GOTTLIEB GROUPS AND SUBGROUPS OF THE
GROUP OF SELF-HOMOTOPY EQUIVALENCES

JAE-RYONG KIM, NOBUYUKI ODA,
JIANGZHONG PAN, AND MOO HA WOO

Abstract. Let $\mathcal{E}_\#(X)$ be the subgroups of $\mathcal{E}(X)$ consisting of homotopy classes of self-homotopy equivalences that fix homotopy groups through the dimension of $X$ and $\mathcal{E}_*(X)$ be the subgroup of $\mathcal{E}(X)$ that fix homology groups for all dimension. In this paper, we establish some connections between the homotopy group of $X$ and the subgroup $\mathcal{E}_\#(X) \cap \mathcal{E}_*(X)$ of $\mathcal{E}(X)$. We also give some relations between $\pi_n(W)$, as well as a generalized Gottlieb group $G^s_n(W, X)$, and a subset $\mathcal{M}_\#^n(X, W)$ of $[X, W]$. Finally we establish a connection between the coGottlieb group of $X$ and the subgroup of $\mathcal{E}(X)$ consisting of homotopy classes of self-homotopy equivalences that fix cohomology groups.

1. Introduction and preliminaries

By a space, we mean a connected $CW$-complex of finite type. We mainly consider finite dimensional $CW$-complexes with base point. We begin this section with remarks about the set of all base point preserving continuous maps from a space $X$ to a space $W$. This set of maps splits up into disjoint equivalence classes, called homotopy classes. We write $[X, W]$ for the set of all base point preserving homotopy classes of the maps from $X$ to $W$; by keeping $X$ fixed and varying $W$, this set is an invariant of the homotopy type of $W$ in the sense that it is determined by the homotopy equivalence of spaces: the set $[X, W]$ can often be endowed, in a natural way, with some algebraic structure, and we obtain
exactly the algebraic invariant. Alternatively, we can keep \( W \) fixed and vary \( X \): once again a homotopy invariant results. Let \( W \) be a \( CW \)-complex of dimension \( N \) and \( E(W) \) the group of homotopy classes of self-homotopy equivalences of \( W \). In this paper, we give results about some subgroups of \( E(W) \) and we will extend these results to more general cases.

We now review some standard material that we will use. A cofibration sequence \( Z \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} \Sigma Z \) where \( X \) is the mapping cone of \( \gamma \), gives a homotopy coaction \( c : X \rightarrow X \vee \Sigma Z \), obtained by pinching the ‘equator’ of the cone of \( Z \) to a point. This coaction induces an action of \([\Sigma Z, W] \) on \([X, W] \) for any space \( W \). That is,

\[
\mu : [X, W] \times [\Sigma Z, W] \rightarrow [X, W], \quad \mu(f, \alpha) = f^\alpha
\]

for any \( \alpha \in [\Sigma Z, W] \) and \( f \in [X, W] \), where

\[
f^\alpha : X \xrightarrow{c} X \vee \Sigma Z \xrightarrow{f^\alpha} W \vee W \xrightarrow{\nabla} W.
\]

The following properties of this action are mentioned on p. 174 of Hilton [6]:

1. If \( h : W \rightarrow W' \), then \( h(f^\alpha) = (h f)^{\alpha} \),
2. If \( \alpha, \beta \in [\Sigma Z, W] \), then \( (f^\alpha)^\beta = f^{(\alpha + \beta)} \).

We are interested in the effect that \( f^\alpha \) has on homology and homotopy groups. This is described in the following results by Proposition 2.1 of [1].

**Proposition 1.1.** For the above cofibration sequence, suppose \( f \in [X, W] \) and \( \alpha \in [\Sigma Z, W] \). Then we have the following for any \( i > 0 \):

1. The induced homology homomorphism \( (f^\alpha)_* : H_i(X) \rightarrow H_i(W) \) is given by

\[
(f^\alpha)_*(x) = f_*(x) + \alpha_* q_*(x)
\]

for each \( x \in H_i(X) \).

2. Suppose that \( (f, \alpha) : X \vee \Sigma Z \rightarrow W \) factors through the product \( X \times \Sigma Z \). Then the induced homotopy homomorphism \( (f^\alpha)_\#: \pi_i(X) \rightarrow \pi_i(W) \) is given by

\[
(f^\alpha)_\#(x) = f_\#(x) + \alpha_\# q_\#(x)
\]

for each \( x \in \pi_i(X) \).

We now specialize to a mapping cone sequence of the form

\[
S^{n-1} \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} \Sigma S^{n-1} \equiv S^n,
\]
i.e. $X = Y \cup_{\gamma} e^n$. Then we have an action of $\pi_n(W)$ on $[X, W]$. We will consider elements of the form $f^\alpha \in [X, W]$ for $f \in [X, W]$ and $\alpha \in \pi_n(W)$.

Let $X$ be a CW-complex of dimension $N$ and $\mathcal{E}(X)$ the group of homotopy classes of self-equivalences of $X$. If we consider elements of the form $e^\alpha \in [X, X]$ for the identity map $i$ of $X$ and $\alpha \in \pi_n(X)$, these are not self-homotopy equivalences in general.

In [1], Arkowitz, Lupton and Murillo studied the subgroup $\mathcal{E}_\#(X)$ and $\mathcal{E}_*(X)$ of $\mathcal{E}(X)$ as follows (cf. Dror and Zabrodsky [4]; Maruyama [8] and [9]):

$$\mathcal{E}_\#(X) = \{ f \in \mathcal{E}(X) \mid f^i = 1 : \pi_i(X) \to \pi_i(X) \text{ for } i \leq N \},$$

$$\mathcal{E}_\#_{\infty}(X) = \{ f \in \mathcal{E}(X) \mid f^i = 1 : \pi_i(X) \to \pi_i(X) \text{ for all } i \},$$

$$\mathcal{E}_*(X) = \{ f \in \mathcal{E}(X) \mid f_* = 1 : H_i(X) \to H_i(X) \text{ for all } i \}.$$

Maruyama [10] introduces a subset of $[X, W]$ as follows:

$$Z^\alpha_n(X, W) = \{ \alpha \in [X, W] \mid \alpha^i = 0 : \pi_i(X) \to \pi_i(W) \text{ for } i \leq n \}$$

If we consider $\mathcal{E}_\#(X) \subset [X, X]$ and $Z^\alpha_n(X, X) \subset [X, X]$ as two special subsets in $[X, X]$, we can give some definitions which generalize them and which we will study in the next section.

**Definition 1.1.** Let $f \in [X, W]$. We define

$$\mathcal{M}_f^i(X, W) = \{ g \in [X, W] \mid g_* = f_* : H_i(X) \to H_i(W) \text{ for all } i \}.$$

Similarly we denote

$$\mathcal{M}_{\#}^f(X, W) = \{ g \in [X, W] \mid g^i = f^i : \pi_i(X) \to \pi_i(W) \text{ for all } i \leq N \}.$$

Especially, we denote $\mathcal{M}_f^i(X, W) = \mathcal{M}_{\#}^i(X, W)$ if $W$ is a $N$-dimensional CW-complex. Thus we can get

$$\mathcal{M}_{\#}^0(X, W) = Z^\alpha_n(X, W)$$

$$\mathcal{M}_{\#}^i(W, W) = \mathcal{E}_\#(W)$$

$$\mathcal{M}_{\#}^i_{\infty}(W, W) = \mathcal{E}_{\#\infty}(W),$$

where $0$ denotes the constant map.

Recall the $n$'th Gottlieb group [5] of $\pi_n(W)$, denoted by $G_n(W)$, consists of those $\alpha \in \pi_n(W)$ for which there is an associated map $F : W \times S^n \to W$ such that the following diagram is homotopy commutative:
2. Gottlieb groups and subgroups of self-homotopy equivalences

It was shown in Theorem 2.3 of [1] that there is a homomorphism between the Gottlieb group of $X$ and $E_\#(X)$ and $E_\ast(X)$ as follows.

**Theorem 2.1.** Let $X = Y \cup \gamma e^n$ be a 1-connected $n$-dimensional complex. Suppose that $q_\# = 0 : \pi_n(X) \to \pi_n(S^n)$. Then there is a homomorphism

$$\theta : G_n(X) \to E_\#(X)$$

defined by $\theta(\alpha) = \iota^\alpha$ for $\alpha \in G_n(X)$. This homomorphism restricts to

$$\theta' : G_n(X) \cap \text{Ker } h_n \to E_\ast(X) \cap E_\#(X),$$

where $h_n : \pi_n(X) \to H_n(X)$ denotes the Hurewicz homomorphism.

For the map $\theta : \pi_n(X) \to [X, X]$ given by $\theta(\alpha) = \iota^\alpha$ for $\alpha \in \pi_n(X)$ and the group $E_\#(X) \subset [X, X]$, this theorem gives a condition to be $G_n(X) \subset \theta^{-1}(E_\#(X))$. The objects of this section is to find conditions such that $\pi_n(X) \subset \theta^{-1}(E_\#(X))$ and $\pi_n(X) \subset \zeta^{-1}(Z_\#^n(X, X))$ for a function $\zeta : \pi_n(X) \to [X, X]$ defined later.

First we will find some conditions for $\theta$ to be a homomorphism from the homotopy group $\pi_n(X)$ to $E_\#(X)$. Consider the cofibration sequence

$$S^{n-1} \xrightarrow{\gamma} Y \xrightarrow{j} X \xrightarrow{q} S^n \quad (n \geq 2).$$

**Theorem 2.2.** Let $n \geq 2$ and $Y$ be 1-connected. Let $X = Y \cup \gamma e^n$ with $X^{n-1} = Y$. If the Hurewicz homomorphism $h_n = 0 : \pi_n(X) \to H_n(X)$, then there is a homomorphism

$$\theta : \pi_n(X) \to E_\ast(X) \cap E_\#(X)$$

defined by $\theta(\alpha) = \iota^\alpha$ for any $\alpha \in \pi_n(X)$.

**Proof.** First we prove that $j_\# : \pi_k(Y) \to \pi_k(X)$ is surjective for all $k \leq n$. 
For $n \geq 1$, the Whitehead's $\Gamma$-group $\Gamma_n(X)$ is defined by the following image group

$$
\Gamma_n(X) = \text{image } (j^n_\# : \pi_n(X^{n-1}) \to \pi_n(X^n)),
$$

where $j : Y = X^{n-1} \to X = X^n$ is the inclusion of the $(n-1)$-skeleton to the $n$-skeleton of $X$. Let $j_n : \pi_n(Y) \to \Gamma_n(X)$ be the map induced by $j_\#: \pi_n(Y) = \pi_n(X^{n-1}) \to \pi_n(X) = \pi_n(X^n)$. Then we have the following commutative diagram:

\[
\begin{array}{cccccc}
\pi_n(Y) & & & & \pi_n(X) & \to \pi_n(X) \\
\downarrow{j_n} & \downarrow{j_\#} & \downarrow{i_n} & \downarrow{h_n} & \downarrow{} & \to \pi_n(X) \\
\Gamma_n(X) & \to & \pi_n(X) & \to & H_n(X) \\
\end{array}
\]

where $n \geq 2$. The row in the above diagram is the exact sequence of J.H.C. Whitehead (see [3]). Then we see that $j_\#: \pi_n(Y) \to \pi_n(X)$ is surjective since $j_n : \pi_n(Y) \to \Gamma_n(X)$ is surjective and $h_n = 0 : \pi_n(X) \to H_n(X)$. It is clear that $j_\#: \pi_k(Y) \to \pi_k(X)$ is surjective for all $k < n$. Therefore $j_\#: \pi_k(Y) \to \pi_k(X)$ is surjective for all $k \leq n$.

Since $h_n = 0 : \pi_n(X) \to H_n(X)$, we have $h_n(\alpha) = 0$ in $H_n(X)$ for any $\alpha \in \pi_n(X)$. This implies $\alpha_* = 0 : H_n(S^n) \to H_n(X)$ and hence $\alpha_* q_* = 0 : H_k(X) \to H_k(X)$ for any $k > 0$. We remark that $X$ is a 1-connected $n$-dimensional complex by the assumption. Hence by Proposition 1.1 (1), we have $\iota^\alpha \in \mathcal{E}_\#(X)$ for any element $\alpha$ of $\pi_n(X)$.

Let $c : X \to X \vee S^n$ be the co-action and $i_1 : X \to X \vee S^n$ be the inclusion map to the first factor. Then we have

$$
cj = i_1 j : Y \to X \vee S^n.
$$

Let $\iota : X \to X$ be the identity map. It follows that for any element $\delta \in \pi_k(X)$ for $k \leq n$, there exists an element $\beta \in \pi_k(Y)$ such that $j_\#(\beta) = \delta$ by the discussion above. Therefore we have

$$
(\iota^\alpha)_\#(\delta) = \nabla(\iota \vee \alpha) c \delta \\
= \nabla(\iota \vee \alpha) c j \beta \\
= \nabla(\iota \vee \alpha) i_1 j \beta \\
= 1_X j \beta = \delta.
$$

Thus $\theta(\alpha) \in \mathcal{E}_\#(X)$. Suppose that $\alpha$ and $\beta$ are two elements in $\pi_n(X)$. We remark that $(\iota^\alpha)_\#(\beta) = \beta$ since $\iota^\alpha = \theta(\alpha) \in \mathcal{E}_\#(X)$ as is shown above. It follows that

$$
\theta(\alpha + \beta) = \iota^{\alpha + \beta} = \iota^{\alpha + (\iota^\alpha)_\#(\beta)} = (\iota^\alpha)(\iota^\alpha) \iota(\beta) = (\iota^\alpha \iota^\alpha)^{\iota^\alpha \beta} = \iota^\alpha \iota^\beta = \theta(\alpha) \theta(\beta)
$$

Therefore, $\theta$ is a homomorphism.
by the formulas $f^{\alpha+\beta} = (f^\alpha)^\beta$ and $(hf)^{\rho} = h(f)^{\rho}$. This completes the proof. \hfill \Box

P. J. Kahn [7] has proved the following:

**Theorem 2.3.** Let $X$ be a homotopy type of the mapping cone of a map $\gamma : S^{2n-1} \to Y = X_n$, $n \geq 2$, where $X_n$ is the $r$-fold bouquet of $n$-spheres, $r = \text{rank}H_n(X) \geq 1$. Then there exists an exact sequence

$$[\Sigma Y, X] \xrightarrow{\Sigma \gamma^* + \Psi} \pi_{2n}(X) \xrightarrow{\theta} \mathcal{E}_\#(X) \xrightarrow{R} \mathcal{E}_\#(Y),$$

where $R$ is given by restriction to $Y$.

We will slightly extend above theorem.

**Theorem 2.4.** Let $n \geq 2$ and $Y$ be 1-connected homotopy associative $co$-$H$-space. Let $X = Y \cup_\gamma e^n$ with finite $n$'th homotopy group and $X^{n-2} = Y$. Then there exists an exact sequence

$$[\Sigma Y, X] \xrightarrow{\Gamma} \pi_n(X) \xrightarrow{\theta} \mathcal{E}_\#(X) \xrightarrow{R} \mathcal{E}_\#(Y),$$

where $R$ is given by restriction to $Y$.

**Proof.** By Corollary 3.2.2 of Rutter [13] (cf. also Lemmas 2.7, 2.8 and 2.9 of [12]), we have the following exact sequence:

$$[\Sigma Y, X] \xrightarrow{\Gamma(j, \gamma)} [S^n, X] \xrightarrow{\theta} [X, X]_{1X} \xrightarrow{j^*} [Y, X]_j \xrightarrow{\gamma^*} [S^{n-1}, X],$$

where $[X, X]_{1X}$ and $[Y, X]_j$ show that $1_X$ and $j$ are base points of each homotopy sets.

By Theorem 2.2, the image of $\theta$ is contained $\mathcal{E}_\#(X)$ because the Hurewicz map $h_n = 0 : \pi_n(X) \to H_n(X)$ by the assumption that $\pi_n(X)$ is a finite group. Next we show that the image of $R (= j^*)$ is contained in $\mathcal{E}_\#(Y)$. Consider the following commutative diagram

$$\begin{array}{ccc}
\pi_i(X) & \xrightarrow{f_k = 1} & \pi_i(X) \\
\downarrow j_{\pi} & & \downarrow j_{\pi} \\
\pi_i(Y) & \xrightarrow{R(f)_{\#}} & \pi_i(Y) \\
\downarrow \partial & & \downarrow \partial \\
\pi_{i+1}(X, Y) = 0 & & \pi_{i+1}(X, Y) = 0
\end{array}$$

for any $f \in \mathcal{E}_\#(X)$ and each $k \leq n - 2$. From the above diagram, we have

$$j_{\#}(\alpha) = f_{\#}j_{\#}(\alpha) = j_{\#}R(f)_{\#}(\alpha) = j_{\#}(R(f)_{\#}(\alpha)).$$
Since \( j_1 \) is injective, we have \( \alpha = R(f)_1(\alpha) \) for any \( k \leq \dim Y \).

Then the exactness of the sequence is obtained by the exact sequence of Rutter.

Hence if we use the above Theorem and [7, Lemma 4], we have the following

**Corollary 2.1.** Let \( X \) be a closed compact, oriented, \( C^\infty \), \((n - 1)\)-connected \( 2n \)-manifold, \( n \geq 2 \). Then there is an exact sequence

\[
[\Sigma Y, X] \to \pi_{2n}(X) \stackrel{\theta}{\to} \mathcal{E}_\#(X) \stackrel{R}{\to} 0.
\]

Especially, if we let \( X = S^n \times S^n \), then we have

\( \mathcal{E}_\#(X) \) is isomorphic to \( \pi_{2n}(X) \) if \( n = 2, 6 \) or \( 3 \) (mod 4)

\( \mathcal{E}_\#(X) \) is isomorphic to \( \pi_{2n}(X)/(Z_2 \oplus Z_2) \) if \( n \neq 2, 6 \) or \( 3 \) (mod 4).

**Proof.** If we consider the following exact sequence

\[
[\Sigma Y, X] \to \pi_{2n}(X) \to \mathcal{E}_\#(X) \to \mathcal{E}_\#(Y)
\]

from the above theorem, it is sufficient to show that the image of \( R \) is trivial. It will be proved by the same method of [2, Proposition 6.1] that the kernel of \( R \) is \( \mathcal{E}_\#(X) \).

Next we will show a condition that there exists a morphism

\[
\theta : \pi_n(X) \to Z^n \#(X, X).
\]

Recall the generalized Gottlieb group [16] of \( \pi_n(W) \), denoted by \( G_n^f(W, X) \), consists of those \( \alpha \in \pi_n(W) \) for which there is an associated map \( F : X \times S^n \to W \) such that the following diagram is homotopy commutative:

\[
\begin{array}{ccc}
X \times S^n & \xrightarrow{F} & W \\
j \downarrow & & \downarrow \psi \\
X \vee S^n & \xrightarrow{f \vee \alpha} & W \vee W
\end{array}
\]

Since \( G_n(W) = G_n^i(W, W) \), this group is a generalization of Gottlieb group and it is clear that \( G_n(W) \subset G_n^f(W, X) \subset \pi_n(W) \).

By the new notations and Proposition 1.1, we have the following:

**Corollary 2.2.** Let \( X = Y \cup_{\gamma} e^n \) be a 1-connected CW-complex, \( \alpha \in \pi_n(W) \) and \( f \in [X, W] \). Then the following results hold.

1. \( f^\alpha \in M^f_\#(X, W) \) if and only if \( \alpha_* q_* = 0 : H_n(X) \to H_n(W) \).

2. Suppose that \( \alpha \in G_n^f(W, X) \). Then \( f^\alpha \in M^f_\#(X, W) \) if and only if \( \alpha q \in M^0_\#(X, W) \).
Here we want to show a condition for $\pi_n(W)$ to be contained in $\theta^{-1}(\mathcal{M}_\#^f(X, W))$.

**Theorem 2.5.** Let $n \geq 2$ and $Y$ be 1-connected. Let $X = Y \cup \tau \epsilon^n$ with $X^{n-1} = Y$ and $\dim W \leq n$ and $f \in [X, W]$. If $q_\#: 0 : \pi_n(X) \twoheadrightarrow \pi_n(S^n)$, then there exists a morphism

$$\Theta : \pi_n(W) \twoheadrightarrow \mathcal{M}_\#^f(X, W)$$

which is defined by $\Theta(\alpha) = f^\alpha$ for any $\alpha \in \pi_n(W)$. This morphism restricts to

$$\Theta' : \ker \{ h_n : \pi_n(W) \twoheadrightarrow H_n(W) \} \to \mathcal{M}_*^f(X, W) \cap \mathcal{M}_\#^f(X, W),$$

where $h_n : \pi_n(W) \twoheadrightarrow H_n(W)$ is the Hurewicz homomorphism.

**Proof.** By Blaker-Massey Theorem (7.12) on p. 368 (Chapter VII) of [15], we see that

$$p_\#: \pi_n(X, Y) \twoheadrightarrow \pi_n(X/Y) = \pi_n(S^n)$$

is an isomorphism when $Y$ is 1-connected. We consider the long homotopy exact sequence

$$\cdots \to \pi_n(Y) \xrightarrow{j_\#} \pi_n(X) \xrightarrow{k_\#} \pi_n(X, Y) \to \cdots$$

Since $p_\#k_\# = q_\# = 0 : \pi_n(X) \to \pi_n(S^n)$ and $p_\#$ is an isomorphism, we see that $k_\# = 0 : \pi_n(X) \to \pi_n(X, Y)$ and hence $j_\#: \pi_n(Y) \to \pi_n(X)$ is surjective and hence $j_\#: \pi_i(Y) \to \pi_i(X)$ is surjective for any $i \leq n$. Hence for any element $\delta \in \pi_i(X)$ for $i \leq \dim W \leq n$, there exists an element $\beta \in \pi_i(Y)$ such that $j_\#(\beta) = \delta$. Let $c : X \to X \vee S^n$ be the co-action and $i_1 : X \to X \vee S^n$ be the inclusion map to the first factor. Then as in the proof of Theorem 2.2 we have

$$cj = i_1j : Y \to X \vee S^n.$$ 

Hence we have

$$(f^\alpha)_\#(\delta) = \nabla(f \vee \alpha)c\delta$$

$$= \nabla(f \vee \alpha)cj\beta$$

$$= \nabla(f \vee \alpha)i_1j\beta$$

$$= fj\beta = f_\#(\delta).$$

It follows that $\Theta(\alpha) = f^\alpha \in \mathcal{M}_\#^f(X, W)$.

Now suppose that $h_n(\alpha) = 0$ for an element $\alpha \in \pi_n(W)$. Then $\alpha_* = 0 : H_n(S^n) \to H_n(W)$. It follows that $\alpha_*q_\# = 0 : H_i(X) \to H_i(S^n) \to H_i(W)$ for any $i > 0$ and hence $f^\alpha \in \mathcal{M}_*^f(X, W)$ by Proposition 1.1 (1).
The following result is (partly) a generalization of Theorem 2.3 of [1]. (In Theorem 2.3 of [1], the condition \( X^{n-1} = Y \) is not assumed.)

**COROLLARY 2.3.** Let \( n \geq 2 \) and \( Y \) be 1-connected. Let \( X = Y \cup \gamma e^n \) with \( X^{n-1} = Y \). If \( q_\# = 0 : \pi_n(X) \rightarrow \pi_n(S^n) \), then there exists a homomorphism

\[
\theta : \pi_n(X) \rightarrow \mathcal{E}_*(X) \cap \mathcal{E}_\#(X)
\]

defined by \( \theta(\alpha) = \iota^\alpha \) for any \( \alpha \in \pi_n(X) \).

**Proof.** If \( q_\# = 0 : \pi_n(X) \rightarrow \pi_n(S^n) \), then \( j_\# : \pi_n(Y) \rightarrow \pi_n(X) \) is surjective as is shown in the proof of Theorem 2.5, and hence we see that \( i_n : \Gamma_n(X) \rightarrow \pi_n(X) \) is surjective by the assumption that \( X^{n-1} = Y \). It follows that the Hurewicz homomorphism \( h_n : 0 : \pi_n(X) \rightarrow H_n(X) \), and hence Theorem 2.2 implies the result.

**COROLLARY 2.4.** Let \( n \geq 2 \) and \( Y \) be 1-connected. Let \( X = Y \cup \gamma e^n \) with \( X^{n-1} = Y \). If \( q_\# = 0 : \pi_n(X) \rightarrow \pi_n(S^n) \), then there exists a morphism

\[
\zeta : \pi_n(X) \rightarrow \mathcal{Z}_\#^0(X, X) = \mathcal{M}_\#^0(X, X)
\]

defined by \( \zeta(\alpha) = 0^\alpha = \alpha q \) for any \( \alpha \in \pi_n(X) \). This morphism restricts to

\[
\zeta' : \text{Ker}\{ h_n : \pi_n(X) \rightarrow H_n(X) \} \rightarrow \mathcal{M}_\#^0(X, X) \cap \mathcal{Z}_\#^0(X, X).
\]

The function \( \zeta \) defined above satisfies \( \zeta(\alpha)\zeta(\beta) = 0 \) for any \( \alpha, \beta \in \pi_n(X) \).

**Proof.** By Proposition 2.6 of [11], we see that \( \zeta(\alpha) = 0^\alpha = \ast + \alpha = q^*(\alpha) = \alpha q \). Then the result follows by putting \( X = W \) in Theorem 2.5.

Arkowitz, Lupton and Murillo [1] showed a space \( X \) with the homomorphism \( \theta : G_n(X) \neq 0 \rightarrow \mathcal{E}_\#(X) \) but they didn’t show that the homomorphism is nontrivial. Next theorem gives a condition for \( X \) which does not have a nontrivial homomorphism given by \( \theta(\alpha) = \iota^\alpha \).

**THEOREM 2.6.** Let \( X = Y \cup \gamma e^n \) be a 1-connected \( n \)-dimensional CW-complex. Suppose that \( q : X \rightarrow S^n \) has a right homotopy inverse. If \( \theta : G_n(X) \rightarrow \mathcal{E}_\#(X) \) given by \( \theta(\alpha) = \iota^\alpha \) is a well-defined homomorphism, then \( G_n(X) \) is trivial.

**Proof.** Let \( S^{n-1} \rightarrow Y \rightarrow X \xrightarrow{q} \Sigma S^{n-1} = S^n \) be the mapping cone sequence for \( X = Y \cup \gamma e^n \). For any \( \alpha \in G_n(X) \), we have \( \iota^\alpha \in \mathcal{E}_\#(X) \) if and only if \( \alpha_\# q_\# = 0 : \pi_i(X) \rightarrow \pi_i(X) \) for \( i \leq n \) by Corollary 2.2.
(2) of [1]. Let \( s \in \pi_n(X) \) be a right homotopy inverse of \( q \). Then for \( \iota S^n \in \pi_n(S^n) \), we have
\[
\alpha = \alpha \iota S^n = \alpha \iota (\iota S^n) = (\alpha q s) \iota (\iota S^n) = \alpha \iota q (s) = 0.
\]
Therefore \( G_n(X) \) is trivial. \( \square \)

**Corollary 2.5.** Let \( X = Y \cup_r e^n \) be a 1-connected \( n \)-dimensional CW-complex. If \( \theta : G_n(X) \to E \#(X) \) given by \( \theta(\alpha) = \iota^\alpha \) is a well-defined homomorphism and nontrivial, then \( G_n(X) \neq 0 \) and consequently the map \( q : X \to S^n \) does not have a right homotopy inverse.

The following example has been considered in [1], but here we extend the domain from \( G_n(X) \) to \( \pi_n(X) \) and reduce the codomain from \( E \#(X) \) to \( E_*(X) \cap E \#(X) \).

**Example.** Let \( X = S^2 \times S^3 = S^2 \vee S^3 \cup_{[i_1, i_2]} e^5 \). Since \( H_5(X) \) is infinite cyclic, the Hurewicz homomorphism \( h_5 : \pi_5(X) \to H_5(X) \) is zero. Now \( Y = S^2 \vee S^3 \) is \( X^4 \) and hence \( X \) satisfies all the hypothesis in Theorem 2.2. Therefore we have a homomorphism
\[
\theta : \pi_5(X) \to E_*(X) \cap E \#(X).
\]
Let \( S^4 \xrightarrow{[i_1, i_2]} S^2 \vee S^3 \to X \xrightarrow{q} \Sigma S^4 = S^5 \) be the mapping cone sequence for \( X = S^2 \times S^3 = S^2 \vee S^3 \cup_{[i_1, i_2]} e^5 \) and \( F_q \) be the homotopy fibre of \( q : X \to S^5 \). Since \( G_5(X) \) is nontrivial, \( G_4(F_q) \) is nontrivial by Theorem 2.6. Because if \( G_4(F_q) = 0 \), the map \( q : X \to S^5 \) has a right homotopy inverse (see [5, Corollary 2.7]).

3. CoGottlieb groups and a subgroup of self-homotopy equivalences

Let \((f, \alpha) : W \xrightarrow{\Delta} W \times W \xrightarrow{f \times \alpha} X \times \Omega Z\) be the composite map of the maps \( f \in [W, X] \) and \( \alpha \in [W, \Omega Z] \). A map \( f : W \to X \) is said to be *cocyclic* [14] if there exists a map \( \Phi : W \to W \vee X \) such that the following diagram is homotopy commutative.

\[
\begin{array}{ccc}
W & \xrightarrow{\Phi} & W \vee X \\
\downarrow_{(\iota, f)} \quad & \quad \downarrow j \quad & \quad \\
W \times X & \quad & \\
\end{array}
\]
We denote by $H^n(X; G)$ the cohomology group of $X$ with coefficient group $G$, and we simply write $H^n(X) = H^n(X; Z)$ for the integral cohomology group of $X$.

**Definition 3.1.** \( G^n(X) = \{ \alpha \in H^n(X) \mid \alpha \text{ is cocyclic} \} \) is said to be a coGottlieb group.

Let \( \Omega Z \xrightarrow{q} X \xrightarrow{h} Y \xrightarrow{\gamma} Z \) be a fibration sequence. This sequence gives an action of \( \Omega Z \) on \( X \) by \( \mu : X \times \Omega Z \to X \). The induced action of \( [W, \Omega Z] \) on \( [W, X] \) is given by

\[
f^\alpha = \mu(f \times \alpha) \Delta : W \xrightarrow{\Delta} W \times W \xrightarrow{f \times \alpha} X \times \Omega Z \xrightarrow{\mu} X
\]

for any \( \alpha \in [W, \Omega Z] \) and \( f \in [W, X] \).

**Remark.** We will use the same notation \( f^\alpha \) for the composite maps \( f^\alpha = \mu(f \times \alpha) \Delta \) and the \( f^\alpha \) used in the previous sections, for the convenience, because we can easily distinguish them.

**Lemma 3.1.** Let \( \alpha \in [W, \Omega Z] \) and \( f \in [W, X] \). Then this action satisfies the following:

1. If \( h : W' \to W \), then \( (f^\alpha)_h = (fh)^{\alpha h} \).
2. If \( \alpha, \beta \in [W, \Omega Z] \), then \( (f^\alpha)^\beta = f^{(\alpha + \beta)} \).

**Proof.** The first is clear, so we will prove the second case: (2) We have the following relation

\[
(f^\alpha)^\beta = \mu(f^\alpha \times \beta) \Delta = \mu((\mu(f \times \alpha) \Delta) \times \beta) \Delta
\]

\[
= \mu(\mu \times 1_{\Omega Z})((f \times \alpha) \times \beta)((\Delta \times 1_W) \Delta
\]

\[
= \mu(1_X \times m)((f \times (\alpha \times \beta)))(1_W \times (\Delta) \Delta
\]

\[
= \mu((f \times (\alpha + \beta)))\Delta = f^{(\alpha + \beta)}
\]

by using the following homotopy commutative diagram:

\[
\begin{array}{cccccc}
W \times W & \xrightarrow{\Delta \times 1_W} & W \times W \times W & \xrightarrow{(f \times \alpha) \times \beta} & X \times \Omega Z \times \Omega Z & \xrightarrow{\mu \times 1_{\Omega Z}} & X \\
& & & & & & X \\
W & \xrightarrow{\Delta} & W \times W & \xrightarrow{f \times (\alpha \times \beta)} & X \times \Omega Z \times \Omega Z & \xrightarrow{1_X \times m} & X \times \Omega Z
\end{array}
\]

**Theorem 3.1.** Let \( \Omega Z \xrightarrow{q} X \xrightarrow{h} Y \xrightarrow{\gamma} Z \) be a fibration sequence. Let \( f \in [W, X] \) and \( \alpha \in [W, \Omega Z] \). Then the following formulas hold for any \( i > 0 \).
(1) The induced homotopy homomorphism $f^\alpha: \pi_i(W) \to \pi_i(X)$ satisfies
\[ f^\alpha(x) = f_\#(x) + q_\# \alpha_\#(x) \]
for any $x \in \pi_i(W)$.

(2) If $(f, \alpha): W \to X \times \Omega Z$ factors through $X \vee \Omega Z$, then the induced cohomology homomorphism $f^{\alpha*}: H^i(X; G) \to H^i(W; G)$ satisfies
\[ f^{\alpha*}(x) = f^*(x) + \alpha^* q^*(x) \]
for any $x \in H^i(X; G)$ and any abelian group $G$.

Proof. (1) Consider the following diagram:

\[
\begin{array}{c}
\pi_i(W) \xrightarrow{\Delta} \pi_i(W \times W) \xrightarrow{(f \times \alpha)_\#} \pi_i(X \times \Omega Z) \xrightarrow{\mu_\#} \pi_i(X) \\
\pi_i(W) \oplus \pi_i(W) \xrightarrow{\Delta} \pi_i(X) \oplus \pi_i(\Omega Z) \xrightarrow{\mu_\#} \pi_i(X) \oplus \pi_i(\Omega Z)
\end{array}
\]

The right-most triangle commutes since $\mu|X \times \{\ast\} \simeq 1_X$ and $\mu|\{\ast\} \times \Omega Z \simeq q$ (cf. [11, Proposition 3.4 (2)]). The fact that the other two parts of the diagram commute is obvious. By definition, the composite of the homomorphisms on the top line is the homomorphism induced by $f^\alpha$. The other way to go around the diagram gives the desired formula.

(2) By definition, $(f^\alpha)^*(x)$ is represented by
\[ W \xrightarrow{(f, \alpha)} X \times \Omega Z \xrightarrow{\mu} X \xrightarrow{x} K(G, i) \]

On the other hand, by definition of the sum of cohomology classes in $H^*(W; Z)$, $f^*(x) + \alpha^* q^*(x)$ is represented by
\[ W \xrightarrow{(f, \alpha)} X \times \Omega Z \xrightarrow{x \times q} K(G, i) \times K(G, i) \xrightarrow{\mu} K(G, i). \]

Now, by the given condition, $(f, \alpha)$ is the composite $W \xrightarrow{\Phi} X \vee \Omega Z \xrightarrow{j} X \times \Omega Z$, where $j : X \vee \Omega Z \to X \times \Omega Z$ is the natural inclusion. The second part follows from the fact that the following two maps are homotopic:
\[ X \vee \Omega Z \xrightarrow{j} X \times \Omega Z \xrightarrow{x \times q} K(G, i) \times K(G, i) \xrightarrow{\mu} K(G, i). \]

\[ X \vee \Omega Z \xrightarrow{j} X \times \Omega Z \xrightarrow{x \times q} K(G, i) \times K(G, i) \xrightarrow{\mu} K(G, i). \]

\[ X \vee \Omega Z \xrightarrow{j} X \times \Omega Z \xrightarrow{x \times q} K(G, i) \times K(G, i) \xrightarrow{\mu} K(G, i). \]

Let $E^*(X) = \{ f \in E(X) \mid f^* = 1 : H^i(X) \to H^i(X) \text{ for } i \leq N \}$, where $N$ is the homotopical dimension of $X$ denoted by $h$-dim $X$. One use this subgroup instead of requiring $f^* = 1$ for all $i$ because only the
present subgroup is nilpotent and commutes with the rationalization operation when \( X \) has finite h-dim \( X \).

Let \( N = \text{h-dim} \ X = \max \{ i \mid \pi_i(X) \neq 0 \} \). We define
\[
\mathcal{E}'(X) = \{ f \in \mathcal{E}(X) \mid f^* = 1 : H^i(X) \to H^i(X) \text{ for any } i \leq N \}.
\]

**Corollary 3.1.** Let \( X \) be the homotopy fibre of a map \( \gamma : Y \to K(Z, n + 1) \) and \( \alpha \in H^n(X) \). Then
\[
(1) \quad \iota^\alpha \in \mathcal{E}_{\#\infty}(X) \text{ if and only if } q_{\#} \alpha = 0 : \pi_n(X) \to \pi_n(K(Z, n)) \to \pi_n(X).
\]
\[
(2) \quad \text{Let } \alpha \in G^n(X) \subset H^n(X) \text{ and } N = \text{h-dim} \ X. \text{ Then } \iota^\alpha \in \mathcal{E}'(X)
\text{ if and only if } \alpha^* q^* = 0 : H^i(X) \to H^i(K(Z, n)) \to H^i(X) \text{ for any } i \leq N.
\]

**Proof.** (1) is obtained by Theorem 3.1 (1).

(2) By Theorem 3.1 (2), we see that the condition \( \iota^\alpha \in \mathcal{E}'(X) \) implies \( \alpha^* q^* = 0 : H^i(X) \to H^i(X) \) for any \( i \leq N \).

To prove the converse, we consider two cases separately: The case \( N < n \): Consider the following composite of homomorphisms.

\[
q_{\#} \alpha : \pi_i(X) \to \pi_i(K(Z, n)) \to \pi_i(X).
\]

Since \( \pi_i(X) = 0 \) for any \( i > N \), we see that \( q_{\#} \alpha = 0 : \pi_i(X) \to \pi_i(X) \) for any \( i > 0 \). Then \( (\iota^\alpha)_{\#} = \text{id} : \pi_i(X) \to \pi_i(X) \) for any \( i > 0 \) by Theorem 3.1 (1) and hence \( (\iota^\alpha)_{\#} \) is a homotopy equivalence.

The case \( N \geq n \): By the assumption we see
\[
\alpha^* q^* = 0 : H^n(X) \to H^n(K(Z, n)) \to H^n(X),
\]
and hence \( (\iota^\alpha)^*(x) = x \) for any \( x \in H^n(X) \) by Theorem 3.1 (2). It follows that
\[
\iota^{-\alpha} \iota^\alpha = (\iota^\alpha)^{-\alpha} \iota^\alpha = (\iota^\alpha)^{-1} \alpha \iota^\alpha = (\iota^\alpha)^{-\alpha} \iota^\alpha = \iota^{-\alpha} \iota^\alpha = \iota^0 = \iota;
\]
\[
\iota^\alpha \iota^{-\alpha} = (\iota^\alpha)^{\alpha} \iota^{-\alpha} = (\iota^\alpha)^{-1} \alpha \iota^{-\alpha} = (\iota^\alpha)^{-\alpha} \iota^{-\alpha} = \iota^{-\alpha} \iota^\alpha = \iota^0 = \iota.
\]

by Lemma 3.1. Hence \( \iota^\alpha \) is a homotopy equivalence. \( \square \)

**Theorem 3.2.** Let \( X \) be the homotopy fiber of a map \( \gamma : Y \to K(Z, n + 1) \) and \( n \) is h-dim \( X \). Suppose \( q^* = 0 : H^n(X) \to H^n(K(Z, n)) \).

Then there is a homomorphism \( \theta : G^n(X) \to \mathcal{E}'(X) \).

**Proof.** The condition \( q^* = 0 : H^n(X) \to H^n(K(Z, n)) \) implies that \( q^* = 0 : H^i(X) \to H^i(K(Z, n)) \) for \( i \leq n \). It follows that \( \iota^\alpha \in \mathcal{E}'(X) \) by Corollary 3.1. Let \( \alpha, \beta \in G^n(X) \) be any elements. We see
\[
\alpha \beta = (\iota^\beta)^*(\alpha) = \iota^* \alpha + \beta^* q^*(\alpha) = \alpha
\]
by Theorem 3.1 (2). Since
\[(\iota^\alpha)^\beta = \iota^{\alpha+\beta}\]
and
\[\iota^\alpha \iota^\beta = (\iota^\beta)\iota^\alpha = \iota^{\beta+\alpha} = \iota^{\beta+\alpha},\]
we have
\[\theta(\alpha + \beta) = \iota^{\alpha+\beta} = \iota^{\beta+\alpha} = \iota^\alpha \iota^\beta = \theta(\alpha) \theta(\beta).\]
Therefore \(\theta\) is a homomorphism.

We define a map
\[DH : H^n(X) = [X, K(Z, n)] \longrightarrow \text{Hom}(\pi_n(X), Z)\]
by \(DH(x) = x_\sharp : \pi_n(X) \rightarrow \pi_n(K(Z, n)).\)

**Corollary 3.2.** Let \(X\) be the homotopy fiber of a map \(\gamma : Y \rightarrow K(Z, n+1)\) and \(n\) is h-dim \(X\). Suppose \(q^* = 0 : H^n(X) \rightarrow H^n(K(Z, n)).\) Then there is a homomorphism
\[\theta' : \text{Ker}(DH) \cap G^n(X) \rightarrow \mathcal{E}_n'(X) \cap \mathcal{E}'_{\#\infty}(X)\]
given by \(\theta'(\alpha) = \iota^\alpha\).

**Proof.** Let \(\alpha\) be an element of \(\text{Ker}(DH)\). Then \(\alpha_\sharp = 0 : \pi_n(X) \rightarrow \pi_n(K(Z, n)).\) Since \(\pi_i(K(Z, n)) = 0\) for all \(i \neq n\), we see \(q_\sharp \alpha_\sharp = 0 : \pi_i(X) \rightarrow \pi_i(X)\) for all \(i\). Therefore \(\theta(\alpha) = \iota^\alpha \in \mathcal{E}_{\#\infty}(X)\) by Corollary 3.1 and the proof is completed.

**Theorem 3.3.** Let \(X\) be the homotopy fiber of a map \(\gamma : Y \rightarrow K(Z, n+1)\) and \(n\) is h-dim \(X\). Suppose \(q : K(Z, n) \rightarrow X\) has a left homotopy inverse. If \(\theta : G^n(X) \rightarrow \mathcal{E}_n'(X)\) given by \(\theta(\alpha) = \iota^\alpha\) is a well-defined homomorphism, then \(G^n(X) = 0\).

**Proof.** Suppose that \(\theta : G^n(X) \rightarrow \mathcal{E}_n'(X)\) given by \(\theta(\alpha) = \iota^\alpha\) is a well-defined homomorphism. Then for any \(\alpha \in G^n(X)\), we see \(\alpha^* q^* = 0 : H^i(X) \rightarrow H^i(X)\) for \(i \leq n\) by Corollary 3.1. Let \(s \in H^n(X)\) be a left homotopy inverse of \(q\). Then for \(\iota_{K(Z, n)} \in H^n(K(Z, n))\), we have
\[\alpha = \iota_{K(Z, n)} \alpha = \alpha^*(\iota_{K(Z, n)}) = \alpha^*(sq)^*(\iota_{K(Z, n)}) = \alpha^* q^*(s) = 0.\]
Therefore \(G^n(X)\) is trivial.

**Corollary 3.3.** Let \(X\) be the homotopy fiber of a map \(\gamma : Y \rightarrow K(Z, n+1)\) and \(n\) is h-dim \(X\). Suppose the homomorphism \(\theta : G^n(X) \rightarrow \mathcal{E}_n'(X)\) given by \(\theta(\alpha) = \iota^\alpha\) is well-defined and nontrivial. Then \(G^n(X) \neq 0\) and consequently the map \(q : K(Z, n) \rightarrow X\) does not have a left homotopy inverse.
To work rationally, one has to define $D_{Q}$ instead of $D_{H}$ and one needs the following rational version of Corollary 3.1.

We now assume that all the spaces are 1-connected rational spaces. We define

$$D_{Q}: H^{n}(X; Q) \rightarrow \text{Hom}(\pi_{n}(X), Q)$$

by $D_{Q}(x) = x^{\#} : \pi_{n}(X) \rightarrow \pi_{n}(K(Q, n))$ for any $x \in H^{n}(X; Q) = [X, K(Q, n)]$. Moreover we define

$$G_{Q}^{n}(X) = \{ \alpha \in H^{n}(X; Q) \mid \alpha \text{ is cocyclic} \},$$

$$\mathcal{E}_{Q}^{*}(X) = \{ f \in \mathcal{E}(X) \mid f^{*} = 1 : H^{i}(X; Q) \rightarrow H^{i}(X; Q) \}
\text{for any} \ i \leq N = \text{h-dim}(X) \}.$$

**Corollary 3.1'.** (Rational case) Let $X$ be the homotopy fibre of a map $\gamma : Y \rightarrow K(Q, n + 1)$ and $\alpha \in H^{n}(X; Q)$. Then

1. $\iota^{\alpha} \in \mathcal{E}_{\#}(X)$ if and only if $q_{\#}\alpha^{\#} = 0 : \pi_{n}(X) \rightarrow \pi_{n}(K(Q, n)) \rightarrow \pi_{n}(X)$.
2. Let $\alpha \in G_{Q}^{n}(X) \subset H^{n}(X)$ and $N = \text{h-dim}(X)$. Then $\iota^{\alpha} \in \mathcal{E}_{Q}^{*}(X)$ if and only if $\alpha^{*}q^{*} = 0 : H^{i}(X; Q) \rightarrow H^{i}(K(Q, n); Q) \rightarrow H^{i}(X; Q)$ for any $i \leq N$.

**Proof.** The proof of Corollary 3.1 can be applied changing the coefficient group $Z$ to $G$. \qed

**Lemma 3.2.** Suppose that we have a fibration sequence

$$X \xrightarrow{j} Y \xrightarrow{\gamma} K(Q, n + 1)$$

with $X$ and $Y$ 1-connected. If $\gamma : Y \rightarrow K(Q, n + 1)$ is nontrivial, then $j^{*} : H^{n}(Y; Q) \rightarrow H^{n}(X; Q)$ is surjective.

**Proof.** The Lemma is a direct consequence of the “Wang” sequence of the given fibration. The original Wang sequence is an infinite long exact sequence and applies only to fibrations with sphere as a base space. A usual argument using the Serre spectral sequence gives a part of Wang sequence which is enough for our purpose. \qed

**Theorem 3.4.** Let $X$ be the homotopy fibre of a map $\gamma : Y \rightarrow K(Q, n + 1)$ and assume that $n \equiv 1 \pmod{2}$. If $\gamma$ is nontrivial, then there exists a homomorphism

$$\theta : G_{Q}^{n}(X) \rightarrow \mathcal{E}_{Q}^{*}(X).$$

The homomorphism $\theta$ defined above restricts to

$$\theta' : \text{Ker}(D_{Q}) \cap G_{Q}^{n}(X) \rightarrow \mathcal{E}_{Q}^{*}(X) \cap \mathcal{E}_{\#}(X).$$
Proof. By Lemma 3.2, we see that $q^* = 0 : H^n(X; Q) \to H^n(K(Q, n); Q)$. Since $K(Q, n) \simeq S^n$ when $n \equiv 1 \pmod{2}$, we see that $q^* = 0 : H^i(X; Q) \to H^i(K(Q, n); Q)$ for any $i > 0$. Hence by Theorem 3.1 (2), we have $\iota^\alpha \in \mathcal{E}^*_Q(X)$ for any $\alpha \in G^0_Q(X)$.

If $\alpha \in \text{Ker} \{ DH_Q : H^n(X; Q) \to \text{Hom}(\pi_n(X), Q) \}$, then we see that $q_2 \alpha = 0 : \pi_n(X) \to \pi_n(K(Q, n)) \to \pi_n(X)$, and hence $\iota^\alpha \in \mathcal{E}_{#\infty}(X)$ by Theorem 3.1 (1). \hfill \Box

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References


Jae-Ryong Kim  
Department of Mathematics  
Kookmin University  
Seoul 136-702, Korea  
E-mail: kimjr@kookmin.ac.kr

Nobuyuki Oda  
Department of Applied Mathematics  
Fukuoka University  
Fukuoka 814-0180, Japan  
E-mail: odanobu@cis.fukuoka-u.ac.jp

Jianzhong Pan  
Institute of Mathematics  
Academy Sciences of China  
Beijing, China  
E-mail: pjz@mail.amss.ac.cn

Moo Ha Woo  
Department of Mathematics Education  
Korea University  
Seoul 136-701, Korea  
E-mail: woomh@korea.ac.kr