proof of Lemma 2.5, we obtain

$$ P \left\{ \sup_{I_k} \sup_{0 \leq t \leq \theta} \sup_{\theta \geq 1} \frac{\|X_d(t + \theta^j) - X_d(t + s)\|}{\sigma(d, \theta^j - \theta^j-1)} \geq \theta^{-1} R\beta_{kj} \right\} $$

$$ \leq C_\varepsilon \sum_{j=I_k}^{I_k'} \theta^k \exp \left\{ -\frac{8}{2 + \varepsilon} \log \left( \theta^{k-j} \log \theta^k \right) \right\} $$

$$ \leq C_\varepsilon k^{-2}, $$

where $C_\varepsilon > 0$ is a constant.

By the Borel-Cantelli lemma, we obtain (11).

Consider $H_1$. Let $i_0 = i_0(\theta^j)$ and

$$ W_0(j; l) = \frac{X_{i_0}(LM^j \theta^j + \theta^j) - X_{i_0}(LM^j \theta^j)}{\sigma_{i_0}(\theta^j)}, \quad 1 \leq l \leq N_{k,j}, $$

then $W_0(j; l)$ is a standard normal random variable. We have

$$ H_1 \geq \liminf_{k \to \infty} \min_{I_k \leq j \leq I_k'} \max_{1 \leq l \leq N_{k,j}} \frac{\|X_d(LM^j \theta^j + \theta^j) - X_d(LM^j \theta^j)\|}{\sigma(d, \theta^j) \left(2 \log N_{k,j}ight)^{1/2}} $$

$$ \geq \liminf_{k \to \infty} \min_{I_k \leq j \leq I_k'} \max_{1 \leq l \leq N_{k,j}} \max_{1 \leq t \leq d} \frac{|X_t(LM^j \theta^j + \theta^j) - X_t(LM^j \theta^j)|}{\sigma(d, \theta^j) \left(2 \log N_{k,j}ight)^{1/2}} $$

$$ \geq \liminf_{k \to \infty} \min_{I_k \leq j \leq I_k'} \max_{1 \leq l \leq N_{k,j}} \frac{W_0(j; l)}{\left(2 \log N_{k,j}\right)^{1/2}} =: H_3. $$

Let us estimate a lower bound of $H_3$. Using the elementary relation $ab = (a^2 + b^2 - (a - b)^2)/2$ and Lemma 2.3, it follows that for $l > l'$

$$ |r_{ll'}| := |\text{Cov}(W_0(j; l), W_0(j; l'))| $$

$$ = \frac{1}{\sigma_{i_0}^2(\theta^j)} \left| E \left\{ X_{i_0}(LM^j \theta^j + \theta^j) X_{i_0}(LM^j \theta^j + \theta^j) \right. $$

$$ - X_{i_0}(LM^j \theta^j + \theta^j) X_{i_0}(LM^j \theta^j) $$

$$ - X_{i_0}(LM^j \theta^j) X_{i_0}(LM^j \theta^j + \theta^j) + X_{i_0}(LM^j \theta^j) X_{i_0}(LM^j \theta^j) \right\} \right| $$

$$ \leq \frac{1}{2 \sigma_{i_0}^2(\theta^j)} \left| \sigma_{i_0}^2(M(l - l') \theta^j + \theta^j) - \sigma_{i_0}^2(M(l - l') \theta^j) \right| $$

$$ - \left( \sigma_{i_0}^2(M(l - l') \theta^j) - \sigma_{i_0}^2(M(l - l') \theta^j - \theta^j) \right) $$

$$ (13) $$
\[
\begin{align*}
&= \frac{1}{2\sigma_{l,0}^2(\theta^j)} \left| \int_{M(l-l')\theta^j}^{M(l-l')\theta^j + \theta^j} d(\sigma_{l,0}^2(x)) - \int_{M(l-l')\theta^j - \theta^j}^{M(l-l')\theta^j} d(\sigma_{l,0}^2(x)) \right| \\
&\leq \frac{c_2}{2\sigma_{l,0}^2(\theta^j)} \frac{\sigma_{l,0}^2(M(l-l')\theta^j - \theta^j)}{(M(l-l') - 1)^2} \\
&\leq c_5 |M(l-l') - 1|^{2\alpha_0 - 2} < \delta |l - l'|^{-\nu}
\end{align*}
\]

for any given \(\delta > 0\) provided \(M\) is large enough, where \(0 < \nu = 2 - 2\alpha_0\).

Let \(\eta_l, l = 1, \ldots, N_{k,j}\), and \(\xi\) be independent normal variables with \(E\eta_l = E\xi = 0\) and \(E\eta_l^2 = 1 - \delta\) and \(E\xi^2 = \delta\). Then by Slepian’s lemma, we have

\[
P \left\{ \max_{1 \leq l \leq N_{k,j}} \frac{W_0(j;l)}{(2 \log N_{k,j})^{1/2}} \leq 1 - 3\varepsilon \right\}
\]

\[
\leq P \left\{ \max_{1 \leq l \leq N_{k,j}} \left( \eta_l + \xi \right) \leq (1 - 3\varepsilon)(2 \log N_{k,j})^{1/2} \right\}
\]

\[
\leq P \left\{ \max_{1 \leq l \leq N_{k,j}} \eta_l \leq (1 - 2\varepsilon)(2 \log N_{k,j})^{1/2} \right\} + P \left\{ \xi \geq\varepsilon(2 \log N_{k,j})^{1/2} \right\}
\]

\[
\leq \{ 1 - \exp\left( -(1 - \varepsilon) \log N_{k,j} \right) \}^{N_{k,j}} + \exp\left\{ -\frac{\varepsilon^2}{\delta} \log N_{k,j} \right\}
\]

\[
\leq \exp\left\{ -N_{k,j}^\varepsilon \right\} + N_{k,j}^{-\varepsilon^2/\delta}
\]

\[
\leq 2(\theta^{k-j-1}/M)^{-\varepsilon^2/\delta}
\]

and taking \(\delta \leq \varepsilon^2 r(1 - \tau)/2\), we obtain

\[
P \left\{ \min_{I_k \leq j \leq I_k'} \max_{1 \leq l \leq N_{k,j}} \frac{W_0(j;l)}{(2 \log N_{k,j})^{1/2}} \leq \sqrt{1 - \varepsilon} \right\}
\]

\[
\leq 2 \sum_{I_k \leq j \leq I_k'} (\theta^{k-1-j}/M)^{-\varepsilon^2/\delta} \leq c_6 \theta^{-2(\log \theta)^{-1}} \log k = c_6 k^{-2},
\]

which implies

\[
\sum_{k=1}^{\infty} P \left\{ \min_{I_k \leq j \leq I_k'} \max_{1 \leq l \leq N_{k,j}} \frac{W_0(j;l)}{(2 \log (N_{k,j}))^{1/2}} \leq 1 - 3\varepsilon \right\} < \infty
\]

and hence by the Borel-Cantelli lemma, we have

\[
\lim_{k \to \infty} \min_{I_k \leq j \leq I_k'} \max_{1 \leq l \leq N_{k,j}} \frac{W_0(j;l)}{(2 \log (N_{k,j}))^{1/2}} > 1 - 3\varepsilon \quad \text{a.s.}
\]
Since $c$ is arbitrary, we obtain
\begin{equation}
H_3 \geq 1 \quad \text{a.s.}
\end{equation}
Combining (11), (12), (14) with (10), the proof is completed. \qed

References


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