SOME EXISTENCE THEOREMS FOR FUNCTIONAL EQUATIONS ARISING IN DYNAMIC PROGRAMMING

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Abstract. The existence, uniqueness and iterative approximation of solutions for a few classes of functional equations arising in dynamic programming of multistage decision processes are discussed. The results presented in this paper extend, improve and unify the results due to Bellman [2, 3], Bhakta-Choudhury [6], Bhakta-Mitra [7], and Liu [12].

1. Introduction and preliminaries


\[ f(x) = \sup_{y \in D} H(x, y, f(T(x, y))), \quad x \in S, \]

where \( x \) and \( y \) represent the state and decision vectors, respectively, \( T \) represents the transformation of the process, and \( f(x) \) represents the optimal return function with initial state \( x \). Baskaran and Subrahmanyam [1], Bhakta and Choudhury [6], Bhakta and Mitra [7], Chang [8], Chang and Ma [9], Liu [10]-[12] and others extended the results of [2]-[5] in
various directions. Bhakta and Mitra [7] established the existence and uniqueness of solutions for the functional equation:

\[(1.2) \quad f(x) = \sup_{y \in D} \{ u(x,y) + f(T(x,y)) \}, \ x \in S. \]

Under suitable conditions, Bellman [2], Bhakta and Choudhury [6] and Liu [12] obtained the existence or uniqueness of solutions for the functional equations:

\[(1.3) \quad f(x) = \inf_{y \in D} \max \{ u(x,y), f(T(x,y)) \}, \ x \in S. \]

\[(1.4) \quad f(x) = \inf_{y \geq x} \{ u(x,y) + v(x,y) \left[ \int_{y}^{+\infty} p(s-y)q(s)ds \right. \\
+ f(0) \int_{y}^{+\infty} q(s)ds + \int_{0}^{y} f(y-s)q(s)ds \left. \right] \}. \]

Inspired and motivated by the work in [1]-[12], in this paper, we prove the existence, uniqueness and iterative approximation of solutions for the functional equations (1.4)-(1.6):

\[(1.5) \quad f(x) = \sup_{y \in D} \{ u(x,y) + f(T(x,y)) \}, \ x \in S; \]

\[(1.6) \quad f(x) = \sup_{y \in D} \max \{ u(x,y), f(T(x,y)) \}, \ x \in S, \]

where the \( \sup \) denotes the sup or inf. The results presented in this paper extend, improve and unify the corresponding results of Bellman [2, 3], Bhakta-Choudhury [6], Bhakta-Mitra [7], and Liu [12].

Throughout this paper, \( N \) denotes the set of all positive integers, \( R = (-\infty, +\infty) \) and \( R^+ = [0, +\infty) \). Define

\[ \Phi_1 = \{ \varphi : \varphi : R^+ \rightarrow R^+ \text{ is nondecreasing} \}, \]

\[ \Phi_2 = \{ \varphi : \varphi \in \Phi_1 \text{ and } \lim_{n \rightarrow -\infty} \varphi^n(t) = 0 \text{ for } t > 0 \}, \]

\[ \Phi_3 = \{ \varphi : \varphi \in \Phi_1, \varphi(0) = 0 \text{ and } \varphi \text{ is right continuous at } 0 \}. \]

**Remark 1.1.** It is easy to see that \( \varphi \in \Phi_2 \) implies \( \varphi(t) < t \) for any \( t > 0 \).
Let us recall the following concept. Let $X$ be a nonempty set and let \( \{d_n\}_{n \in \mathbb{N}} \) be a countable family of pseudometrics on $X$ such that for any distinct $x, y \in X$, $d_k(x, y) \neq 0$ for some $k \in \mathbb{N}$. Define
\[
    d(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x, y)}{1 + d_k(x, y)} \quad \text{for all } x, y \in X.
\]
It is clear that $d$ is a metric on $X$. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ converges to a point $x \in X$ if and only if $d_k(x_n, x) \to 0$ as $n \to \infty$ and $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if and only if $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$ for each $k \in \mathbb{N}$.

**Lemma 1.1.** ([12]) Let $a, b, c$ be in $\mathbb{R}$. Then
\[
|\text{opt} \{a, c\} - \text{opt} \{b, c\}| \leq |a - b|.
\]

2. Existence and uniqueness theorems

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be real Banach spaces, $S \subseteq X$ be the state space, and $D \subseteq Y$ be the decision space. Denote by $BB(S)$ the set of all real-valued mappings on $S$ that are bounded on bounded subsets of $S$. It is easy to verify that $BB(S)$ is a linear space over $\mathbb{R}$ under usual definitions of addition and multiplication by scalars. For any $k \in \mathbb{N}$ and $a, b \in BB(S)$, let
\[
    d_k(a, b) = \sup \{\|a(x) - b(x)\| : x \in \overline{B}(0, k)\},
\]
\[
    d(a, b) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(a, b)}{1 + d_k(a, b)},
\]
where $\overline{B}(0, k) = \{x : x \in S \text{ and } \|x\| \leq k\}$. Clearly, $\{d_k\}_{k \in \mathbb{N}}$ is a countable family of pseudometrics on $BB(S)$ and $(BB(S), d)$ is a complete metric space.

**Theorem 2.1.** Let $u : S \times D \to \mathbb{R}$, $T : S \times D \to S$ be mappings and
\[
a_0(x) = \sup_{y \in D} u(x, y),
\]
\[
a_n(x) = \sup_{y \in D} \{u(x, y) + a_{n-1}(T(x, y))\}, \quad x \in S, \quad n \in \mathbb{N}.
\]
If there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_1$ such that

\begin{equation}
\|T(x, y)\| \leq \varphi(\|x\|) \text{ for all } (x, y) \in S \times D,
\end{equation}

\begin{equation}
|u(x, y)| \leq \psi(\|x\|) \text{ for all } (x, y) \in S \times D,
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} \psi(\varphi^n(t)) < \infty \text{ for all } t > 0,
\end{equation}

then the functional equation (1.2) possesses a solution $w \in BB(S)$ such that

\begin{equation}
\lim_{n \to \infty} w(x_n) = 0 \text{ for any } x_0 \in S,
\end{equation}

\begin{equation}
\{y_n\}_{n \in N} \subseteq D, \ x_n = T(x_{n-1}, y_n), \ n \in N.
\end{equation}

Moreover, the solution $w$ of the functional equation (1.2) is also unique with respect to (2.5).

Proof. Put

$$
H(x, y, a) = u(x, y) + a(T(x, y)) \text{ for all } (x, y, a) \in S \times D \times BB(S),
$$

$$
f(a) = \sup_{y \in D} H(x, y, a) \text{ for all } (x, a) \in S \times BB(S).
$$

For any $k \in N$, $y \in D$ and $x \in \overline{B}(0, k)$, by (2.2), (2.3), and Remark 1.1, we have

\begin{equation}
|u(x, y)| \leq \psi(\|x\|) \leq \psi(k), \|T(x, y)\| \leq \varphi(\|x\|) \leq \varphi(k).
\end{equation}

Using (2.6) and the definition of $f$, we infer that $f a \in BB(S)$ for any $a \in BB(S)$. That is, $f$ maps $BB(S)$ into $BB(S)$.

Now we prove that $f$ is a nonexpansive mapping in $BB(S)$. For any $a, b \in BB(S)$, $\varepsilon > 0$, $k \in N$ and $x \in \overline{B}(0, k)$, there exist $y, z \in D$ such that

\begin{equation}
f(a) - \varepsilon < H(x, y, a), \ f(b) - \varepsilon < H(x, z, b),
\end{equation}

\begin{equation}
f(a) \geq H(x, z, a), \ f(b) \geq H(x, y, b).
\end{equation}
It follows from (2.7) that

\[ |fa(x) - fb(x)| \\
< \max\{|H(x, z, a) - H(x, z, b)|, |H(x, y, a) - H(x, y, b)|\} + \varepsilon \\
= \max\{|a(T(x, z) - b(T(x, z))|, |a(T(x, y)) - b(T(x, y))|\} + \varepsilon \\
\leq d_k(a, b) + \varepsilon, \]

which implies that \(d_k(fa, fb) \leq d_k(a, b) + \varepsilon\). Letting \(\varepsilon \to 0\), we have

\[ d_k(fa, fb) \leq d_k(a, b) \text{ for all } a, b \in BB(S), \ k \in N, \]

which yields that

\[ d(fa, fb) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(fa, fb)}{1 + d_k(fa, fb)} \leq \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(a, b)}{1 + d_k(a, b)} = d(a, b). \]

That is,

\[ (2.8) \quad d(fa, fb) \leq d(a, b) \text{ for all } a, b \in BB(S). \]

We claim that for any \(n \geq 0\),

\[ (2.9) \quad |a_n(x)| \leq \sum_{i=0}^{n} \psi(\varphi^i(||x||)) \text{ for all } x \in S. \]

In fact, by (2.3) we conclude that

\[ -\psi(||x||) \leq u(x, y) \leq \psi(||x||) \text{ for all } (x, y) \in S \times D, \]

which means that

\[ |a_0(x)| = |\sup_{y \in D} u(x, y)| \leq \psi(||x||) \text{ for all } x \in S. \]

Assume that (2.9) holds for some \(n \geq 0\). It follows from (2.2) that

\[ (2.10) \quad |a_n(T(x, y))| \leq \sum_{i=0}^{n} \psi(\varphi^i(||T(x, y)||)) \leq \sum_{i=0}^{n} \psi(\varphi^{i+1}(||x||)) \]
for all \((x, y) \in S \times D\). From (2.3) and (2.10) we know that

\[
- \sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) \leq u(x, y) + a_n(T(x, y))
\]

\[
\leq \sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) \quad \text{for all } (x, y) \in S \times D.
\]

This yields that

\[
|a_{n+1}(x)| = \left| \sup_{y \in D} \{u(x, y) + a_n(T(x, y))\} \right|
\]

\[
\leq \sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) \quad \text{for all } x \in S.
\]

That is, (2.9) holds for all \(n \geq 0\).

Next we prove that \(\{a_n\}_{n \geq 0}\) is a Cauchy sequence in \(BB(S)\). Let \(k \in N, \varepsilon > 0\) and \(x_0 \in B(0, k)\) be given. (2.4) ensures that there exists some \(m \in N\) such that

\[
(2.11) \quad \sum_{i=n}^{n+p} \psi(\varphi^i(k)) < \varepsilon \quad \text{for all } n \geq m \text{ and } p \in N.
\]

For any \(n \geq m\) and \(p \in N\), by (2.1) we know that there exist \(v_1, w_1 \in D\) and \(y_1 = T(x_0, v_1), z_1 = T(x_0, w_1)\) satisfying

\[
(2.12) \quad a_{n+p}(x_0) < H(x_0, v_1, a_{n+p-1}) + 2^{-1}\varepsilon, \quad a_n(x_0) \geq H(x_0, v_1, a_{n-1}),
\]

\[
a_n(x_0) < H(x_0, w_1, a_{n-1}) + 2^{-1}\varepsilon, \quad a_{n+p}(x_0) \geq H(x_0, w_1, a_{n+p-1}).
\]

From (2.12) we have

\[
(2.13) \quad |a_{n+p}(x_0) - a_n(x_0)|
\]

\[
< \max\{|H(x_0, v_1, a_{n+p-1}) - H(x_0, v_1, a_{n-1})|, |H(x_0, w_1, a_{n+p-1}) - H(x_0, w_1, a_{n-1})|\} + 2^{-1}\varepsilon
\]

\[
= \max\{|a_{n-1}(y_1) - a_{n+p-1}(y_1)|, |a_{n-1}(z_1) - a_{n+p-1}(z_1)|\} + 2^{-1}\varepsilon
\]

\[
= |a_{n-1}(x_1) - a_{n+p-1}(x_1)| + 2^{-1}\varepsilon,
\]
where \( x_1 = y_1 \) or \( z_1 \) and

\[
|a_{n-1}(x_1) - a_{n+p-1}(x_1)| \\
= \max\{|a_{n-1}(y_1) - a_{n+p-1}(y_1)|, |a_{n-1}(z_1) - a_{n+p-1}(z_1)|\}.
\]

Similarly, we conclude that there exist \( v_i, w_i \in D, y_i = T(x_{i-1}, v_i), z_i = T(x_{i-1}, w_i) \), \( x_i = y_i \) or \( z_i \) for \( 2 \leq i \leq n \) satisfying

\[
|a_{n+p-1}(x_1) - a_{n-1}(x_1)| < |a_{n+p-2}(x_2) - a_{n-2}(x_2)| + 2^{-2} \varepsilon, \\
|a_{n+p-2}(x_2) - a_{n-2}(x_2)| < |a_{n+p-3}(x_3) - a_{n-3}(x_3)| + 2^{-3} \varepsilon, \\
\vdots \\
|a_{p+1}(x_{n-1}) - a_1(x_{n-1})| < |a_p(x_n) - a_0(x_n)| + 2^{-n} \varepsilon.
\]

(2.14)

It follows from \( \varphi \in \Phi_2, (2.2) \) and Remark 1.1 that

\[
\|x_n\| \leq \varphi(\|x_{n-1}\|) \leq \varphi^2(\|x_{n-2}\|) \leq \cdots \leq \varphi^n(\|x_0\|) \leq \|x_0\| \leq k
\]

(2.15)

for any \( n \in N \). In the light of (2.9), (2.11), and (2.13)-(2.15), we obtain that

\[
|a_{n+p}(x_0) - a_n(x_0)| < |a_p(x_n) - a_0(x_n)| + \varepsilon, \\
\leq |a_p(x_n)| + |a_0(x_n)| + \varepsilon \\
\leq \psi(\|x_n\|) + \sum_{i=0}^{p} \psi(\varphi^i(\|x_n\|)) + \varepsilon \\
\leq \psi(\varphi^n(\|x_0\|)) + \sum_{i=0}^{p} \psi(\varphi^{i+n}(\|x_0\|)) + \varepsilon \\
\leq 2 \sum_{i=n}^{n+p} \psi(\varphi^i(\|k\|)) + \varepsilon \\
< 3\varepsilon
\]

(2.16)

for any \( n \geq m \) and \( p \in N \). Thus (2.15) and (2.16) yield that \( d_k(a_n, a_{n+p}) \leq 3\varepsilon \). That is, \( \{a_n\}_{n \geq 0} \) is a Cauchy sequence in \( (BB(S), d) \) and hence it converges to some \( w \in BB(S) \). By virtue of (2.8), we get that

\[
d(fw, w) \leq d(fw, fa_n) + d(a_{n+1}, w) \leq d(w, a_n) + d(a_{n+1}, w) \to 0
\]
as $n \to \infty$. That is, $w = f w$ is a fixed point of $f$ and hence the functional equation (1.2) possesses a solution $w \in BB(S)$.

We prove that (2.5) holds. For any $x_0 \in S$, \{\(y_n\)\}_{n \in \mathbb{N}} \subseteq D, x_n = T(x_{n-1}, y_n), n \in \mathbb{N}$, we have by (2.2)

\begin{equation}
\|x_n\| = \|T(x_{n-1}, y_n)\| \leq \varphi(\|x_{n-1}\|) \leq \cdots \leq \varphi^n(\|x_0\|) \to 0, \text{ as } n \to \infty.
\end{equation}

Put $k = [\|x_0\|] + 1$, where $[t]$ denotes the largest integer not exceeding $t$. From Remark 1.1 and (2.17) we conclude that \{\(x_n\)\}_{n \geq 0} \subseteq \overline{B}(0, k)$. Let $k$ be in $N$. Note that $\lim_{m \to \infty} d_k(w, a_m) = 0$. For given $\varepsilon > 0$, by (2.4) and (2.17) we know that there exists some $m \in N$ such that

\begin{equation}
\max \left\{d_k(w, a_m), \sum_{i=n}^{m+n} \psi(\varphi^i(\|x_0\|)) \right\} < \varepsilon \text{ for any } n \geq m.
\end{equation}

By virtue of (2.9) and (2.18), we infer that

\begin{align*}
|w(x_n)| &\leq |w(x_n) - a_m(x_n)| + |a_m(x_n)| \\
&\leq d_k(w, a_m) + \sum_{i=0}^{m} \psi(\varphi^i(\|x_n\|)) \\
&\leq d_k(w, a_m) + \sum_{i=n}^{m+n} \psi(\varphi^i(\|x_0\|)) \\
&< 2\varepsilon
\end{align*}

for all $n \geq m$. Therefore $\lim_{m \to \infty} w(x_n) = 0$.

Finally we prove that $w$ is a unique solution of the functional equation (1.2) in $BB(S)$ satisfying (2.5). Suppose that $v$ is also a solution of the functional equation (1.2) in $BB(S)$ satisfying (2.5). Let $x_0 = t_0 \in S$ and $\varepsilon > 0$ be given. By the definition of $w$ and $v$, we conclude that there exist \{\(y_n\)\}_{n \geq 1} \subseteq D, \{z_n\}_{n \geq 1} \subseteq D, \{x_n\}_{n \geq 1} \subseteq S$ and \{\(t_n\)\}_{n \geq 1} \subseteq S with $x_n = T(x_{n-1}, y_n), t_n = T(t_{n-1}, z_n)$ for all $n \in N$, such that

\begin{align}
&w(x_i) < u(x_i, y_{i+1}) + w(x_{i+1}) + 2^{-i-1}\varepsilon, \\
&v(t_i) < u(t_i, z_{i+1}) + v(t_{i+1}) + 2^{-i-1}\varepsilon,
\end{align}

\begin{align}
&w(t_i) \geq u(t_i, z_{i+1}) + w(t_{i+1}), \ v(x_i) \geq u(x_i, y_{i+1}) + v(x_{i+1})
\end{align}
for all \( i \geq 0 \). By (2.19) we easily deduce that

\[
\begin{align*}
  w(x_0) &< u(x_0, y_1) + u(x_1, y_2) + \cdots + u(x_{n-1}, y_n) \\
  &\quad + w(x_n) + (1 - 2^{-n})\varepsilon, \\
  v(t_0) &< u(t_0, z_1) + u(t_1, z_2) + \cdots + u(t_{n-1}, z_n) \\
  &\quad + v(t_n) + (1 - 2^{-n})\varepsilon, \\
  w(t_0) &\geq u(t_0, z_1) + u(t_1, z_2) + \cdots + u(t_{n-1}, z_n) + w(t_n), \\
  v(x_0) &\geq u(x_0, y_1) + u(x_1, y_2) + \cdots + u(x_{n-1}, y_n) + v(x_n)
\end{align*}
\]

(2.20)

for any \( n \in \mathbb{N} \). Using (2.20) and \( x_0 = t_0 \), we have

\[
|w(x_0) - v(x_0)| < |w(x_n) - v(x_n)| + |w(t_n) - v(t_n)| + (1 - 2^{-n})\varepsilon.
\]

Letting \( n \to \infty \) in the above inequality, we obtain that \( |w(x_0) - v(x_0)| \leq \varepsilon \), which implies that \( w(x_0) = v(x_0) \) by letting \( \varepsilon \to 0 \). This completes the proof.

\[\square\]

**Remark 2.1.** Theorem 2.4 in [7] is a special case of Theorem 2.1 with \( \psi(t) = Mt \) for all \( t \in \mathbb{R}^+ \), where \( M \) is a positive constant. The following example reveals that Theorem 2.1 generalizes properly Theorem 2.4 in [7].

**Example 2.1.** Let \( X = Y = \mathbb{R}, S = D = \mathbb{R}^+ \). Define \( u : S \times D \to \mathbb{R}, T : S \times D \to S \) by

\[
u(x, y) = \frac{x^2(1 - xy)}{1 + xy}, \quad T(x, y) = \frac{x\sin^2(x + y)}{2 + y^2} \quad \text{for all } (x, y) \in S \times D.
\]

Choose \( \varphi(t) = 2^{-1}t \) and \( \psi(t) = t^2 \) for all \( t \in \mathbb{R}^+ \). It is easy to verify that the conditions of Theorem 2.1 are satisfied. Hence the functional equation (1.2) possesses a solution in \( \mathcal{BB}(S) \). But Theorem 2.4 in [7] is not applicable since

\[
|u(x, y)| = |u(M + 1, 0)| = (M + 1)^2 > M|x|
\]

for any \( M > 0 \) with \( (x, y) = (M + 1, 0) \in S \times D \).

A proof similar to that of Theorem 2.1 gives the following result and is thus omitted.
THEOREM 2.2. Let $u : S \times D \to R$, $T : S \times D \to S$ be mappings and

$$a_0(x) = \inf_{y \in D} u(x, y), a_n(x)$$

$$= \inf_{y \in D} \{u(x, y) + a_{n-1}(T(x, y))\}, \quad x \in S, \quad n \in N. \quad (2.21)$$

Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_1$ satisfying (2.2)-(2.4). Then the functional equation

$$f(x) = \inf_{y \in D} \{u(x, y) + f(T(x, y))\}, \quad x \in S, \quad (2.22)$$

possesses a solution $w \in BB(S)$ such that (2.5) holds. Moreover, the solution $w$ of the functional equation (2.22) is also unique with respect to (2.5).

THEOREM 2.3. Let $u : S \times D \to R$, $T : S \times D \to S$ be mappings and

$$a_0(x) = \sup_{y \in D} u(x, y), a_n(x)$$

$$= \sup_{y \in D} \max\{u(x, y), a_{n-1}(T(x, y))\}, \quad x \in S, \quad n \in N. \quad (2.23)$$

Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_3$ satisfying (2.2) and (2.3). Then the functional equation

$$f(x) = \sup_{y \in D} \max\{u(x, y), f(T(x, y))\}, \quad x \in S, \quad (2.24)$$

possesses a solution $w \in BB(S)$ such that (2.5) holds and

$$w(x) \geq 0 \quad \text{for all} \quad x \in S. \quad (2.25)$$

Moreover, the solution $w$ of the functional equation (2.24) is also unique with respect to (2.5).

Proof. Set

$$H(x, y, a) = \max\{u(x, y), a(T(x, y))\} \quad \text{for all} \quad (x, y, a) \in S \times D \times BB(S),$$

$$fa(x) = \sup_{y \in D} H(x, y, a) \quad \text{for all} \quad (x, a) \in S \times BB(S).$$

As in the proof of Theorem 2.1, we can conclude that $f$ maps $BB(S)$ into $BB(S)$ and (2.8) holds. Now we claim that for all $n \geq 0$,

$$|a_n(x)| \leq \psi(\|x\|) \quad \text{for all} \quad x \in S. \quad (2.26)$$
It is easy to verify that (2.3) implies that (2.26) holds for \( n = 0 \). Suppose that (2.26) holds for some \( n \geq 0 \). From (2.2), \( \varphi \in \Phi_2 \) and Remark 1.1, we infer that

\[
|a_n(T(x, y))| \leq \psi(\|T(x, y)\|) \\
\leq \psi(\varphi(\|x\|)) \leq \psi(\|x\|) \text{ for all } (x, y) \in S \times D.
\]

Using (2.3) and (2.27), we have

\[-\psi(\|x\|) \leq \max\{u(x, y), a_n(T(x, y))\} \leq \psi(\|x\|) \text{ for all } (x, y) \in S \times D,
\]

which implies that

\[|a_{n+1}(x)| = |\sup_{y \in D} \max\{u(x, y), a_n(T(x, y))\}| \leq \psi(\|x\|) \text{ for all } x \in S.
\]

Hence (2.26) holds for all \( n \geq 0 \). On the other hand, (2.23) ensures that

\[a_0(x) \leq a_1(x) \leq \ldots \leq a_n(x) \leq a_{n+1}(x) \leq \ldots \text{ for all } x \in S.
\]

Next we show that \( \{a_n\}_{n \geq 0} \) is a Cauchy sequence in \( BB(S) \). Let \( k \in N, \varepsilon > 0 \) and \( x_0 \in \overline{B}(0, k) \) be given. Since \( \varphi \in \Phi_2 \) and \( \psi \in \Phi_3 \), it follows that there exists some \( m \in N \) such that

\[\psi(\varphi^n(k)) < \varepsilon \text{ for all } n \geq m.
\]

For any \( n \geq m \) and \( p \in N \), by (2.23) we easily conclude that there exist \( y_1 \in D \) and \( x_1 = T(x_0, y_1) \) satisfying

\[a_{n+p}(x_0) < H(x_0, y_1, a_{n+p-1}) + 2^{-1}\varepsilon, \ a_n(x_0) \geq H(x_0, y_1, a_{n-1}).
\]

By virtue of (2.28), (2.30), and Lemma 1.1, we have

\[0 \leq a_{n+p}(x_0) - a_n(x_0)
\]

\[< H(x_0, y_1, a_{n+p-1}) - H(x_0, y_1, a_{n-1}) + 2^{-1}\varepsilon
\]

\[= \max\{u(x_0, y_1), a_{n+p-1}(x_1)\} - \max\{u(x_0, y_1), a_{n-1}(x_1)\} + 2^{-1}\varepsilon
\]

\[\leq a_{n+p-1}(x_1) - a_{n-1}(x_1) + 2^{-1}\varepsilon.
\]
Similarly, we conclude that there exist \( y_i \in D, x_i = T(x_{i-1}, y_i) \in S, \) \( 2 \leq i \leq n \) satisfying

\[
0 \leq a_{n+p-1}(x_1) - a_{n-1}(x_1) < a_{n+p-2}(x_2) - a_{n-2}(x_2) + 2^{-2}\varepsilon, \\
0 \leq a_{n+p-2}(x_2) - a_{n-2}(x_2) < a_{n+p-3}(x_3) - a_{n-3}(x_3) + 2^{-3}\varepsilon, \\
\vdots \\
0 \leq a_{p+1}(x_{n-1}) - a_1(x_{n-1}) < a_p(x_n) - a_0(x_n) + 2^{-n}\varepsilon.
\]

It follows from (2.2), (2.3), (2.26), (2.29), (2.31), and (2.32) that

\[
0 \leq a_{n+p}(x_0) - a_n(x_0) < a_p(x_n) - a_0(x_n) + \varepsilon \\
\leq |a_p(x_n)| + |a_0(x_n)| + \varepsilon \leq 2\psi(\|x_n\|) + \varepsilon \\
= 2\psi(\|T(x_{n-1}, y_n)\|) + \varepsilon \leq 2\psi(\varphi(\|x_{n-1}\|)) + \varepsilon \\
\leq 2\psi(\varphi^n(\|x_0\|)) + \varepsilon \leq 2\psi(\varphi^n(k)) + \varepsilon < 3\varepsilon
\]

for any \( n \geq m \) and \( p \in N \). This gives that \( d_k(a_n, a_{n+p}) \leq 3\varepsilon \) for any \( n \geq m \) and \( p \in N \). Consequently, \( \{a_n\}_{n \geq 0} \) is a Cauchy sequence in \( (BB(S), d) \) and it converges to some \( w \in BB(S) \). From (2.8), we deduce that \( w = fw \). That is, the functional equation (2.24) possesses a solution \( w \in BB(S) \).

We prove that (2.5) holds. For any \( x_0 \in S, \{y_n\}_{n \in N} \subseteq D, x_n = T(x_{n-1}, y_n), n \in N \), (2.2) yields that (2.17) holds. Note that \( \psi(0) = 0 \) and \( \psi \) is right continuous at 0. Thus (2.17) means that

\[
\lim_{n \to \infty} \psi(\|x_n\|) = \psi(0) = 0.
\]

Put \( k = [\|x_0\|] + 1 \). It is easy to verify that \( \{x_n\}_{n \in N} \subseteq \overline{B}(0, k) \). Let \( k \) be in \( N \) and \( \varepsilon > 0 \). Since \( \{a_n\}_{n \in N} \) converges to \( w \), by (2.33) we know that there exists some \( m \in N \) such that

\[
\max\{d_k(w, a_m), \psi(\|x_n\|)\} < \varepsilon \text{ for any } n \geq m.
\]

By virtue of (2.26) and (2.34), we have

\[
|w(x_n)| \leq |w(x_n) - a_m(x_n)| + |a_m(x_n)| \leq d_k(w, a_m) + \psi(\|x_n\|) < 2\varepsilon
\]

for all \( n \geq m \). That is, \( \lim_{n \to \infty} w(x_n) = 0 \).
Given $x_0 \in S$ and $\{y_n\}_{n \in \mathbb{N}} \subseteq D$, take $x_n = T(x_{n-1}, y_n)$ for all $n \in \mathbb{N}$. Since $w$ is a solution of the functional equation (2.24), by (2.5) we immediately infer that

$$w(x_0) \geq \max \{u(x_0, y_1), w(T(x_0, y_1))\} \geq w(x_1) \geq \ldots \geq w(x_n) \to 0$$

as $n \to \infty$. That is, $w(x_0) \geq 0$ for all $x_0 \in S$.

Finally we prove that $w$ is a unique solution of the functional equation (2.24) in $BB(S)$ satisfying (2.5). Suppose that $v$ is also a solution of the functional equation (2.24) in $BB(S)$ satisfying (2.5). Let $x_0 = t_0 \in S$ and $\varepsilon > 0$ be given. By the definition of $w$ and $v$, we conclude that there exist $\{y_n\}_{n \geq 1} \subseteq D$, $\{z_n\}_{n \geq 1} \subseteq D$, $\{x_n\}_{n \geq 1} \subseteq S$ and $\{t_n\}_{n \geq 1} \subseteq S$ with $x_n = T(x_{n-1}, y_n)$, $t_n = T(t_{n-1}, z_n)$ for all $n \in \mathbb{N}$, such that

(2.35)

$$w(x_i) < \max \{u(x_i, y_{i+1}), w(x_{i+1})\} + 2^{-i-1} \varepsilon,$$

$$v(t_i) < \max \{u(t_i, z_{i+1}), v(t_{i+1})\} + 2^{-i-1} \varepsilon,$$

$$w(t_i) \geq \max \{u(t_i, z_{i+1}), w(t_{i+1})\}, \quad v(x_i) \geq \max \{u(x_i, y_{i+1}), v(x_{i+1})\}$$

for all $i \geq 0$. By (2.35) we easily deduce that

(2.36)

$$w(x_0) < \max \{u(x_0, y_1), u(x_1, y_2), \ldots, u(x_{n-1}, y_n), w(x_n)\} + (1 - 2^{-n}) \varepsilon,$$

$$v(t_0) < \max \{u(t_0, z_1), u(t_1, z_2), \ldots, u(t_{n-1}, z_n), v(t_n)\} + (1 - 2^{-n}) \varepsilon,$$

$$w(t_0) \geq \max \{u(t_0, z_1), u(t_1, z_2), \ldots, u(t_{n-1}, z_n), w(t_n)\},$$

$$v(x_0) \geq \max \{u(x_0, y_1), u(x_1, y_2), \ldots, u(x_{n-1}, y_n), v(x_n)\}$$

for any $n \in \mathbb{N}$. Using (2.36), Lemma 1.1 and $x_0 = t_0$, we have

$$|w(x_0) - v(x_0)| < |w(x_n) - v(x_n)| + |w(t_n) - v(t_n)| + (1 - 2^{-n}) \varepsilon.$$

Letting $n \to \infty$ in the above inequality, we obtain that $|w(x_0) - v(x_0)| \leq \varepsilon$, which implies that $w(x_0) = v(x_0)$ by letting $\varepsilon \to 0$. This completes the proof. $\square$

Following a similar argument as in the proof of Theorem 2.3, we have the following.
THEOREM 2.4. Let $u : S \times D \to R, T : S \times D \to S$ be mappings and
\begin{equation}
    a_0(x) = \inf_{y \in D} u(x, y), a_n(x) = \inf_{y \in D} \max\{u(x, y), a_{n-1}(T(x, y))\}, \quad x \in S, \; n \in \mathbb{N}.
\end{equation}
Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_3$ satisfying (2.2) and (2.3). Then the functional equation
\begin{equation}
    f(x) = \inf_{y \in D} \max\{u(x, y), f(T(x, y))\}, \quad x \in S,
\end{equation}
possesses a solution $w \in BB(S)$ such that (2.5) and (2.25) hold. Moreover, the solution $w$ of the functional equation (2.38) is also unique with respect to (2.5).

REMARK 2.2. Theorem 2.4 extends, improves and unifies Theorem 3.5 of Bhakta and Choudhury [6], Theorem 3.5 of Liu [12] and a result of Bullman [2, p.135]. The example below shows that Theorem 2.4 is indeed a generalization of the results due to Bhakta and Choudhury [6], Liu [12], and Bullman [2].

EXAMPLE 2.2. Let $X, Y, S, D$ be as in Example 2.1. Define $u : S \times D \to R, T : S \times D \to S$ by
\[
    u(x, y) = \frac{x^4(1 + xy)}{1 + x^2 + y^2}, \quad T(x, y) = \frac{x|\sin(x + y)|}{1 + x} \quad \text{for all } (x, y) \in S \times D.
\]
Put $\varphi(t) = \frac{t}{1+t^4}$, $\psi(t) = t^4$ for all $t \in R^+$. Then the assumptions of Theorem 2.4 are fulfilled. However, we cannot invoke the results of Bhakta and Choudhury [6], Liu [12], and Bullman [2] to establish that the functional equation (2.38) possesses a solution in $BB(S)$ because
\[
    |u(x, y)| = \left|u\left(\frac{3}{2}(M + 1), \frac{3}{2}(M + 1)\right)\right| \geq \frac{2}{3} \left[\frac{3}{2}(M + 1)\right]^2 > M|x|
\]
for any $M > 0$ with $(x, y) = \left(\frac{3}{2}(M + 1), \frac{3}{2}(M + 1)\right) \in S \times D$.

Let $BC(R^+)$ denote the set of all bounded continuous real-valued functions on $R^+$. Put $d(a, b) = \sup\{|a(x) - b(x)| : x \in R^+\}$ for all $a, b \in BC(R^+)$. It is easily seen that $(BC(R^+), d)$ is a complete metric space.
Theorem 2.5. Let $X = Y = R$, $S = D = R^+$. Let $u, v : S \times D \to R^+$ be continuous, $u(x, x)$ be bounded on $S$, $\lim_{y \to +\infty} u(x, y) = +\infty$, $u(x, \cdot)$ and $v(x, \cdot)$ be nondecreasing with respect to the second argument on $[x, +\infty)$ for every $x \in S$, and

$$(2.39) \quad 0 \leq v(x, y) \leq r \text{ for all } (x, y) \in S \times D,$$

where $r$ is a positive constant. Let $p, q : S \to R^+$ satisfy that $p$ is continuous, nondecreasing, $\int_0^{+\infty} p(s)q(s)ds < +\infty$ and

$$(2.40) \quad \int_0^{+\infty} q(s)ds = t > 0.$$

Assume that

$$(2.41) \quad a_0(x) = \inf_{y \geq x} u(x, y), \quad x \in S,$$

$$a_{n+1}(x) = \inf_{y \geq x} \left\{ u(x, y) + v(x, y) \left[ \int_y^{+\infty} p(s-y)q(s)ds \right. \right.$$  

$$+ a_n(0) \int_y^{+\infty} q(s)ds + \int_0^y a_n(y-s)q(s)ds \bigg]\}, \quad x \in S, n \geq 0.$$

If $rt < 1$, then the functional equation (1.4) possesses a unique solution $w \in BC(R^+)$ and

$$(2.42) \quad d(a_{n+1}, w) \leq (rt)^{n+1}(1 - rt)^{-1} d(a_0, a_1) \text{ for all } n \geq 0.$$

Proof. For all $(x, y, b) \in S \times D \times BC(R^+)$, set

$$H(x, y, b) = u(x, y) + v(x, y) \left[ \int_y^{+\infty} p(s-y)q(s)ds \right. \right.$$  

$$+ b(0) \int_y^{+\infty} q(s)ds + \int_0^y b(y-s)q(s)ds \bigg].$$

Let

$$(2.43) \quad fb(x) = \inf_{y \geq x} H(x, y, b) \text{ for all } (x, b) \in S \times BC(R^+).$$
It is easy to see that $f$ maps $BC(R^+)$ into itself. Let $\varepsilon > 0$, $x \in S$ and $b, c \in BC(R^+)$ be given. It follows from (2.43) that there exist $y_1 \geq x$ and $y_2 \geq x$ such that

\begin{equation}
fb(x) > H(x, y_1, b) - \varepsilon, \quad fc(x) > H(x, y_2, c) - \varepsilon,
\end{equation}

\begin{equation}
f(x) \leq H(x, y_2, b), \quad f(x) \leq H(x, y_1, c).
\end{equation}

By virtue of (2.39), (2.40), and (2.44), we deduced that

\[
|fb(x) - fc(x)| \\
< \max \left\{ |H(x, y_i, b) - H(x, y_i, c)| : i = 1, 2 \right\} + \varepsilon \\
\leq \max \left\{ \left| v(x, y_i) \left[ |b(0) - c(0)| \int_{y_i}^{+\infty} q(s)ds \right. \right. \\
+ \left. \left. \int_0^{y_i} |b(y_i - s) - c(y_i - s)|q(s)ds \right| : i = 1, 2 \right\} + \varepsilon \\
\leq \max \left\{ rd(b, c) \left[ \int_{y_i}^{+\infty} q(s)ds + \int_0^{y_i} q(s)ds \right] : i = 1, 2 \right\} + \varepsilon \\
= rtd(b, c) + \varepsilon,
\]

which implies that

\[
d(fb, fc) = \sup \{|fb(x) - fc(x)| : x \in S\} \leq rtd(b, c) + \varepsilon.
\]

Letting $\varepsilon \to 0$, we easily conclude that

\[
d(fb, fc) \leq rtd(b, c) \text{ for all } b, c \in BC(R^+).
\]

It follows from Banach fixed-point theorem that $f$ has a unique fixed point $w \in BC(R^+)$ and (2.42) holds. Obviously, $w$ is a unique solution of the functional equation (1.4). This completes the proof. □

**Remark 2.3.** Theorem 2.5 generalizes Theorem 3.6 of Bhakta and Choudhury [6] and a result of Bellman [2, p.129].

**Problem 2.1.** If $rt < 1$ is replaced by $rt = 1$ in Theorem 2.5, does the functional equation (1.4) possess a solution in $BC(R^+)$?

**Problem 2.2.** If the answer to Problem 2.1 is no, then what additional hypotheses on $u, v, p, q$ are needed to guarantee the existence of a solution of the functional equation (1.4)?
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