MAXIMUM SUBSPACES RELATED TO A-CONTRACTIONS
AND QUASINORMAL OPERATORS

Laurian Suciu

Abstract. It is shown that if $A \geq 0$ and $T$ are two bounded linear operators on a complex Hilbert space $\mathcal{H}$ satisfying the inequality $T^*AT \leq A$ and the condition $AT = A^{1/2}TA^{1/2}$, then there exists the maximum reducing subspace for $A$ and $A^{1/2}T$ on which the equality $T^*AT = A$ is satisfied. We concretely express this subspace in two ways, and as applications, we derive certain decompositions for quasinormal contractions. Also, some facts concerning the quasi-isometries are obtained.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$, $I = I_\mathcal{H}$ being the identity operator in $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$ we denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and the kernel of $T$, respectively.

Recall that $T$ is a quasinormal operator if $T$ and $T^*T$ commute, where $T^*$ is the adjoint operator of $T$.

Throughout in this paper $A \in \mathcal{B}(\mathcal{H})$ is a non zero positive fixed operator. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an $A$-contraction on $\mathcal{H}$ if it satisfies the following operator inequality

$$T^*AT \leq A.$$  \hfill (1.1)

According to [1], this means that $A$ is a lower $T$-Toeplitz operator. If the equality in (1.1) occurs, then $T$ is called an $A$-isometry on $\mathcal{H}$. In the terminology of [1], [2] the fact that $T$ is an $A$-isometry means that $A$ is a $T$-Toeplitz operator.

We say that $T$ is an $A$-weighted contraction on $\mathcal{H}$ if $T^*T \leq A$, and we call $T$ an $A$-weighted isometry on $\mathcal{H}$ if the equality $T^*T = A$ holds. Clearly, $T$ is an $A$-weighted contraction ($A$-weighted isometry) if and only if there is a contraction (respectively, an isometry) $V$ from $\overline{\mathcal{R}(A)}$ into $\overline{\mathcal{R}(T)}$ such that $T = VA^{1/2}$, where $A^{1/2}$ is the square root of $A$. In this case, $V$ is uniquely determined by $A$ and $T$.

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It immediately follows from (1.1) that $T$ is an $A$-contraction ($A$-isometry) if and only if $A^{1/2}T$ is an $A$-weighted contraction ($A$-weighted isometry), on $\mathcal{H}$. If $T$ is an $A$-contraction, we denote by $\widehat{T}$ the (unique) contraction on $\mathcal{R}(A)$ satisfying

$$\widehat{T}A^{1/2}h = A^{1/2}Th \quad (h \in \mathcal{H}).$$

Clearly, $\widehat{T}$ is an isometry if and only if $T$ is an $A$-isometry.

For an $A$-contraction $T$ and any integer $n \geq 0$ we denote $\mathcal{N}_n = \mathcal{N}(A - T^{*}(n+1)AT^{n+1})$. It was shown in Proposition 2.1 [11] that the subspace

$$\mathcal{N}_\infty = \bigcap_{n=0}^{\infty} \mathcal{N}_n$$

is invariant for $T$, and if $\mathcal{N}_\infty$ reduces $A$ then $\mathcal{N}_\infty$ is the maximum invariant subspace for $A$ and $T$ on which $T$ is an $A$-isometry. For instance, if either the range $\mathcal{R}(A)$ is closed, or $T$ is a regular $A$-contraction, which means that $AT = A^{1/2}TA^{1/2}$, then $\mathcal{N}_\infty$ reduces $A$. In these cases $\mathcal{N}_\infty$ is also invariant for $A^{1/2}T$ and furthermore, if $T$ is a regular $A$-contraction then $\mathcal{N}_\infty$ is the maximum subspace into $\mathcal{N}_0$ which is invariant for $A$ and $A^{1/2}T$ (Proposition 2.3 [11]), and obviously, $A^{1/2}T$ is an $A$-weighted isometry on $\mathcal{N}_\infty$.

This last meaning of $\mathcal{N}_\infty$ suggests us to investigate the existence of the maximum reducing subspace for $A$ and $A^{1/2}T$ on which $A^{1/2}T$ is an $A$-weighted isometry. We find such a subspace in Section 2, which will be denoted $\mathcal{M}_\infty$, and we show that $\mathcal{M}_\infty$ can be concretely obtained in two ways, using the subspace $\mathcal{N}_\infty$ and the minimal isometric dilation of the contraction $\widehat{T}$, respectively.

Recall ([4]) that the minimal isometric dilation of a contraction $S$ on $\mathcal{H}$ is an isometry $V$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ with the property

$$P_\mathcal{H}V = SP_\mathcal{H},$$

$P_\mathcal{H}$ being the orthogonal projection onto $\mathcal{H}$, such that

$$\mathcal{K} = \bigvee_{n \geq 0} V^n \mathcal{H}.$$

Remark that the property (1.4) means that $V$ is a lifting for $S$.

The space $\mathcal{M}_\infty$ has interesting meanings in the context of quasinormal operators, where different natural $A$-contractions appear. We consider this context in Section 3, and we obtain an orthogonal decomposition of $\mathcal{H}$ relative to a quasinormal contraction $T$, where all reducing subspaces can be intrinsically expressed in terms of $T$ and of the partial isometry from the polar decomposition of $T$. It is a complete description of the decompositions from [8, 9] concerning the quasinormal contractions. Also, we derive some consequences involving the quasi-isometries, that is the operators $T \in \mathcal{B}(\mathcal{H})$ which are $T^*T$-isometries (see [6, 11]).
2. Maximum $A$-weighted isometric part

Let $T$ be an $A$-contraction on $\mathcal{H}$. Using a standard argument which involves Zorn’s lemma, one can obtain the existence of the maximum subspace $\mathcal{M}_0$ of $\mathcal{H}$ which reduces $A$ and $T$ such that $T|_{\mathcal{M}_0}$ is a $A|_{\mathcal{M}_0}$-isometry (see [12]). In the case when $\mathcal{H}$ is separable, $\mathcal{M}_0$ can be also obtained from Theorem 1 [5], which in particular gives that any $S \in \mathcal{B}(\mathcal{H})$ has a maximum subspace which reduces $A$ and $S$ on which $S$ is an $A$-contraction, or an $A$-isometry, respectively.

In our setting one has $\mathcal{M}_0 \subset \mathcal{N}_\infty$ and $\mathcal{M}_0$ reduces $A^{1/2}T$ to an $A$-weighted isometry, but $\mathcal{M}_0$ is not the maximum subspace having this property relative to $A^{1/2}T$, in general (see [12]). In the regular case it is possible to get more invariant (or reducing) subspaces in $\mathcal{N}_\infty$ for $A$ and $A^{1/2}T$ on which $A^{1/2}T$ is an $A$-weighted isometry (see [9-11]), but the maximum such subspace between $\mathcal{M}_0$ and $\mathcal{N}_\infty$ is now obtained in the following.

Theorem 2.1. Let $T$ be a regular $A$-contraction on $\mathcal{H}$. Then the maximum subspace which reduces $A$ and $A^{1/2}T$ on which $A^{1/2}T$ is an $A$-weighted isometry is

$$\mathcal{M}_\infty = \mathcal{H} \oplus \bigvee_{n \geq 0} A^{1/2}T^n \mathcal{N}_\infty^\perp.$$  

Moreover, $A^{1/2}T$ is a quasinormal operator on $\mathcal{M}_\infty$, and $\mathcal{M}_\infty$ is an invariant subspace for $T$ such that $T$ is an $A$-isometry on $\mathcal{M}_\infty$. In particular, if $A^{1/2}T$ is a quasinormal operator on $\mathcal{H}$, then $\mathcal{M}_\infty = \mathcal{N}_0$.

Proof. Let $\mathcal{M}_\infty$ be the subspace defined in (2.1). Since $T$ is a regular $A$-contraction, $\mathcal{N}_\infty^\perp$ is invariant for $A$ and we have for $n \geq 0$,

$$AA^{1/2}T^n \mathcal{N}_\infty^\perp = A^{1/2}T^n A\mathcal{N}_\infty^\perp \subset A^{1/2}T^n \mathcal{N}_\infty^\perp \subset \mathcal{M}_\infty^\perp,$$

and also

$$A^{1/2}TA^{1/2}T^n \mathcal{N}_\infty^\perp = AT^{n+1} \mathcal{N}_\infty^\perp = A^{1/2}T^{n+1} A^{1/2}T^{n} \mathcal{N}_\infty^\perp \subset \mathcal{M}_\infty^\perp.$$

It follows that $\mathcal{M}_\infty^\perp$ is an invariant subspace for the operators $A$ and $A^{1/2}T$. Next, since $\mathcal{N}_\infty^\perp$ is invariant for $A^{1/2}T$, we have firstly $T^*A^{1/2} \mathcal{N}_\infty^\perp \subset \mathcal{N}_\infty^\perp$. On the other hand, for $n \geq 1$ we obtain

$$T^*A^{1/2}T^n \mathcal{N}_\infty^\perp = T^*A^{1/2}T(T^{n-1} \mathcal{N}_\infty^\perp)$$

$$\subset (T^*A^{1/2}T - A^{1/2})T^{n-1} \mathcal{N}_\infty^\perp + A^{1/2}T^{n-1} \mathcal{N}_\infty^\perp$$

$$\subset \mathcal{N}_\infty^\perp + \mathcal{M}_\infty^\perp \subset \mathcal{M}_\infty^\perp.$$ 

Here we used the fact that $\mathcal{N}_0 = \mathcal{N}(A^{1/2} - T^*A^{1/2}T)$ (by Theorem 2.6 [11]), which gives that $\mathcal{N}(A^{1/2} - T^*A^{1/2}T) \subset \mathcal{N}_0^\perp \subset \mathcal{N}_\infty^\perp$. Thus, we infer that $\mathcal{M}_\infty^\perp$ is invariant for $T^*$ and, because $\mathcal{M}_\infty$ is reducing for $A^{1/2}$, $\mathcal{M}_\infty^\perp$ is also invariant for $T^*A^{1/2}$. Hence $\mathcal{M}_\infty$ is reducing for $A$ and $A^{1/2}T$, and also $\mathcal{M}_\infty$ is invariant for $T$. 
Now, since $\mathcal{M}_\infty \subset \mathcal{N}_\infty \subset \mathcal{N}_0$, we have

$$(A^{1/2}T|_{\mathcal{M}_\infty})^*(A^{1/2}T|_{\mathcal{M}_\infty}) = (T^*AT)|_{\mathcal{M}_\infty} = A|_{\mathcal{M}_\infty},$$

which just means that $A^{1/2}T$ is an $A$-weighted isometry on $\mathcal{M}_\infty$. As $\mathcal{M}_\infty$ is invariant for $A$ and $T$, the above relation also gives that $T$ is an $A$-isometry on $\mathcal{M}_\infty$, and having in view the fact that $T$ is a regular $A$-contraction on $\mathcal{H}$, and particularly on $\mathcal{M}_\infty$, it follows from Proposition 2.3 [11] that $A^{1/2}T$ is quasinormal on $\mathcal{M}_\infty$.

It remains to prove that $\mathcal{M}_\infty$ is the maximum subspace reducing $A$ and $A^{1/2}T$ on which $A^{1/2}T$ is an $A$-weighted isometry. Let $\mathcal{M} \subset \mathcal{H}$ be another subspace having these properties. Firstly, for any $h \in \mathcal{M}$ we have $T^*A\mathcal{M} = Ah$. Since $T$ is also a regular $A^{1/4}$-contraction (by Theorem 2.6 [11]), the previous relation implies $A^{3/4}T^*A^{1/4}T^*Ah = Ah$ and later $T^*A^{1/2}T^*Ah = A^{1/2}h$, because $A^{1/4}$ is injective on its range. Then using the fact that $\mathcal{M}$ is invariant for $A^{1/2}T$, we obtain

$$T^*A^{1/2}h = T^*(T^*A^{1/2}T)^{1/2}T^*Ah = T^*A^{1/2}A^{1/2}T^*Ah = Ah,$$

and by induction we infer that $T^*A^nT^*Ah = Ah$, for $n \geq 1$ and $h \in \mathcal{M}$. So $\mathcal{M} \subset \mathcal{N}(A - T^*A^nT^*)$ for $n \geq 1$, hence $\mathcal{M} \subset \mathcal{N}_\infty$ (by (1.3)). To prove $\mathcal{M} \subset \mathcal{M}_\infty$ we show that $T^*A^{1/2}A^m \subset \mathcal{M}$ for $m \geq 2$.

Let $\{p_n(A)\}$ be an approximation polynomial for $A^{1/2}$ with $p_n(0) = 0$ (as in [7], p. 261). If $p_n(A) = \sum_{j \geq 1} c_j A^{2j}$ (a finite sum, $c_j$ being positive scalars), then for $h \in \mathcal{M}$ we have $T^*A^{1/2}h \in \mathcal{M}$ and also

$$T^*A^{1/2}h = \lim_n \sum_{j \geq 1} c_j T^*A^{1/2}T^*(A^{1/2})^{2j-1}h \in \mathcal{M},$$

because in each term $2j - 1 \geq 1$. Using this fact, we obtain

$$T^*A^{1/2}h = \lim_n \sum_{j \geq 1} c_j T^*A^{1/2}T^*A^{1/2}(A^{1/2})^{2j-1}h \in \mathcal{M},$$

and by induction we get that $T^*A^{1/2}h \in \mathcal{M}$ for any $m \geq 1$. Thus, for $m \geq 1$ we have $T^*A^{1/2}A^m \subset \mathcal{M} \subset \mathcal{N}_\infty$, whence it follows that $\mathcal{M}$ is orthogonal to $A^{1/2}T^*A^m \subset \mathcal{M}_\infty$. Hence $\mathcal{M}$ is orthogonal to $\mathcal{M}_\infty$, that is $\mathcal{M} \subset \mathcal{M}_\infty$.

Finally, we suppose that $A^{1/2}T$ is quasinormal on $\mathcal{H}$. Then by Corollary 2.7 [11] one has $\mathcal{N}_0 = \mathcal{N}_\infty$ and this subspace reduces $A$ and $A^{1/2}T$. Clearly, $A^{1/2}T$ will be an $A$-weighted isometry on $\mathcal{N}_0$ and consequently, by the maximality of $\mathcal{M}_\infty$ one obtains $\mathcal{M}_\infty = \mathcal{N}_0$. The proof is finished.

Another description of the subspace $\mathcal{M}_\infty$ is given by the following.

**Proposition 2.2.** If $T$ is a regular $A$-contraction on $\mathcal{H}$ then

$$\mathcal{M}_\infty = \{h \in \mathcal{H} : V^nT^*A^{1/2}h \in \overline{R(A)}, \; n, m \geq 0, \; j \geq 1\},$$

where $V$ is the minimal isometric dilation of the contraction $T$ defined by (1.2).
Proof. We know from Theorem 2.5 [12] that the subspace \( \widetilde{M}_0 \) defined by the right side in (2.2) reduces \( A \) and \( A^{1/2}T \), \( \widetilde{M}_0 \) is invariant for \( T \) and \( T \) is an \( A \)-isometry on \( \widetilde{M}_0 \). Thus, \( A^{1/2}T \) is an \( A \)-weighted isometry on \( \widetilde{M}_0 \), and from Theorem 2.1 we have \( \widetilde{M}_0 \subset M_\infty \).

To prove the converse inclusion, let \( h \in M_\infty \) be arbitrary. Since \( A^{1/2}T \) is an \( A \)-weighted isometry on \( M_\infty \), there is an isometry \( S \) from \( \mathcal{R}_\infty := A^{1/2}M_\infty \) into \( A^{1/2}TM_\infty \) such that \( SA^{1/2}h = A^{1/2}Th \). Then \( \hat{T}A^{1/2}h = A^{1/2}Th = JSA^{1/2}h \), where \( J \) is the natural injection of \( A^{1/2}TM_\infty \) into \( \mathcal{R}_\infty \). Therefore \( \mathcal{R}_\infty \) is invariant for \( \hat{T} \) and \( \hat{T}|_{\mathcal{R}_\infty} = JS \) is an isometry on \( \mathcal{R}_\infty \). In fact, \( \mathcal{R}_\infty \) even reduces \( \hat{T} \) because for \( h' \in \mathcal{N}_\infty^\perp \) and \( n \geq 0 \) one has

\[
\langle \hat{T}^n A^{1/2}h, A^{1/2}T^n h' \rangle = \langle h, AT^{n+1}h' \rangle = \langle h, A^{1/2}T^{n+1}A^{1/2}h' \rangle = 0,
\]

having in view that \( \mathcal{N}_\infty \) reduces \( A \), and \( T \) is a regular \( A \)-contraction. This shows that \( \hat{T}^* A^{1/2}M_\infty \) is orthogonal to \( A^{1/2}T^n \mathcal{N}_\infty^\perp \) for \( n \geq 0 \), therefore \( \hat{T}^* A^{1/2}M_\infty \subset M_\infty \). Thus one obtains that

\[
\hat{T}^* \mathcal{R}_\infty \subset M_\infty \cap \overline{\mathcal{R}(A)} = \mathcal{R}_\infty.
\]

To see the previous equality, we first remark that \( \mathcal{R}_\infty \subset M_\infty \cap \overline{\mathcal{R}(A)} \). Next let \( k \in M_\infty \cap \overline{\mathcal{R}(A)} \) and we write \( k = k_0 + k_1 \) with \( k_0 \in \mathcal{R}_\infty \) and \( k_1 \in \overline{\mathcal{R}(A)} \cap \mathcal{R}_\infty \). Then \( k_1 = k - k_0 \in M_\infty \) and \( k_1 \) is orthogonal on \( \mathcal{R}_\infty \) and so to \( AM_\infty \). Hence \( k_1 \) is orthogonal to \( AK_1 \), that is \( A^{1/2}k_1 = 0 \). This means \( k_1 \in \mathcal{N}(A) \), therefore \( k_1 = 0 \) because we also have \( k_1 \in \overline{\mathcal{R}(A)} \). Thus \( k = k_0 \in \mathcal{R}_\infty \), which gives the required equality. Consequently, \( \mathcal{R}_\infty \) reduces \( \hat{T} \).

Next, if \( V \) is the minimal isometric dilation of \( \hat{T} \) then, since \( V \) is a lifting for \( \hat{T} \), \( V \) will be also a lifting for the isometry \( \hat{T}|_{\mathcal{R}_\infty} \), hence \( V \) is an extension for \( \hat{T}|_{\mathcal{R}_\infty} \) (see [4]). Thus, for \( h \in M_\infty \), \( n, m \geq 0 \) and \( j \geq 1 \) we obtain

\[
V^n \hat{T}^* A^{j/2}h = \hat{T}^n \hat{T}^* A^{j/2}h \in \overline{\mathcal{R}(A)}
\]

because \( A^{j/2}h \in \mathcal{R}_\infty \). Consequently, \( M_\infty \subset \widetilde{M}_0 \) what ends the proof. \( \Box \)

**Corollary 2.3.** Let \( T \) be a regular \( A \)-contraction on \( \mathcal{H} \). Then \( \mathcal{H} \) admits a unique orthogonal decomposition of the form

\[
\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1,
\]

where both subspaces reduce \( A \) and \( A^{1/2}T \), such that \( A^{1/2}T \) is an \( A \)-weighted isometry on \( \mathcal{H}_0 \) and \( A^{1/2}T \) is a completely non \( A \)-weighted isometry on \( \mathcal{H}_1 \). Furthermore, one has \( \mathcal{H}_0 = M_\infty \) and

\[
\mathcal{H}_1 = \bigvee_{n,j \geq 0} A^{1/2}T^n \overline{\mathcal{R}(A - T^* A T^j)}.
\]

**Proof.** This follows from Theorem 2.1, and (2.4) is obtained from (1.3) and (2.1). \( \Box \)
Having in view this decomposition, we call $\mathcal{M}_\infty$ the maximum $A$-weighted isometric part of $\mathcal{H}$ relative to the $A$-contraction $T$.

**Remark 2.4.** From the corresponding maximality properties of the subspaces $\mathcal{M}_\infty$ and $\mathcal{N}_\infty$ we have immediately the inclusions

$$\mathcal{M}_0 \subset \mathcal{M}_\infty \subset \mathcal{N}_\infty.$$  

In addition, one has $\mathcal{M}_0 = \mathcal{M}_\infty$ if and only if $\mathcal{M}_\infty$ is invariant for $T^*$, and $\mathcal{M}_\infty = \mathcal{N}_\infty$ if and only if $\mathcal{N}_\infty$ is invariant for $T^*A^{1/2}$.

**Proposition 2.5.** Let $T$ be an $A$-contraction on $\mathcal{H}$ such that $AT = TA$. Then the maximum subspace which reduces $A$ and $T$ on which $T$ is an $A$-isometry is

$$\mathcal{M}_0 = \mathcal{H} \oplus \bigvee_{n,j \geq 0} T^n(I - T^{*j}T^j)\overline{\mathcal{R}(A)}.$$  

**Proof.** Since $A$ and $T$ commute, the $A$-contraction $T$ is regular. Then from (2.3) and (2.4) we infer that the corresponding $A$-weighted isometric part of $\mathcal{H}$ is the subspace

$$\mathcal{M}_\infty = \mathcal{H} \oplus \bigvee_{n,j \geq 0} T^nA^{1/2}\overline{\mathcal{R}(A - T^{*j}AT^j)}.$$  

This shows that $\mathcal{M}_\infty$ is invariant for $T^*$ and by Remark 2.4 we have $\mathcal{M}_\infty = \mathcal{M}_0$, this being the maximum subspace which reduces $T$ to an $A$-isometry.

To prove the formula (2.5), we firstly remark that for $j \geq 1$,

$$\overline{\mathcal{R}(A - T^{*j}AT^j)} = (I - T^{*j}T^j)\overline{A^{1/2}\mathcal{H}} = (I - T^{*j}T^j)A^{1/2}\mathcal{H}.$$  

Thus for $n, j \geq 0$ one has

$$[T^nA^{1/2}\overline{\mathcal{R}(A - T^{*j}AT^j)}]^{\perp} = [T^n(I - T^{*j}T^j)A\mathcal{H}]^{\perp}$$  

$$= [T^n(I - T^{*j}T^j)A\mathcal{H}] = [T^n(I - T^{*j}T^j)\overline{A\mathcal{H}}]^{\perp},$$

whence it follows that

$$[\bigvee_{n,j \geq 0} T^nA^{1/2}\overline{\mathcal{R}(A - T^{*j}AT^j)}]^{\perp} = \{h \in \mathcal{H} : V^nT^*h \in \mathcal{H}, \ n, m \geq 0\}.$$  

Hence the subspace $\mathcal{M}_0 = \mathcal{M}_\infty$ has the form (2.5). \qed

We remark that the above proposition completes the Proposition 2.8 [12]. In general we cannot obtain $\mathcal{M}_\infty = \mathcal{N}_\infty$, but it is possible to have this equality in certain cases, as we see below. First we recover a usual decomposition of a contraction.

**Corollary 2.6.** If $T$ is a contraction on $\mathcal{H}$ and $V$ is the minimal isometric dilation of $T$, then the maximum subspace which reduces $T$ to an isometry is

$$\mathcal{H}_t = \mathcal{H} \oplus \bigvee_{n,j \geq 0} T^n(I - T^{*j}T^j)\mathcal{H}$$  

$$\mathcal{H}_t = \{h \in \mathcal{H} : V^nT^*h \in \mathcal{H}, \ n, m \geq 0\}.$$
Hence, $\mathcal{H}$ admits a unique orthogonal decomposition of the form
\begin{equation}
\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s \oplus \mathcal{H}_c,
\end{equation}
where the three subspaces reduce $T$, such that $T$ is unitary on $\mathcal{H}_u$, $T$ is a shift on $\mathcal{H}_s$ and $T$ is completely nonisometric on $\mathcal{H}_c$.

Proof. One applies Proposition 2.5 and Proposition 2.2 in the case $A = I$. Also, the decomposition (2.7) is obtained by combining in this case the decomposition (2.3) with the Wold decomposition of $\mathcal{H}_i$ in a unitary part and a shift part. \hfill \square

The following corollary completes Corollary 2.9 [12].

Corollary 2.7. Let $T$ be a regular $A$-contraction on $\mathcal{H}$. Then the maximum subspace $\overline{\mathcal{M}_0}$ which reduces $\hat{T}$ to an $A|_{\overline{R(A)}}$-isometry coincides with the maximum subspace $\overline{\mathcal{H}_0}$ which reduces $\hat{T}$ to an isometry. In fact, one has
\begin{equation}
\overline{\mathcal{H}_0} = \overline{\mathcal{M}_0} = M_\infty \cap \overline{R(A)} = A^{1/2} M_\infty,
\end{equation}
and
\begin{equation}
\mathcal{M}_\infty = \mathcal{H}_0 \oplus N(A).
\end{equation}

Proof. Since $\hat{T}$ and $A|_{\overline{R(A)}}$ commute, the relations (2.5) and (2.6) give
\begin{align*}
\overline{\mathcal{M}_0} &= \overline{R(A)} \cap \bigvee_{n,j \geq 0} \hat{T}^n (I - \hat{T}^* j ) \overline{R(A)} \\
&= \overline{R(A)} \cap \bigvee_{n,j \geq 0} \hat{T}^n (I - \hat{T}^* j \hat{T}^j) \overline{R(A)} = \mathcal{H}_0.
\end{align*}
The second equality in (2.8) is quoted in [12], and the last equality in (2.8) follows from the proof of Proposition 2.2. Finally (2.9) is derived from (2.8) because $M_\infty$ contains $N(A)$. \hfill \square

Notice that if $A$ is injective in this corollary, then
\begin{equation}
\overline{\mathcal{H}_0} = \overline{\mathcal{M}_0} = M_\infty.
\end{equation}

Remark 2.8. Suppose that $T$ is a regular $A$-contraction such that $T^2 = 0$. Then one infers that $N_\infty \subseteq N(A)$, hence
\begin{equation}
N(A) = N = M_0 = M_\infty = N_\infty,
\end{equation}
where $N = N(A - AT)$. But in general, $N_\infty \nsubseteq N_0$ (as in Example 4.3 [9]), and in this case, $A^{1/2} T$ is not quasinormal on $\mathcal{H}$ (by Theorem 2.1).

Remark 2.9. Assume that $T$ is a regular $A$-contraction with $T^2 = T$. Then $A^{1/2} T$ is quasinormal on $\mathcal{H}$ because
\begin{equation}
A^{1/2} T T^* A T = A^{1/2} T A T = A T A^{1/2} T = T^* A A^{1/2} T^2 = T^* A T A^{1/2} T,
\end{equation}
where we used the fact that $AT = T^*A$ (see [3]). Then both Theorem 2.1 and the fact that $T^*AT = AT$ imply in this case

$$\mathcal{N} = \mathcal{M}_\infty = \mathcal{N}_\infty = \mathcal{N}_0.$$

Furthermore, for $\hat{T}$ as in Corollary 2.7 we have

$$\hat{\mathcal{M}}_0 = \mathcal{N} \cap \overline{\mathcal{R}(A)} = \mathcal{N}(I - \hat{T}),$$

hence $\hat{T}|_{\hat{\mathcal{M}}_0} = I_{\hat{\mathcal{M}}_0}$. But $\hat{T}^2 = \hat{T}$ (as $T^2 = T$) and so $\hat{T} = 0$ on $\overline{\mathcal{M}_0} = \overline{\mathcal{R}(I - \hat{T})}$. So, $\hat{T}$ is the orthogonal projection onto $\mathcal{N}(I - \hat{T})$. Moreover, when $A$ is injective we have $T = \hat{T}$. Indeed, in this case $\hat{T}^*A^{1/2} = A^{1/2}\hat{T}^* = T^*A^{1/2}$ and $\hat{T}^* = \hat{T}^*$ on $\mathcal{H} = \overline{\mathcal{R}(A)}$, we obtain

$$T^*(I - \hat{T}^*)A^{1/2} = T^*A^{1/2}(I - \hat{T}^*) = A^{1/2}\hat{T}^*(I - \hat{T}^*) = \{0\},$$

hence $T^*|_{\overline{\mathcal{R}(I - \hat{T})}} = 0$. On the other hand, for $k = A^{1/2}h$ where $h \in \mathcal{N} = \mathcal{N}(I - \hat{T})$, we have $T^*k = A^{1/2}\hat{T}^*h = A^{1/2}h = k$. As $\mathcal{N}(I - \hat{T}) = A^{1/2}\mathcal{N}$ (see [10]), we deduce that $T^*$ is the identity on $\mathcal{N}(I - \hat{T})$. Thus, $T^*$ is the orthogonal projection onto $\mathcal{N}(I - \hat{T})$, and consequently, $T = T^* = \hat{T}$ on $\mathcal{H} = \overline{\mathcal{R}(A)}$.

Finally we remark that if $A$ is invertible and $T$ is an $A$-contraction with $T^2 = T$, then $T$ is an orthogonal projection if and only if the $A$-contraction $T$ is regular, or equivalently $AT = TA$.

3. Applications to quasinormal operators and quasi-isometries

We derive from the above results some facts concerning the quasinormal operators.

**Proposition 3.1.** Let $T$ be a quasinormal contraction on $\mathcal{H}$. The following statements hold:

(i) $\mathcal{N}(I - T^*T)$ is the maximum subspace which reduces $T$ to an isometry.

(ii) $\mathcal{H}_q = \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T)$ is the maximum subspace which reduces $T$ to a quasi-isometry, or equivalently, to a partial isometry. Also, $\mathcal{H}_q$ is the maximum $T^*T$-weighted isometric part of $\mathcal{H}$ relative to the $T^*T$-contraction $T$. In addition, $T$ is normal on $\mathcal{H}_q$ if and only if $|T|T$ is normal on $\mathcal{H}_q$.

**Proof.** Assertion (i) follows applying Theorem 2.1 to the quasinormal $I$-contraction $T$, which gives in this case

$$\mathcal{M}_0 = \mathcal{M}_\infty = \mathcal{N}_0 = \mathcal{N}(I - T^*T).$$

For (ii), we apply Theorem 2.1 to the regular $T^*T$-contraction $T$, having in view that $|T|T$ is quasinormal. In this case one has

$$\mathcal{M}_\infty = \mathcal{N}_0 = \mathcal{N}(T^*T - T^*T^2) = \mathcal{N}(T^*T - (T^*T)^2) = \mathcal{N}(I - T^*T) \oplus \mathcal{N}(T),$$

the orthogonal decomposition being obvious. This subspace, denoted $\mathcal{H}_q$, is the maximum $T^*T$-weighted isometric part of $\mathcal{H}$ relative to $T$. On the other
hand, since $T^*$ and $T^*T$ commute, $\mathcal{H}_q = \mathcal{M}_\infty$ is also invariant for $T^*$, and by Remark 2.4 we have $\mathcal{H}_q = \mathcal{M}_0$. So, $\mathcal{H}_q$ is the maximum subspace which reduces $T$ to a $T^*T$-isometry, that is to a quasi-isometry, or equivalently, to a partial isometry (see [6], [11]).

The last assertion in (ii) follows from Theorem 2.9 [6], since $T$ is a quasinormal quasi-isometry on $\mathcal{H}_q$. \hfill \Box

**Proposition 3.2.** Let $T$ be a quasinormal contraction on $\mathcal{H}$. The following statements hold:

(i) $\mathcal{H}_* = \mathcal{H} \oplus \bigvee_{n,j \geq 0} T^n(I - T^j T^{*j}) \overline{\mathcal{R}(T^*)}$ is the maximum subspace which reduces $T$ to a normal quasi-isometry, or equivalently, on which $T^*$ is a $T^*T$-isometry. Furthermore, $\mathcal{H}_*$ is the maximum $T^*T$-weighted isometric part of $\mathcal{H}$ relative to the $T^*T$-contraction $T^*$.

(ii) $\mathcal{H} = \bigcap_{n \geq 0} T^n \mathcal{N}(I - T T^*) \oplus \bigvee_{n,j \geq 0} T^n(I - T^j T^{*j}) \overline{\mathcal{R}(T^*)} \oplus \mathcal{N}(T)$, where the first subspace reduce $T$ to a unitary operator, and the second subspace contains no nonzero subspace which reduces $T$ to a normal quasi-isometry.

**Proof.** Since $T$ is a quasinormal contraction one has $TT^* \leq T^*T \leq I$, whence $TT^* T^* \leq TT^* \leq T^*T$. Hence $T^*$ is a $T^*T$-contraction, which is regular because $T^*$ commutes with $T^*T$. In this case, the maximum $T^*T$-weighted isometric part relative to $T^*$ is just the subspace $\mathcal{H}_*$ from (i), having in view (2.5) and the fact that $\overline{\mathcal{R}(T^*T)} = \overline{\mathcal{R}(T^*)}$. Now the form of $\mathcal{H}_*$ immediately gives that $\mathcal{H}_*$ is invariant for $T$. But by Theorem 2.1, $\mathcal{H}_*$ is also invariant for $T^*$, and $T^*$ is a $T^*T$-isometry on $\mathcal{H}_*$. Consequently, by Remark 2.4 we have that $\mathcal{H}_*$ is the maximum subspace which reduces $T$, on which $T^*$ is a $T^*T$-isometry. On the other hand, we have

$\mathcal{H}_* \subset \mathcal{N}(T^*T - TT^*T^*) \subset \mathcal{N}(T^*T - TT^*) \cap \mathcal{N}(T^*T - T^2T^2)$,

the second inclusion being proved in Theorem 3.4 [11]. So $\mathcal{H}_*$ reduces $T$ to a normal quasi-isometry, and furthermore, it is the maximum subspace with this property. Indeed, if $\mathcal{M} \subset \mathcal{H}$ is another subspace reducing $T$ to a normal quasi-isometry, then $T^*$ will be a $T^*T$-isometry on $\mathcal{M}$, hence $\mathcal{M} \subset \mathcal{H}_*$ by the above remark. Hence, $\mathcal{H}_*$ is the maximum subspace which reduces $T$ to a normal quasi-isometry, and all assertions from (i) are proved.

Next, by Corollary 3.5 [11] we have

$\mathcal{H}_* = \bigcap_{n=0}^\infty T^n \mathcal{N}(I - TT^*) \oplus \mathcal{N}(T)$,

which leads to the decomposition (3.1). The required properties of the subspaces from (3.1) are obtained from the above remarks on $\mathcal{H}_*$ and from Theorem 3.1 [11]. \hfill \Box
In the sequel we denote as usually $|T| = (T^*T)^{1/2}$.

**Corollary 3.3.** Let $T$ be an injective quasinormal operator on $\mathcal{H}$, and $T = W|T|$ be the polar decomposition of $T$. Then the maximum subspace $\mathcal{H}_u$ which reduces $W$ to a unitary operator is the maximum $T^*T$-weighted isometric part of $\mathcal{H}$ relative to the $T^*T$-contraction $W^*$. Moreover, $\mathcal{H}_u$ reduces $T$ to a normal operator, and $\mathcal{H} \cap \mathcal{H}_u$ reduces $T$ to a pure quasinormal operator.

**Proof.** Since $T$ is quasinormal injective, $W$ is an isometry which commutes with $|T|$. Then the decomposition (3.1) with $W$ instead of $T$ gives $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$, where

$$\mathcal{H}_u = \bigcap_{n \geq 1} W^n \mathcal{H}$$

is the unitary part of $\mathcal{H}$ relative to $W$. But (by Proposition 3.2) $\mathcal{H}_u$ is just the maximum $T^*T$-weighted isometric subspace of $\mathcal{H}$ relative to $W^*$ as a $T^*T$-contraction. Then (by Theorem 2.1) $\mathcal{H}_u$ reduces $T^*T$, hence $\mathcal{H}_u$ reduces $T = W|T|$ and clearly, $T$ is normal ($W$ being unitary) on $\mathcal{H}_u$. Finally, $\mathcal{H}_s = \mathcal{H} \cap \mathcal{H}_u$ reduces $W$ to a shift ($W$ being an isometry), and $\mathcal{H}_s$ reduces $T$. But $\mathcal{H}_s$ contains no non zero subspace which reduces $|T|W^* = T^*$ to a $T^*T$-weighted isometry, that is with property $TT^* = T^*$. This means that $\mathcal{H}_s$ reduces $T$ to a pure quasinormal operator. \(\Box\)

As an application one obtains the following result.

**Corollary 3.4.** Let $T$ be a quasinormal operator on $\mathcal{H}$ and $T = W|T|$ be the polar decomposition of $T$. Then the maximum subspace which reduces $T$ to a normal operator is

$$\mathcal{H}_n = \mathcal{H}_u \oplus \mathcal{N}(T),$$

where $\mathcal{H}_u$ is the unitary part of $\mathcal{H}$ relative to $W$.

Moreover, $\mathcal{H}_n$ is the maximum subspace which reduces $W$ to a normal partial isometry, and $W$ is a shift on $\mathcal{H} \cap \mathcal{H}_n$.

**Proof.** Since $\mathcal{N}(T)$ reduces $T$, one can defines the operator $T_0 = T|_{\overline{\mathcal{R}(T^*)}}$ in $\mathcal{B}(\overline{\mathcal{R}(T^*)})$, and $T_0$ is an injective quasinormal operator. Then the polar decomposition of $T_0$ is $T_0 = W_0|T_0|$, where $W_0 = W|_{\overline{\mathcal{R}(T^*)}}$. Clearly, $\overline{\mathcal{R}(T^*)}$ reduces $W$ because $W$ and $|T|$ commute.

Now, if we consider the decomposition $\overline{\mathcal{R}(T^*)} = \mathcal{H}_u \oplus \mathcal{H}_s$ in the unitary part $\mathcal{H}_u$ and the completely non-unitary part $\mathcal{H}_s$ relative to $W_0$, then (by Corollary 3.3) $\mathcal{H}_u$ reduces $T_0$ to a normal operator, while $\mathcal{H}_s$ reduces $T_0$ to a pure quasinormal. Clearly, one has

$$\mathcal{H}_u = \bigcap_{n=1}^{\infty} W_0^n \overline{\mathcal{R}(T^*)} = \bigcap_{n=1}^{\infty} W^n \mathcal{H} \subset \mathcal{W} \subset \overline{\mathcal{R}(T)},$$

hence $\mathcal{H}_u$ and $\mathcal{N}(T^*)$ are orthogonal. As $\mathcal{N}(T) \subset \mathcal{N}(T^*)$, $T$ being quasinormal, it follows that $\mathcal{H}_u$ and $\mathcal{N}(T)$ are orthogonal. Thus $\mathcal{H}_n = \mathcal{H}_u \oplus \mathcal{N}(T)$ is the
maximum subspace which reduces $T$ to a unitary operator, because $\mathcal{H} \ominus \mathcal{H}_n = \mathcal{R}(T^*) \ominus \mathcal{H}_n$ reduces $T$ to a pure operator. But $W$ is unitary on $\mathcal{H}_n$, and so $\mathcal{H}_n$ reduces $W$ to a normal operator.

Finally, since $\mathcal{R}(T^*)$ reduces $W$, we infer that $W|_{\mathcal{H} \ominus \mathcal{H}_n} = W_0|_{\mathcal{R}(T^*) \ominus \mathcal{H}_n}$ is a shift, and in particular $W$ is pure on $\mathcal{H} \ominus \mathcal{H}_n$ (the unitary part of $W$ being $\mathcal{H}_n$). Consequently, $\mathcal{H}_n$ is the maximum subspace on which $W$ is a normal operator.

Now we obtain, as application, an orthogonal decomposition of $\mathcal{H}$ induced by a quasinormal contraction $T$, where all reducing subspaces can be completely described in terms of $T$ and $W$.

**Theorem 3.5.** Let $T$ be a quasinormal contraction on $\mathcal{H}$ with the polar decomposition $T = W|T|$. Then $\mathcal{H}$ has the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4,$$

where

$$\mathcal{H}_0 = \bigcap_{n=0}^{\infty} T^n \mathcal{N}(I - TT^*),$$

$$\mathcal{H}_1 = \bigoplus_{n=0}^{\infty} T^n [\mathcal{N}(I - T^*T) \cap \mathcal{N}(T^*)],$$

$$\mathcal{H}_2 = \mathcal{N}(T - |T|) \ominus \mathcal{N}(I - T) = [\mathcal{N}(I - W) \ominus \mathcal{N}(I - T)] \ominus \mathcal{N}(T),$$

$$\mathcal{H}_3 = \bigcap_{n=1}^{\infty} W^n \mathcal{H} \ominus \mathcal{H}_0 \ominus [\mathcal{N}(I - W) \ominus \mathcal{N}(I - T)],$$

$$\mathcal{H}_4 = \bigoplus_{n=0}^{\infty} W^n [\mathcal{N}(T^*) \ominus \mathcal{N}(T)] \ominus \mathcal{H}_1.$$  

Furthermore, all subspaces in (3.3) reduce $T$ such that $T|_{\mathcal{H}_0}$ is unitary, $T|_{\mathcal{H}_1}$ is a shift, $T|_{\mathcal{H}_2}$ is positive completely nonisometric quasinormal, $T|_{\mathcal{H}_3}$ is normal completely nonpositive and completely nonisometric quasinormal, and $T|_{\mathcal{H}_4}$ is a completely nonisometric pure quasinormal contraction. Also, one has

$$\mathcal{N}(T - |T|) = \mathcal{N}(I - W) \oplus \mathcal{N}(T),$$

$$\mathcal{N}(|T| - T|T|) = \mathcal{N}(I - T) \oplus \mathcal{N}(T).$$

**Proof.** Using the notations from the previous proof we have

$$\mathcal{H} = \mathcal{H}_u \ominus \mathcal{H}_s \ominus \mathcal{N}(T),$$

where $\mathcal{H}_u = \bigcap_{n=1}^{\infty} W^n \mathcal{H}$ and $\mathcal{H}_s = \bigoplus_{n=0}^{\infty} W^n \mathcal{N}(W^*_n)$. Let $\mathcal{H}_0$ be the maximum unitary part for the contraction $T$. As $T$ and $W$ commute, $\mathcal{H}_0$ will reduces $W$, and $T = W$ on $\mathcal{H}_0$, $T$ being unitary on $\mathcal{H}_0$. So $W$ is unitary on $\mathcal{H}_0$, hence $\mathcal{H}_0 \subset \mathcal{H}_u$, and the above form of $\mathcal{H}_0$ is given in [8] (see also [11, 12]). We also remark that $\mathcal{H}_u \ominus \mathcal{H}_s \subset \mathcal{H}_n \ominus \mathcal{H}_0$, therefore $T$ is a completely nonisometric normal contraction.
Now, we know from Theorem 3.1 [9] that the maximum subspace which reduces $T$ to a positive operator is $\mathcal{N}_T = \mathcal{N}(T - |T|)$. We even get the decomposition (3.5) for $\mathcal{N}_T$. Indeed, $W$ being a $|T|^2$-contraction which commutes with $|T|$, we have $\widehat{W} = W|\mathcal{N}(T)|$. Also, $\mathcal{N}(I - \widehat{W}) = \mathcal{N}(I - W)$ because $\mathcal{N}(W) = \mathcal{N}(T)$ and

$$\mathcal{N}(I - W) = \mathcal{N}(I - W^*) \subset \mathcal{N}(T)^\perp.$$  

Thus, for the regular $|T|^2$-contraction $W$ we obtain

$$\mathcal{N}_T = \mathcal{N}(|T| - |T|W) = \mathcal{N}(I - \widehat{W}) \oplus \mathcal{N}(|T|) = \mathcal{N}(I - W) \oplus \mathcal{N}(T).$$

Clearly, $\mathcal{N}(I - W)$ reduces $T$ to a positive contraction, and since $T$ is unitary on $\mathcal{N}(I - T)$ and $\mathcal{N}(I - T) = \mathcal{N}(I - T^*) \subset \mathcal{N}(T)^\perp$, we have $T = W$ on $\mathcal{N}(I - T)$. Hence $\mathcal{N}(I - T) \subset \mathcal{N}(I - W)$. Then the operator $T|_{\mathcal{N}_T \ominus \mathcal{N}(I - T)}$ being positive, it is completely nonisometric, or equivalently, a completely non unitary contraction. Hence $T$ has the required properties on the subspace

$$\mathcal{H}_2 := \mathcal{N}_T \ominus \mathcal{N}(I - T) = [\mathcal{N}(I - W) \ominus \mathcal{N}(I - T)] \oplus \mathcal{N}(T),$$

and we have, in addition,

$$\mathcal{H}_2 \subset (\mathcal{H}_u \ominus \mathcal{H}_0) \oplus \mathcal{N}(T) = \mathcal{H}_n \ominus \mathcal{H}_0.$$

Next, it is clear that the subspace

$$\mathcal{H}_3 := (\mathcal{H}_n \ominus \mathcal{H}_0) \ominus [\mathcal{N}_T \ominus \mathcal{N}(I - T)] = (\mathcal{H}_u \ominus \mathcal{H}_0) \ominus [\mathcal{N}(I - W) \ominus \mathcal{N}(I - T)]$$

reduces $T$ to a normal, completely nonisometric and completely nonpositive contraction. Clearly, $\mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 = \mathcal{H}_n$.

It remains to analyse the subspace $\mathcal{H}_s$. Recall that $W_0 = W|_{\mathcal{N}(T)^\perp}$, therefore $\mathcal{N}(W_0^*) = \mathcal{N}(T^*) \ominus \mathcal{N}(T)$, and also

$$\mathcal{H}_s = \bigoplus_{n=0}^{\infty} W^n[\mathcal{N}(T^*) \ominus \mathcal{N}(T)].$$

It is immediate that the subspace

$$\mathcal{H}_1 := \mathcal{H}_s \cap \mathcal{N}(I - T^*T)$$

reduces $T$ to a completely non unitary isometry, hence $T|_{\mathcal{H}_1}$ is a shift. Thus, we have

$$\mathcal{H}_1 = \bigoplus_{n=0}^{\infty} T^n \mathcal{N}(T^*|_{\mathcal{H}_1}),$$

and it is easy to see that

$$\mathcal{N}(T^*|_{\mathcal{H}_1}) = \mathcal{N}(I - T^*T) \cap [\mathcal{N}(T^*) \ominus \mathcal{N}(T)] = \mathcal{N}(I - T^*T) \cap \mathcal{N}(T^*).$$

Also, we remark that $\mathcal{H}_1$ is the maximum subspace of $\mathcal{H}_s$ which reduces $T$ to an isometry, because $\mathcal{N}(I - T^*T)$ has the same property in $\mathcal{H}$ (by Proposition 3.1). This also gives that the subspace $\mathcal{H}_4 := \mathcal{H}_s \ominus \mathcal{H}_1$ reduces $T$ to a completely nonisometric pure quasinormal contraction.
Finally, the second decomposition from (3.5) can be proved in a similar way as before, having in view that $T$ is a regular $|T|^2$-contraction and $\hat{T} = T|_{\mathcal{N}(T)}$. The proof is finished.

Corollary 3.6. Let $T$ be a quasinormal contraction on $\mathcal{H}$ with the polar decomposition $T = W |T|$. One has:

(i) If $T$ is completely non unitary, then $\mathcal{H}$ has the decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4,$$

where $\mathcal{H}_1$ and $\mathcal{H}_4$ are as in (3.4), and

$$\mathcal{H}_2 = \mathcal{N}(T - |T|), \quad \mathcal{H}_3 = \bigcap_{n=1}^\infty W^n \mathcal{H} \ominus \mathcal{N}(I - W).$$

(ii) If $W$ is completely non unitary, then $T$ is completely non unitary and the decomposition (3.6) one reduces to

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{N}(T) \oplus \{0\} \oplus \mathcal{H}_4.$$

Corollary 3.7. Let $T$ be a quasinormal partial isometry on $\mathcal{H}$. Then $T$ is a quasi-isometry and one has the orthogonal decomposition

$$\mathcal{H} = \bigcap_{n=0}^\infty T^n \mathcal{N}(I - T^*T) \oplus \bigoplus_{n=0}^\infty T^n [\mathcal{N}(T^*) \ominus \mathcal{N}(T)] \oplus \mathcal{N}(T).$$

Proof. Clearly, $T = W$ in Theorem 3.5, therefore $\mathcal{H}_1 = \mathcal{N}(T)$ and $\mathcal{H}_3 = \{0\}$. Also, $T = W$ is a shift on $\mathcal{H}_s = \mathcal{H} \ominus \mathcal{H}_n$ by Corollary 3.4, hence $\mathcal{H}_1 = \mathcal{H}_s$ and $\mathcal{H}_4 = \{0\}$. Therefore, we infer from (3.3)

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{N}(I - T^*T) \ominus \mathcal{N}(T),$$

which means by Proposition 3.1 (ii) that $T$ is a quasi-isometry. Also, this implies that

$$\mathcal{H}_0 = \bigcap_{n=0}^\infty T^n \mathcal{H} = \bigcap_{n=0}^\infty T^n \mathcal{N}(I - T^*T).$$

Thus we obtain $T = V \oplus 0$ where $V = T|_{\mathcal{N}(I - T^*T)}$ is an isometry.

Remark from Corollary 2.3 [6] that a quasi-isometry is quasinormal if and only if it is a partial isometry. Now we can obtain the following.

Corollary 3.8. Let $T \neq 0$ be a quasi-isometry with $|T|T$ a quasinormal operator on $\mathcal{H}$. Then the following assertions are equivalent:

(i) $\|T\| = 1$;

(ii) $T$ is partial isometry;

(iii) $T$ is quasinormal;

(iv) $T$ is hyponormal.

Furthermore, if $|T|T$ is normal then these assertions are also equivalent to each of the following two assertions:
(v) $T$ is normal;
(vi) $N(T) \subset N(T^*)$.

**Proof.** First we suppose $||T|| = 1$. Then $S = |T|T$ is a contraction and $S^*S = T^*TT = T^*T$ because $T$ is a quasi-isometry. Also, we have

$$S^2S^2 = S^*T^*TS = T^*|T|T^*T|T| = T^*TT^*T^2,$$

whence one infers on one hand

$$S^2S^2 - S^*S = T^*TT^2T^2 - T^*T = (T^*T^2 - T)(T^*T^2 - T) \geq 0.$$

On the other hand, as $S^2 \leq I$ we have $S^2S^2 - S^*S \leq 0$, hence

$$S^*S = S^2,$$

or equivalently, $T^*T^2 - T = 0$. This means $T = T^*T^2$, that is $T = |S|S$. Since by hypothesis $S$ is quasinormal, hence $S$ and $|S|$ commute, it follows that $T$ is also quasinormal, or equivalently, $T$ is a partial isometry. Thus, we have that 

$(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv).$

Now if we assume that $T$ is hyponormal, then for $h \in \mathcal{H}$ we have

$$||T^*Th|| \leq ||T^2h|| = ||Th||,$$

whence $||T||^2 \leq ||T||$ and so $||T|| \leq 1$. Since we also have $||T|| \geq 1$ ($T$ being a quasi-isometry), it follows that $||T|| = 1$. Hence $(iv)$ implies $(i)$.

Next we suppose that $S = |T|T$ is a normal operator. If $||T|| = 1$ then as above we get that $T = |S|S$ is normal, and in this case $N(T) = N(T^*)$. Hence we have $(i) \Rightarrow (v) \Rightarrow (vi)$. The implication $(vi) \Rightarrow (v)$ is even Theorem 2.9 [6], and trivially $(v)$ implies $(iv)$. Consequently, all assertions $(i) - (vi)$ are mutually equivalent, if $|T|T$ is a normal operator. \( \square \)

**Remark 3.9.** From the previous proof we infer that for any quasi-isometry $T$ with $||T|| = 1$ one has

$$T = T^*T^2,$$

this fact being also quoted by S. M. Patel in [6]. Concerning the question from Remark 2.1 [6], namely if the condition $(vi)$ for a quasi-isometry $T$ assures that $T$ is normal, we can see a simple example which shows that this fact need not holds unless the assumption that $|T|T$ is normal. So, we consider the operator $T$ on $\mathcal{H} \oplus \mathcal{H}$ given by

$$T = \begin{pmatrix} V & 0 \\ 0 & Q \end{pmatrix},$$

where $V$ is an isometry and $Q$ is an orthogonal projection on $\mathcal{H}$. Then $T = |T|T$ is not normal, but $T$ is a quasi-isometry and

$$N(T) = \{0\} \oplus N(Q) \subset N(V^*) \oplus N(Q) = N(T^*).$$

So, the answer to Patel's question is negative. In fact, we have the following.

**Corollary 3.10.** A quasi-isometry $T$ is normal if and only if $|T|T$ is normal and $N(T) \subset N(T^*)$. 
As we remarked, a quasi-isometry $T$ with $|T|T$ normal is a normal partial isometry (by Theorem 2.9 [6]). So, the normal quasi-isometries, or equivalently, the normal partial isometries, play a similar role in the general context of $A$-contractions like the unitary operators for contractions. We will refer to this fact in a subsequent paper.

References


Institut Camille Jordan
Université Claude Bernard Lyon 1
21 av. Claude Bernard
69622 Villeurbanne cedex, France
E-mail address: suciu@math.univ-lyon1.fr