SPECTRAL AREA ESTIMATES FOR NORMS OF COMMUTATORS

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ABSTRACT. Let $A$ and $B$ be commuting bounded linear operators on a Hilbert space. In this paper, we study spectral area estimates for norms of $A^*B - BA^*$ when $A$ is subnormal or $p$-hyponormal.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. If $T$ is a hyponormal operator in $\mathcal{B}(\mathcal{H})$ then C. R. Putnam [7] proved that $\| T^*T - TT^* \| \leq \text{Area}(\sigma(T))/\pi$, where $\sigma(T)$ is the spectrum of $T$. The second named author [5] has proved that if $T$ is a hyponormal operator and $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$ then

$$\| T^*K - KT^* \| \leq 2\{\text{Area}(\sigma(T))/\pi\}^{1/2}\|K\|.$$ 

We don’t know whether the constant 2 in the inequality is best possible for a hyponormal operator. In §2, we show that the constant is not best possible for a subnormal operator.

When $T$ is a $p$-hyponormal operator in $\mathcal{B}(\mathcal{H})$, A. Uchiyama [10] generalized the Putnam inequality, that is,

$$\| T^*T - TT^* \| \leq \phi \left(\frac{1}{p}\right) \|T\|^{2(1-p)}\{\text{Area}(\sigma(T))/\pi\}^p.$$

This inequality gives the Putnam inequality when $p = 1$. In §3, we generalize the above inequality for the spectral area estimate of $\| T^*K - KT^* \|$ when $TK = KT$. H. Alexander [1] proved the following inequality for a uniform algebra $A$. If $f$ is in $A$ then

$$\text{dist}(\tilde{f}, A) \leq \{\text{Area}(\sigma(f))/\pi\}^{1/2}.$$ 

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The second named author [5] gave an operator version for the Alexander inequality. This was used in order to estimate \( \| T^*K - KT^* \| \) when \( T \) is a hyponormal operator and \( KT = TK \). We also give an Alexander inequality for a \( p \)-hyponormal and we use it to estimate \( \| T^*K - KT^* \| \).

In §4, we try to estimate \( \| T^*K - KT^* \| \) for arbitrary contraction. In §5, we show a few results about area estimates for \( p \)-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For \( 0 < p \leq 1 \), \( T \) is said to be \( p \)-hyponormal if \( (T^*T)^p - (TT^*)^p \geq 0 \). A 1-hyponormal operator is hyponormal. For an algebra \( \mathcal{A} \) in \( \mathcal{B}(\mathcal{H}) \), let \( \text{lat}\mathcal{A} \) be the lattice of all \( \mathcal{A} \)-invariant projections. For a compact subset \( X \) in \( \mathcal{C} \), \( \text{rat}(X) \) denotes the set of all rational functions on \( X \).

2. Subnormal operators

In order to prove Theorem 1, we use the original Alexander inequality.

**Theorem 1.** Let \( T \) be a subnormal operator in \( \mathcal{B}(\mathcal{H}) \) and \( f \) a rational function on \( \sigma(T) \) whose poles are not on it. Then

\[
\| T^*f(T) - f(T)T^* \| \leq \left\{ \frac{\text{Area}(\sigma(T))}{\pi} \right\}^{1/2}\left\{ \frac{\text{Area}(\sigma(f(T)))}{\pi} \right\}^{1/2}.
\]

**Proof.** Suppose that \( N \in \mathcal{B}(\mathcal{K}) \) is a normal extension of \( T \in \mathcal{B}(\mathcal{H}) \) and \( P \) is an orthogonal projection from \( \mathcal{K} \) to \( \mathcal{H} \). Then \( T = PN \mid \mathcal{H} \) and so

\[
T^*f(T) - f(T)T^* = P(N^*f(N)P - Pf(N)PN^*P = P(N^*f(N)P - f(N)PN^*P = Pf(N)N^*P - f(N)PN^*P = Pf(N)(1 - P)N^*P = Pf(N)(1 - P) \cdot (1 - P)N^*P
\]

because \( f(N)P = Pf(N)P \) and \( f(N)N^* = N^*f(N) \).

Let \( F \) be a rational function in \( \text{rat}(\sigma(T)) \). Put \( \mathcal{B}_F = \) the norm closure of \( \{g(F(N)) ; g \in \text{rat}(\sigma(F(N))) \} \) then \( P \) belongs to \( \text{lat}\mathcal{B}_F \). Hence

\[
\| (1 - P)F(N)^*P \| \leq \text{dist}(F(N)^*, \mathcal{B}_F) \leq \text{dist}(\bar{z}, \text{rat}(\sigma(F(N)))) \leq \left\{ \frac{\text{Area}(\sigma(F(N)))}{\pi} \right\}^{1/2}
\]

by the Alexander's theorem [1]. Hence, applying \( F \) to \( F = z \) or \( F = f \)

\[
\| T^*f(T) - f(T)T^* \| \leq \| (1 - P)f(N)^*P \| \cdot \| (1 - P)N^*P \| \leq \left\{ \frac{\text{Area}(\sigma(f(N)))}{\pi} \right\}^{1/2}\left\{ \frac{\text{Area}(\sigma(N))}{\pi} \right\}^{1/2} \leq \left\{ \frac{\text{Area}(\sigma(f(T)))}{\pi} \right\}^{1/2}\left\{ \frac{\text{Area}(\sigma(T))}{\pi} \right\}^{1/2}.
\]

\[\square\]
If $T$ is a cyclic subnormal operator and $KT = TK$ then using a theorem of T. Yoshino [12] we can prove that
\[ \| T^*K - KT^* \| \leq \{ \text{Area}(\sigma(T))/\pi \}^{1/2} \{ \text{Area}(\sigma(K))/\pi \}^{1/2}. \]

The proof is almost same to one of Theorem 1.

3. $p$-hyponormal operators

In order to prove Theorem 2, we use an operator version of the Alexander inequality for a $p$-hyponormal operator. Unfortunately Lemma 3 is not best possible for $p = 1$ (see [5]). Lemma 1 is due to W. Arveson [2, Lemma 2] and Lemma 2 is due to A. Uchiyama [11, Theorem 3].

We need the following notation to give Theorem 2 and Proposition 1. Let $\phi$ be a positive function on $(0, \infty)$ such that
\[ \phi(t) = \begin{cases} t & \text{if } t \text{ is an integer} \\ t + 2 & \text{if } t \text{ is not an integer}. \end{cases} \]

We write $\ell^2 \otimes \mathcal{H}$ for the Hilbert space direct sum $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$, and $1 \otimes T$ denotes the operator $T \oplus T \oplus \cdots \in B(\ell^2 \otimes \mathcal{H})$ for each operator $T \in B(\mathcal{H})$.

Lemma 1. Let $A$ be an arbitrary ultra-weakly closed subalgebra of $B(\mathcal{H})$ containing 1, and let $T \in B(\mathcal{H})$. Then
\[ \text{dist}(T, A) = \sup\{ \|(1 - P)(1 \otimes T)P\| ; P \in \text{lat}(1 \otimes A) \}. \]

Lemma 2. If $T$ is a $p$-hyponormal operator, then
\[ \| T^*T - TT^* \| \leq \phi \left( \frac{1}{p} \right) \| T \|^{2(1-p)} \{ \text{Area}(\sigma(T))/\pi \}^p. \]

Lemma 3. If $T$ is a $p$-hyponormal operator then
\[ \text{dist}(T^*, A) \leq \sqrt{2\phi \left( \frac{1}{p} \right) \| T \|^{-p} \{ \text{Area}(\sigma(T))/\pi \}^{p/2}}, \]

where $A$ is the strong closure of $\{ f(T) ; f \in \text{rat}(\sigma(T)) \}$.

Proof. Let $S = 1 \otimes T$. Then $S$ is $p$-hyponormal. In order to prove the lemma, by Lemma 1 it is enough to estimate $\sup\{ \|(1 - P)SP\| ; P \in \text{lat}(1 \otimes A) \}$. If $P \in \text{lat}(1 \otimes A)$ then $SP = PSP$ and so
\begin{align*}
\|(1 - P)SP\|^2 &= \|PSS^*P - PSPS^*P\| \\
&= \|PSS^*P - PS^*SP + PS^*SP - PQPS^*P\| \\
&\leq \|P(S^*S - SS^*)P\| + \|PSP^*(PSP) - (PSP)(PSP)^*\| \\
&\leq \|S^*S - SS^*\| + \|PSP^*(PSP) - (PSP)(PSP)^*\|. 
\end{align*}
By [11, Lemma 4], $PSP$ is $p$-hyponormal and so by Lemma 2 we have
\[ \|PSS^*P - PSPS^*P\|^2 \]
\[ \leq \phi \left( \frac{1}{p} \right) \|T\|^2 \{ \text{Area}(\sigma(T))/\pi \}^p \]
\[ + \phi \left( \frac{1}{p} \right) \|PSP\|^2 \{ \text{Area}(\sigma(PSP))/\pi \}^p \]
\[ \leq 2\phi \left( \frac{1}{p} \right) \|T\|^2 \{ \text{Area}(\sigma(T))/\pi \}^p \]
because $\|PSP\| \leq \|S\| = \|T\|$ and $\sigma(PSP) \subset \sigma(S) = \sigma(T)$. By Lemma 1,
\[ \text{dist}(T^*, A) \leq \sqrt{2\phi \left( \frac{1}{p} \right) \|T\|^{1-p} \{ \text{Area}(\sigma(T))/\pi \}^{p/2}}. \]

\[ \square \]

**Theorem 2.** If $T$ is a $p$-hyponormal operator in $B(\mathcal{H})$ and if $K$ is in $B(\mathcal{H})$ with $KT = TK$, then
\[ \|T^*K - KT^*\| \leq 2\sqrt{2\phi \left( \frac{1}{p} \right) \|T\|^{1-p} \{ \text{Area}(\sigma(T))/\pi \}^{p/2}}\|K\|. \]

**Proof.** When $A$ is the strong closure of $\{ f(T) : f \in \text{rat}(\sigma(T)) \}$, for any $A \in A$
\[ ||T^*K - KT^*|| = \|(T^* - A)K + AK - KT^*\| \leq 2\|T^* - A\||\|K\|. \]
Now Lemma 3 implies the theorem. \[ \square \]

In Theorem 2, if $p = 1$, that is, $T$ is hyponormal then $\|T^*K - KT^*\| \leq 2\sqrt{2\{ \text{Area}(\sigma(T))/2 \}^{1/2}}\|K\|$. The constant $2\sqrt{2}$ is not best because the second author [5] proved that $\|T^*K - KT^*\| \leq 2\{ \text{Area}(\sigma(T))/2 \}^{1/2}\|K\|$. If $p = \frac{1}{2}$, that is, $T$ is semi-hyponormal then
\[ ||T^*K - KT^*|| \leq 4\|T\|^{1/2} \{ \text{Area}(\sigma(T))/\pi \}^{1/4}\|K\|. \]

4. **Norm estimates**

In general, it is easy to see that $\|T^*T - TT^*\| \leq \|T\|^2$. By Theorem 1, if $T$ is subnormal and $f$ is an analytic polynomial then
\[ \|T^*f(T) - f(T)T^*\| \leq \|T\| \|f(T)\|. \]

In this section, we will prove that $\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}$ for arbitrary $T$ in $B(\mathcal{H})$.

**Theorem 3.** If $T$ is a contraction on $\mathcal{H}$ and $f$ is an analytic function on the closed unit disc $D$ then $\|T^*f(T) - f(T)T^*\| \leq \sup_{z \in D} |f(z)|$.
Proof. By a theorem of Sz.-Nagy [6], there exists a unitary operator $U$ on $\mathcal{K}$ such that $\mathcal{K}$ is a Hilbert space with $\mathcal{K} \supseteq \mathcal{H}$ and $T^n = PU^n | \mathcal{K}$ for $n \geq 0$, where $P$ is an orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$. Then it is known that $U^*P = PU^*P$ and $f(T) = Pf(U) | \mathcal{H}$. Hence
\[
T^*f(T) - f(T)T^* = PU^*f(U)P - Pf(U)PU^*P = PU^*f(U)P - Pf(U)U^*P = PU^*(I - P)f(U)P,
\]
because $U^*P = PU^*P$ and $f(U)U^* = U^*f(U)$. Therefore
\[
\| T^*f(T) - f(T)T^* \| = \| PU^*(I - P)f(U)P \| \leq \sup_{z \in D} | f(z) |.
\]

Corollary 1. If $T$ is in $\mathcal{B}(\mathcal{H})$ then for any $n \geq 1$ $\| T^*T^n - T^nT^* \| \leq \| T \|^{n+1}$.

Proof. Put $A = T/\| T \|$ then $A$ is a contraction and so by Theorem 2 $\| A^*A^n - A^nA^* \| \leq 1$ and so $\| T^*T^n - T^nT^* \| \leq \| T \|^{n+1}$.

5. Remarks

In this section, we give spectral area estimates for $p$-quasihyponomal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, $T$ is said to be $p$-quasihyponormal if $T^*\{ (T^*T)^p - (TT^*)^p \} T \geq 0$. A $1$-quasihyponormal operator is called quasihyponormal.

Lemma 4. Let $T$ be $p$-quasihyponormal and $P$ be a projection such that $TP = PTP$. Then $PTP$ is also $p$-quasihyponormal.

Proof. Since $T$ is $p$-quasihyponormal, $T^*(T^*T)^p T \geq T^*(TT^*)^p T$. Hence, we have
\[
PT^*(T^*T)^p T P \geq PT^*(TT^*)^p T P.
\]
Since by the Hansen’s inequality [4]
\[
PT^*(T^*T)^p T P = (PTP)^*P(T^*T)^p P(PTP) \leq (PTP)^* (PT^*TP)^p (PTP) = (PTP)^* \{ (PTP)^* (PTP) \}^p (PTP)
\]
and by $0 < p < 1$
\[
PT^*(TT^*)^p T P \geq (PT^*P)(TPT^*)^p (PTP) = (PTP)^* \{ (PTP)(PTP)^* \}^p (PTP),
\]
we have
\[
(PTP)^* \{ (PTP)^* (PTP) \}^p \geq (PTP)^* \{ (PTP)(PTP)^* \}^p (PTP).
\]
Hence, $PTP$ is $p$-quasihyponormal.
Proposition 1. If $T$ is a $p$-quasihyponormal operator in $\mathcal{B}(\mathcal{H})$ and if $K$ is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then
\[
\|T^*K - KT^*\| \leq 4 \left[ \phi \left( \frac{1}{p} \right) \right]^{1/4} \|T\|^{1-p/2} \{\text{Area}(\sigma(T))/\pi\}^{p/4} \|K\|.
\]
In particular, if $T$ is quasihyponormal then
\[
\|T^*K - KT^*\| \leq 4 \|T\|^{1/2} \{\text{Area}(\sigma(T))/\pi\}^{1/4} \|K\|.
\]

Proof. We can prove it as in the proof of Theorem 2. By [11, Theorem 6],
\[
\|T^*T - TT^*\| \leq 2\|T\|^{2-p} \sqrt{\phi \left( \frac{1}{p} \right) \{\text{Area}(\sigma(T))/\pi\}^{p/2}}.
\]
Hence by Lemma 4
\[
\operatorname{dist}(T^*, A) \leq 2\|T\|^{1 - \frac{p}{2}} \phi \left( \frac{1}{p} \right)^{\frac{1}{2}} \{\text{Area}(\sigma(T))/\pi\}^{p/4}.
\]
This implies the proposition. \qed

Let $H^2$ and $H^\infty$ be the usual Hardy spaces on the unit circle and $z$ the coordinate function. $M$ denotes an invariant subspace of $H^2$ under the multiplication by $z$. By the well known Beurling theorem, $M = qH^2$ for some inner function. Suppose $N$ is the orthogonal complement of $M$ in $H^2$. For a function $\phi$ in $H^\infty$, $S_\phi$ is an operator on $N$ such that $S_\phi f = P(\phi f) \ (f \in N)$, where $P$ is the orthogonal projection from $H^2$ to $N$. For a symbol $\phi$ in $L^\infty$, $T_\phi$ denotes the usual Toeplitz operator on $H^2$.

Proposition 2. Suppose $\Phi = q^\Phi$ belongs to $H^\infty$. Then
\[
\begin{align*}
(1) \quad & \|S_\phi S_\phi - S_\phi S_\phi^*\| \leq \{\text{Area}(\Phi(D))/\pi\}; \\
(2) \quad & \|S_\phi^* S_\phi - S_\phi S_\phi^*\| \leq \{\text{Area}(\Phi(D))/\pi\}^{n+1} \text{ for } n \geq 0.
\end{align*}
\]

Proof. By a well known theorem of Sarason [8],
\[
\|S_\phi\| = \|\phi + qH^\infty\| = \|\bar{\phi}\phi + H^\infty\| = \|\Phi + H^\infty\|.
\]
By Nehari’s theorem [6], $\|\Phi + H^\infty\| = \|H_\Phi\|$, where $H_\Phi$ denotes a Hankel operator from $H^2$ to $\bar{z}H^2$. Since $\|H_\Phi\|^2 = \|T_\phi^* T_\Phi - T_\Phi T_\phi^*\|$, where $T_\Phi$ denotes a Toeplitz operator on $H^2$, by the Putnam inequality
\[
\|T_\phi^* T_\Phi - T_\Phi T_\phi^*\| \leq \{\text{Area}(\sigma(T_\Phi))/\pi\} = \{\text{Area}(\Phi(D))/\pi\}.
\]
Now since $\|S_\phi^* S_\phi - S_\phi S_\phi^*\| \leq \|S_\phi\|^2$, (1) follows. (2) is also clear by the proof above and Corollary 1. \qed

Proposition 3. Suppose $f$ and $g$ are in $H^\infty$. Then
\[
\|T_f^* T_g - T_g T_f^*\| \leq \{\text{Area}(f(D))/\pi\}^{1/2} \{\text{Area}(g(D))/\pi\}^{1/2}.
\]
Proof. It is easy to see that \( T_f^*T_g - T_gT_f^* = H_f^*H_f \). Hence
\[
\|T_f^*T_g - T_gT_f^*\| \leq \|H_f\| \cdot \|H_f\|.
\]
Since \( H_f^*H_f = T_f^*T_f - T_fT_f^* \), by the Putnam inequality
\[
\|T_f^*T_g - T_gT_f^*\| \leq \{\text{Area}(f(D))/\pi\}^{1/2}\{\text{Area}(g(D))/\pi\}^{1/2}.
\]
\[
\square
\]

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