ON A FAMILY OF BALANCED GROUPS

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ABSTRACT. A family of balanced groups is introduced. We describe some geometric approach to find these groups in terms of the (orientable) closed 3-manifolds and its fundamental groups.

1. Introduction

Given a group presentation $G = \langle X | R \rangle$, where $X$ is a set of generators and $R$ relators, we may have a question: if $G$ is geometric. In other words, if the canonical 2-dimensional complex associated to $G$ is a spine of a closed connected orientable 3-manifold. In the case of $G$ geometric we can get more information on the structure of the group $G$. For this problem we usually investigate whether $G$ can be realized as the fundamental group of a manifold. As is well known, for $n \geq 4$ every finitely presented group $G = \langle X | R \rangle$ is the fundamental group of $n$-dimensional manifold ([2]). However, it is very difficult (generally impossible) to find such groups in the case of $n = 3$ ([9], [11]). In recent years Helling, Kim and Mennicke showed a very interesting example, A geometric study of Fibonacci groups ([5]). They constructed a family of 2-dimensional complexes on the 2-sphere $S^2$, whose fundamental groups are isomorphic to $F(2, 2n)$ and stated that these groups are “the link between certain objects in 3-dimensional topology, in 3-dimensional hyperbolic geometry, in the theory of discontinuous transformation groups; in rank one Lie groups”. The similar works can be found in the various papers and its references ([1], [4], [5], [9], [12]). In this paper we study a family of balanced groups with equal number of generators and relators, denoted by

$G(n) = \langle x_1, x_2, \ldots, x_n, y \mid x_1 x_2 \cdots x_n = 1, y x_{i+1} x_i = 1, \text{indices } \mod n \rangle$,

which are arising from the identification of opposite faces of a polyhedron. The polyhedron consists of $n$-gons ($n \geq 3$) in the top and bottom faces, and of $2n$ triangles in the side faces. In particular, for $n = 3$ we have a combinatorial octahedral space which is the manifold $M_3$ obtained by opposite face identification of octahedron by a right-helix turn of angle $\pi/3$, corresponding

Received April 25, 2006.
2000 Mathematics Subject Classification. Primary 20F05, 57M05; Secondary 57M12, 57M25.

Key words and phrases. balanced group presentations, 3-manifolds, fundamental groups.

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to a $2\pi/3$ rotation of a tetrahedron. This manifold is a 2-fold covering of the truncated-cube space $S^3/(432)$, where $(432)$ is the binary octahedral group of order 48 ([8], p.120). The recognition problem of 3-manifold groups is of much interest in both algebraic topology and combinatorial group theory. The main theorem of this paper is to show a family of balanced groups $G(n)$ is coincided with the fundamental groups of some closed orientable 3-dimensional manifolds, and is to find its some topological properties related to these groups.

2. The balanced groups $G(n)$

We shall consider the following group presentations

$$G(n) = \langle x_1, x_2, \ldots, x_n, y | R_0, R_4, n \geq 3 \rangle,$$

where $R_0 = x_1 x_2 \cdots x_n, R_4 = y x_{i+1} x_i$ (indices mod $n$). According to the definition of $G(n)$, we have $x_1 x_2 x_3 \cdots x_n = 1$. $y = x_n^{-1} x_1^{-1}, y = x_{i+1}^{-1} x_i^{-1} x_{i+1}, 1 \leq i \leq n-1, n \geq 3$. So, $x_i^{-1} x_{i+1}^{-1} = x_1^{-1} x_2^{-1}$, or equivalently $x_{i+1} x_i = x_2 x_1$, whence $x_{2t} = (x_2 x_1)^{t-1} x_2 (x_2 x_1)^{t-1}, x_{2t+1} = (x_2 x_1)^{t} x_2 (x_2 x_1)^{t-1}$ for $t = 1, 2, 3, \ldots$. Now, the relations $x_2 x_1 = x_1 x_2$ and $x_1 x_2 x_3 \cdots x_n = 1$ lead to the ones between $a := x_1, b := x_2$. At last we eventually obtain the following desirable presentations:

(P1) $G(2m) = \langle a, b | (ba)^m = (ab)^m, a(ba^2)^m b = (ba)^{-m} \rangle$ for $m \geq 2$,

(P2) $G(2m+1) = \langle a, b | b(ab)^m = (ab)^m a, a(ba^2)^m b^2 a = (ba)^{-m} \rangle$ for $m \geq 1$. Now we come to the point where the difference between even and odd values of $n$ is crucial. From (P1) and (P2) it is easily seen that the derived quotients $\overline{G(2m)} = G(2m)/G'(2m)$ and $\overline{G(2m+1)} = G(2m+1)/G'(2m+1)$, written in additive form, have the following:

$$\overline{G(2m)} = Z \oplus Z_m, \quad \overline{G(2m+1)} = Z_{2m+1}.$$

Theorem 2.1. The groups $G(n)$ are infinite for all $n = 2m$ $(m \geq 2)$. There exist an isomorphism $\xi$ of $G(3)$ to the semidirect product $\ast$ of the quaternion group $Q_8$ by the cyclic group of order 3, $Z_3$.

Proof. Since $\overline{G(2m)} = Z \oplus Z_m$, for every $n = 2m$, $m \geq 2$, $G(n)$ is infinite. Secondly, by definition

$$G(3) = \langle a, b | aba = bab, ab^2 a = b \rangle$$

is a balanced group. An obvious epimorphism $\xi : G(3) \longrightarrow SL_2(3) \cong Q_8 \ast Z_3$ such that

$$\xi(a) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array} \right), \quad \xi(b) = \left( \begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array} \right)$$

force to think of the kernel $\xi_{ker}$ of $\xi$. Certainly, it has been known for a long time that the kernel $\xi_{ker}$ is trivial. Anyhow we may consider an implication

$\{aba = bab, ab^2 a = b\} \Rightarrow \{aba^2 = b^2 ab, bab^2 = a^2 ba, ba^2 b = a, b^2 = a^2 ba^2, a^2 = b^2 ab^2, a^2 b^2 a^2 = aba, b^3 = (ab)^3 = (ba)^3 = a^3, a^6 = 1\}$.

So, $G(3)$, as a set, is the following one:

$\{aba = bab, ab^2 a = b\}$
\[ G(3) = \{1, a^3, a^2b, ab^2, aba, a^4ba, a^3ba^2, a^5b, a^2, a^4, a, a^5, b, ab, ba, a^3b, a^2ba, a^4b, a^3ba, a^2ba^2, a^5ba, a^4ba^2, a^5ba^2\}. \] Thus \( |G(3)| = 24 \), and the epimorphism \( \xi \) should be an isomorphism, and we can put \( Q_8 = \{1, a^3, a^2b, ba^2, aba, a^4ba, a^3ba^2, a^5b\} = \langle a^2b, ba^2 \rangle, Z_3 = \{1, a^3, a^4\} \).

Remark 2.2. (i) It is an ordinary method in combinatorial group theory to investigate an epimorphism \( \xi : G \to SL_2(F) \), when we like to verify that \( |G| = \infty \) or at least to obtain an information on \( G \) ([7]). But this is far from being the case that such an epimorphism does exist at all. In particular \( G(2m + 1), m \geq 1 \), do not possess this property.

(ii) The group of outer automorphisms

\[ OutG = AutG/InnG, \ InnG \cong G/Z(G) \]

is an important characteristic of any group \( G \). Certainly, the determination of \( OutG(n) \) is a deep matter beyond the scope of this paper. So we only pay our attention to some special automorphisms. The group \( G(n) \) has an outer automorphism

\[ \psi = (1, 2, 3, \ldots, n)(n + 1). \]

That is easily seen from some geometric considerations. From the former presentations \( (P1) \) and \( (P2) \) we have \( \psi(a) = b, \psi(b) = bab^{-1} \). So it is obvious that \( \psi(ba) = ba \), therefore a cyclic subgroup \( \langle ba \rangle \) is fixed under the action of \( \psi \). If we put \( c = ba, b = ca^{-1} \), then the presentation of the questionable group \( G(5) \) can be rewritten as follows:

\[ G(5) = \langle a, c; ac^2a = c^3, caca^{-1}cac = aca \rangle. \]

So the action of the automorphism \( \psi \) of order 5 will be expressed as \( \psi(c) = c, \psi(a) = ca^{-1} \). Generally, the second relation has a form

\[ aca^{-1}caca^{-1}ca = c^2. \]

But applying \( \psi \) we will come to the form, indicated above.

3. Some geometric realizations of \( G(n) \)

In this paragraph we want to construct a 3-manifold \( M_n \) from a polyhedron \( P_n \) such that

\[ \pi_1(M_n) \cong G(n). \]

For this we shall consider a combinatorial polyhedron \( P_n \) consisting of two \( n \)-gons \( F, F^* \) in the north and south hemisphere, and of 2n triangles \( F_i, F_i^* \) in the equator zone, shown in the figure 1.

The above polyhedron \( P_n \) has 4n edges, 2n + 2 faces, and 2n vertices. The oriented edges can be labeled in the following manner. The oriented edges fall into \( n + 1 \) classes, the verticle edges \( P_iQ_i \) consisting of \( n \) edges and the others of 3 edges. The oriented edges in the same class carry the same label, say, \( x_1, x_2, x_3, \ldots, x_n \). The \( n \)-gons in the north and south hemisphere have a boundary \( x_1x_2 \cdots x_n \) for all \( n \geq 3 \), and each triangle \( yx_{i+1}x_i \) (indices mod \( n \)).
For the pair of \(n\)-gons \((F, F^*)\) and the pairs of triangles \((F_i, F_i^*)\) define identifications \(\alpha, \beta\). The resulting 3-dimensional complex \(K\) is an orientable pseudo manifold. In this case there is a simple criterion, due to H. Seifert and W. Threlfall [10], for \(K\) to be a manifold.

**Theorem 3.1** (Seifert and Threlfall). Let \(K\) be an orientable, closed, 3-dimensional pseudo manifold arising from a (simply connected) polyhedron by identifying pairs of faces on the boundary of \(K\). \(K\) is a manifold if and only if its Euler characteristic vanishes.

So we now define identifications \(\alpha\) and \(\beta\) on the boundary of the above polyhedron \(P_n\) as follows:

\[
\alpha : \begin{align*}
F &= F^* \\
P_i &= Q_{i+2}
\end{align*} \quad \beta : \begin{align*}
F_i &= F_i^* \\
P_i &= P_{i+2} \\
Q_i &= Q_{i+1} \\
Q_{i+1} &= Q_{i+2}
\end{align*} (all indices \(\mod n\)).

These \(\alpha\) and \(\beta\) make all edges of the polyhedron labeled as follows:

\[Q_i P_i = y, P_i P_{i+1} = P_{i+2} Q_{i+1} = Q_{i+2} Q_{i+3} = x_i\] (all indices \(\mod n\)).

These identifications \(\alpha\) and \(\beta\) produce a complex \(M_n\), say, with

\[
\begin{align*}
\alpha^0 &= 1, &\text{vertex} \\
\alpha^1 &= n + 1, &\text{edges} \\
\alpha^2 &= n + 1, &2\text{–cells} \\
\alpha^3 &= 1, &3\text{–cell},
\end{align*}
\]

\[\chi(M_n) = 1 - (n + 1) + (n + 1) - 1 = 0.\]
Hence we can read off the fundamental groups \( \pi_1(M_n) \) of \( M_n \) with generators \( \{x_1, x_2, \ldots, x_n, y\} \) and relators, \( \{x_1 x_2 \cdots x_n = 1, y x_{i+1} x_i = 1, y x_1 x_n = 1, \text{indices mod } n\} \), which is coincided with the balanced groups \( G(n) \). So applying theorem 3.1, we have the following

**Theorem 3.2.** The complex \( M_n \) constructed above is a closed, connected, orientable 3-manifold. The fundamental group \( \pi_1(M_n) \) admits the following group presentations \( \pi_1(M_n) = \langle x_1, x_2, \ldots, x_n, y | x_1 x_2 \cdots x_n = 1, y x_{i+1} x_i = 1, \text{indices mod } n, n \geq 3 \rangle \), which is coincided with \( G(n) \), corresponding to a spine of the manifold \( M_n \) (hence it is geometric).

**Remark 3.3.** (i) In the above, \( G(n) \) has \( n \) distinct group presentations, denoted by \( G_k(n) = \langle x_1, x_2, \ldots, x_n, y | x_1 x_2 \cdots x_n = 1, y x_i x_{i+k-1} = 1, 1 \leq k \leq n, \text{indices mod } n \rangle \). From these presentations a natural question arises: whether these groups are different or not. So we here leave the classification problems of \( G_k(n) \) for all \( n \geq 3 \).

(ii) For \( n = 3 \), \( M_3 \) is the octahedral space \( S^3/\langle 332 \rangle \), where \( \langle 332 \rangle \) is the binary tetrahedral group of order 24. This space is the manifold obtained by an identification of opposite faces of an octahedron by a right-helix turn of angle \( 3/\pi \), which corresponds to a \( 2\pi/3 \) rotations of a tetrahedron. This manifold \( M_3 \) is a 2-fold coverings of the truncated-cube space \( S^3/\langle 432 \rangle \), where \( \langle 432 \rangle \) is the binary octahedral group of order 48 (see: [8], p.122). The fundamental group \( \pi_1(M_3) = G_3 \) admits the following presentations

\[
\pi_1(M_3) = \langle x_1, x_2, \ldots, x_n, y | x_1 x_2 x_3 = 1, x_i x_{i+2} y = 1, \text{indices mod } 3 \rangle.
\]

Let \( H_3 \) be the split extension group of \( G_3 \) by \( Z_3 = \langle \theta; \theta^3 = 1 \rangle \), where \( \theta \) is the automorphism of \( G_3 \) defined by \( \theta(x_i) = x_{i+1} (\text{mod } 3) \), and \( \theta(y) = y \). Then \( H_3 \) has the following presentations:

\[
H_3 = \langle x, y, \theta | \theta^3 = 1, (\theta^{-1})^3 = 1, \theta^2 x \theta^{-2} x = y, \theta y \theta^{-1} = y \rangle \\
\cong \langle x, \theta | \theta^3 = 1, (\theta^{-1} x)^3 = 1, \theta(x \theta x) = (x \theta x \theta) \rangle \\
\cong \langle \theta, \tau | \theta^3 = 1, \tau^3 = 1, \theta(\tau \theta^{-1} \tau) = (\tau \theta^{-1} \tau) \theta \rangle
\]

where \( x_1^{-1} = x, x_1^{-1} = \theta^4 x \theta^{-4}, \theta^{-1} x = \tau \). In this case the presentation \( \langle \theta, \Omega | \theta = \theta \Omega \rangle \) for \( \Omega = \tau \theta^{-1} \tau \) defines the fundamental group of the resulting link \( L \) which is arising from the manifold \( M_3 \), where \( \theta, \tau \) are meridians around its components. In fact, the link \( L \) is the 3-fold cyclic covering of the 3-sphere \( S^3 \) branched over the link \( L \) with branching index 3.

We now study some topological properties of the manifold \( M_n \). As is well known, every closed orientable 3-manifold is a branched covering of the 3-sphere \( S^3 \) over some knots/links. So we describe the manifold \( M_n \) as the branched coverings of the 3-sphere. In the Theorem 3.2, the identification \( \alpha, \beta \) determine the edges and faces identifications for the polyhedron \( P_n \) as follows:

\[
Q_1 P_1 = y, P_i P_{i+1} = P_{i+2} Q_{i+1} = Q_{i+3} Q_{i+3} = x_i, P_1 P_2 \cdots P_n \rightarrow Q_3 Q_4 \cdots Q_2, \\
P_i P_{i+1} Q_i \rightarrow P_{i+2} Q_{i+1} Q_{i+2} (\text{all indices mod } n).
\]

We now define an automorphism \( \phi \) of \( G(n) \) by \( \phi(x_i) = x_{i+1} (\text{indices mod } n) \) and \( \phi(y) = y \). Denote the corresponding homeomorphism of \( M_n \) also by \( \phi \). As
the $\phi$ corresponds to $n$-rotational symmetry of the polyhedron $P_n$, the $1/nP_n$ in the figure 2 is the fundamental domain for the quotient space $M_n/\langle \phi \rangle$. A Heegaard diagram for the quotient space $M_n/\langle \phi \rangle (\cong S^3)$ obtained from $1/nP_n$ by side Pairing of its boundary faces is shown in the Figure 3.

Figure 2: The $1/n P_n$ with side pairing

Figure 3: A Heegaard diagram of the quotient space $(\cong S^3)$ obtained from $1/nP_n$
Then we apply Hilden-Lozano-Montesinos techniques for the figure-8-knot ([6], p.173) and Grunewald-Hirsch methods for link complements ([4], pp.355-362) to modify Fig.3 to Fig.6. The figure 4 is obtained from figure 3 by a simplication along the closed curve $A$, which surrounds the "hole" $F^+$. The figure 5 is obtained from figure 4 by canceling a handle along $X$ between the "holes" $F^+_1, F^-_1$. The result is also a Heegaard diagram for the quotient space ($\cong S^3$) obtained from $1/nP_n$. Finally, we cancel the last handle between the "holes" $A^+, A^-$, and we get the pictured resulting link $L$ in the figure 6, equivalent to the link in the figure 7 (see [3], p. 90).

Figure 4: The simplication along the closed curve $A$

Figure 5: The canceling of a handle along $X$
Figure 6: The resulting link $L$, fibering over $S^3$

On the other hands, if we define the identifications $\alpha$, $\beta$ by $\alpha : F \to F^*$, $P_i \to Q_{i+n-2}$; $\beta : F_i \to F_i^*$, $P_iP_{i+1}Q_i \to P_iQ_{i+n-1}Q_i$, (all indices mod $n$), then the resulting link $L$ will be the Hopf's link $2_1^2$, shown in the figure 8 (sec: [3], p.89). So we summarize all details above in the following Theorem and Corollary:

**Theorem 3.4.** The 3-manifold $M_n$ is $n$-fold cyclic covering of the 3-sphere $S^3$ branched over the link in figure 6. The branching indices on the components of $L$ are equal to $n$.

**Corollary 3.5.** For some identifications $\alpha$, $\beta$ the link $L$ will be the Hopf's link $2_1^2$.

Figure 8: Hopf's link $2_1^2$
References


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