REGULARITY CRITERION ON WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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Abstract. Consider a weak solution $u$ of the Navier-Stokes equations in the class $L^2 \left( (0, T); \tilde{X}_1 (\mathbb{R}^d) \right)$. We establish a new approach to treat the regularity problem for the Navier-Stokes equation in term of the multiplier space $X_1 (\mathbb{R}^d)$.

1. Introduction

Consider the Navier-Stokes equations in $(0, T) \times \mathbb{R}^d$ with $0 < T < \infty$ and $d \geq 3$:

\begin{equation}
\begin{array}{rcl}
\partial_t u + (u, \nabla) u - \Delta u + \nabla p &=& 0, \quad (x, t) \in \mathbb{R}^d \times (0, \infty), \\
\nabla \cdot u &=& 0, \quad (x, t) \in \mathbb{R}^d \times (0, \infty), \\
u(x, 0) &=& a(x), \quad x \in \mathbb{R}^d,
\end{array}
\end{equation}

where $u = u(x, t)$ is the velocity field, $p = p(x, t)$ is the scalar pressure and $a(x)$ with $\text{div } a = 0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

In their famous paper, Leray [12] and Hopf [6] constructed a weak solution $u$ of (1.1) for arbitrary $a \in L^2$. The solution is called the Leray-Hopf weak solution. In the general case the problem on uniqueness and regularity of Leray-Hopf's weak solutions are still open question. Masuda [14] extended Serrin's class for uniqueness of weak solutions and made it clear that the class $L^\infty ((0, T); L^d (\mathbb{R}^d))$ plays an important role for uniqueness of weak solutions. Kozono-Sohr [8] showed that the uniqueness holds in $L^\infty ((0, T); L^d)$.

Foias [4] and Serrin [16] introduced the class $L^\alpha ((0, \infty); L^q)$ and showed that under the additional assumption

$$u \in L^\alpha ((0, \infty); L^q) \quad \text{for} \quad \frac{2}{\alpha} + \frac{d}{q} = 1 \quad \text{with} \quad q > d,$$

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u is the only weak solution.

The purpose of this paper is to improve the criterion on regularity of weak solutions to in the class \( L^2 \left( (0, T); \dot{X}_1(\mathbb{R}^d)^d \right) \). We know that for every \( a \in L^2_\sigma(\mathbb{R}^d) \), there is at least one weak solution \( u \) of (1.1) satisfying the energy inequality:

\[
\|u(t)\|^2_{L^2} + 2 \int_0^t \|\nabla u(\tau)\|^2_{L^2} \, d\tau \leq \|a\|^2_{L^2}.
\]

This is the solution obtained by Leray [12] in the class \( L^\infty((0, T); L^2_\sigma) \cap L^2((0, T); \dot{H}^1_\sigma) \) and satisfying (1.1) in the sense of distributions. The natural regularity obtained from the above energy inequality is that

\[
u \in L^\alpha((0, T); L^q(\mathbb{R}^d)) \quad \text{for} \quad \frac{2}{\alpha} + \frac{d}{q} = 2 \quad \text{with} \quad 2 \leq q \leq \frac{2d}{d - 2}.
\]

If Leray’s weak solution \( u \) satisfies the following

\[
u \in L^\alpha((0, T); L^q(\mathbb{R}^d)) \quad \text{for} \quad \frac{2}{\alpha} + \frac{d}{q} = 1 \quad \text{with} \quad q > d,
\]

then \( u \) is regular on \( (0, T] \). For more facts concerning regularity of weak solutions, we refer to a celebrated paper of Kozono-Sohr [8].

1.1. **BMO and Hardy space \( \mathcal{H}^1(\mathbb{R}^d) \)**

We recall that a locally summable function \( g \) on \( \mathbb{R}^d \) is said to have bounded mean oscillation if

\[
\|g\|_{BMO} = \sup_{x, R} \frac{1}{|B(x, R)|} \int_{B(x, R)} |g(y) - g_{B(x, R)}| \, dy < \infty,
\]

where

\[
g_{B(x, R)} = \frac{1}{|B(x, R)|} \int_{B(x, R)} g(y) \, dy.
\]

The class of functions of bounded mean oscillation is denoted by \( BMO \) and often is refereed as John-Nirenberg space.

Note that

\[
\|g\|_{BMO} = 0 \quad \text{if and only if} \quad g = \text{const}.
\]

It is thus natural to consider the quotient space \( BMO/\mathbb{R} \) with the norm induced by \( \|\cdot\|_{BMO} \). Then \( BMO/\mathbb{R} \) is a Banach space, which will also be denoted \( BMO \) for simplicity. We easily see that \( L^\infty \subset BMO \) with continuous injection. For \( f(x) = \log |x| \), we have \( f \in BMO \) but \( f \notin L^\infty \), so \( BMO \) is strictly larger than \( L^\infty \).

Next, we recall the definition and some of the main properties of Hardy spaces \( \mathcal{H}^p(\mathbb{R}^d) \) introduced by E. Stein and G. Weiss [18] (for more facts on these spaces see C. Fefferman and E. Stein [5]).
Definition 1 ([5]). Let $0 < p < \infty$, and let $\varphi \in S(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} \varphi \, dx = 1$. A tempered distribution $f$ belongs to the Hardy space $\mathcal{H}^p(\mathbb{R}^d)$ if
\begin{equation}
(1.2) \quad f^*(x) = \sup_{t > 0} |(\varphi_t \ast f)(x)| \in L^p(\mathbb{R}^d),
\end{equation}
where $\varphi_t(x) = t^{-d} \varphi(t^{-1} x)$.

It is known that if $f \in \mathcal{H}^p(\mathbb{R}^d)$, then (1.2) holds for all $\varphi \in S(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \varphi \, dx = 1$. The (quasi)-norm of $\mathcal{H}^p(\mathbb{R}^d)$ is defined, up to equivalence, by
\begin{equation*}
||f||_{\mathcal{H}^p(\mathbb{R}^d)} = ||f^*(x)||_{L^p(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |f^*(x)|^p \, dx \right)^{\frac{1}{p}}.
\end{equation*}

We known by ([5], [17]) that if $1 \leq p < \infty$, then $\mathcal{H}^p$ is a Banach space:
\begin{equation*}
\mathcal{H}^p(\mathbb{R}^d) = L^p(\mathbb{R}^d) \quad \text{for} \quad 1 < p < \infty,
\end{equation*}
\begin{equation*}
\mathcal{H}^1(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \quad \text{with continuous injection},
\end{equation*}
and that $\mathcal{H}^p(\mathbb{R}^d)$, $0 < p < 1$, are quasi-Banach spaces in the quasi-norm $||.||_{\mathcal{H}^p(\mathbb{R}^d)}$.

The crucial fact for our purpose is the boundedness of the Riesz transforms $R_j$ on all of the spaces $\mathcal{H}^p$. Furthermore, an $L^1$-function $f$ on $\mathbb{R}^d$ belongs to $\mathcal{H}^1(\mathbb{R}^d)$ if and only if its Riesz transforms $R_j f$ all belong to $L^1(\mathbb{R}^d)$ and
\begin{equation*}
||f||_{\mathcal{H}^1(\mathbb{R}^d)} \cong ||f||_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d ||R_j f||_{L^1(\mathbb{R}^d)} \quad \text{(equivalent norms)}.
\end{equation*}

Notice that all function $f \in \mathcal{H}^1(\mathbb{R}^d)$ satisfy
\begin{equation}
(1.3) \quad \int_{\mathbb{R}^d} f(x) \, dx = 0.
\end{equation}

Indeed, the assumption $f \in \mathcal{H}^1(\mathbb{R}^d)$ implies that the Fourier transforms
\begin{equation*}
\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i x \xi} \, dx \quad \text{and} \quad \widehat{R_j f}(\xi) = \frac{i \xi_j}{|\xi|} \widehat{f}(\xi), \quad (j = 1, \ldots, d),
\end{equation*}
are all continuous on $\mathbb{R}^d$, so $\widehat{f}(0) = 0$, and (1.3) is proved.

A fundamental theorem in the theory of Hardy spaces $\mathcal{H}^1(\mathbb{R}^d)$ developed by C. Fefferman and E. Stein [5] asserts

Theorem 1 (Fefferman). The dual space of $\mathcal{H}^1(\mathbb{R}^d)$ is $BMO$. More precisely, $L$ is a continuous linear functional on $\mathcal{H}^1(\mathbb{R}^d)$ if and only if it can be represented
as
\[ L(f) = \int_{\mathbb{R}^d} fg \]
for some function \( g \) in BMO, moreover for any \( g \in \text{BMO} \) and any \( f \in \mathcal{H}^1(\mathbb{R}^d) \) we have
\[
\left| \int_{\mathbb{R}^d} fg \, dx \right| \leq c(d) \| f \|_{\mathcal{H}^1} \| g \|_{\text{BMO}}.
\]

Let \( \gamma > 1 \). We define the maximal function of \( f \) depending on \( \gamma \),
\[
M_\gamma f(x) = \sup_{t > 0} \left( \frac{1}{|B_t(x)|} \int_{B_t(x)} |f(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}}.
\]

We begin by establishing the following result which is a variant of the Hardy-Littlewood maximal theorem. We need

**Lemma 1.** If \( \gamma < p \leq \infty \), then
\[
M_\gamma : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)
\]
is bounded.

See [17] for the proof.

In [2], Coifman, Lions, Meyer, and Semmes, it was shown that the Hardy spaces can be used to analyze the regularity of the various nonlinear quantities by the compensated compactness theory due to L. Murat [13] and F. Tartar [15]. Since then, these spaces play an important role in studying the regularity of solutions to partial differential equations. In particular, it was shown that for exponents \( p, q \) with \( 1 < p < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \), and vector fields \( u \in L^p(\mathbb{R}^d)^d \), \( v \in L^q(\mathbb{R}^d)^d \) with \( \text{div} \, u = 0 \), \( \text{curl} \, v = 0 \) in the sense of distributions, the scalar product \( u \cdot v \) belongs to the Hardy space \( \mathcal{H}^1(\mathbb{R}^d) \). Moreover, there exists a positive constant \( C \) such that
\[
\| u \cdot v \|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C \| u \|_{L^p} \| v \|_{L^q}.
\]
The main purpose of this subsection is to prove two facts about div-curl lemma without assuming any a priori assumptions on exact cancelation, namely the divergence and curl need not be zero, and which lead to \( \text{div} \, (uv) \) being in the Hardy space \( \mathcal{H}^1(\mathbb{R}^d) \).

The proof will be divided into two parts. In part 1, we consider the case \( u \) and \( v \) are supported on the ball \( |x| \leq R_0 \) where \( R_0 > 1 \) is a positive constant to be determined later, while in part 2, the general case follows by partition of unity. In order to simplify the presentation, we take \( p = q = 2 \).

The Sobolev space \( H^1_0(\mathbb{R}^d) \), \( 1 \leq p < \infty \), consists of functions \( f \in L^p(\mathbb{R}^d) \) such that \( | \nabla f | \in L^p(\mathbb{R}^d) \). It is a Banach space with respect to the norm
\[
\| f \|_{H^1_0} = \| f \|_{L^p} + \| \nabla f \|_{L^p}.
\]
Specifically, we will prove

**Theorem 2.** Let \( u \in H^1_p (\mathbb{R}^d) \) and \( v \in H^1_q (\mathbb{R}^d) \), \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \). Then there exists a positive constant \( C(d) \) such that

\[
\| \text{div} (uv) \|_{H^1(\mathbb{R}^d)} \leq C \left( \| u \|_{L^p} \| \nabla v \|_{L^q} + \| \text{div} u \|_{L^p} \| v \|_{L^q} \right).
\]

**Remark 1.** Such inequalities and their generalizations are useful in hydrodynamics. Reader is referred, in particular to [2], [3].

Theorem 2 is a generalized version of the “div-curl” lemma ([2], Theorem II.1). Observe that when \( \text{div} u = 0 \), Theorem 2 reduces to the classical div-curl lemma [2].

The following result due to [2], shows the importance of the Hardy space theory in estimating the non-linear term \( u.\nabla v \) attached to the Navier-Stokes equations and this produces a useful tool for PDE.

**Lemma 2.** Let \( 1 < p < \infty, 1 < q < d \) and \( \frac{1}{p} = \frac{1}{p} + \frac{1}{q} + 1 \). If \( u \in L^p (\mathbb{R}^d) \) with \( \nabla u = 0 \) and \( \nabla v \in L^q (\mathbb{R}^d) \). Then

\[
u.\nabla v \in H^r (\mathbb{R}^d),
\]

and

\[
\| u.\nabla v \|_{H^r(\mathbb{R}^d)} \leq C \| u \|_{L^p} \| \nabla v \|_{L^q}.
\]

**Proof.** The result is due to [2]; but we give it here a detailed proof for the reader’s convenience. Observe that

\[
f = u.\nabla v = \nabla (u \otimes (v - c))
\]

for an arbitrary constant vector \( c \). So we get

\[
(\varphi_t \ast f)(x) = t^{-d-1} \int_{B_t(x)} (\nabla \varphi) (t^{-1}(x - y)) u(y) (v(y) - m_B(v)) dy,
\]

where

\[
m_B(v) = \frac{1}{|B_t(x)|} \int_{B_t(x)} v(y) dy.
\]

Taking

\[
1 < \gamma < \infty, \quad 1 < \beta < d, \quad \text{with} \quad \frac{1}{\gamma} + \frac{1}{\beta} = 1 + \frac{1}{d},
\]

and writing

\[
\frac{1}{\beta^*} = \frac{1}{\beta} - \frac{1}{d},
\]

we arrive at the following estimate.
we see by Poincaré-Sobolev inequality that

\[
\|(\varphi_t \ast f)(x)\| \leq \frac{C}{t^{d+1}} \left( \int_{B_t(x)} |u(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left( \int_{B_t(x)} |v(y) - m_B(v)|^{\beta^*} \, dy \right)^{\frac{1}{\beta}}
\]

\[
\leq \frac{C}{t^{d+1}} \left( \int_{B_t(x)} |u(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left( \int_{B_t(x)} |\nabla v(y)|^\beta \, dy \right)^{\frac{1}{\beta}}
\]

\[
= C \left( \frac{1}{|B_t(x)|} \int_{B_t(x)} |u(y)|^\gamma \, dy \right)^{\frac{1}{\gamma}} \left( \frac{1}{|B_t(x)|} \int_{B_t(x)} |\nabla v(y)|^\beta \, dy \right)^{\frac{1}{\beta}}
\]

\[
\leq C \left( M_\gamma u \right)(x) \cdot \left( M_\beta (\nabla v) \right)(x).
\]

We thus obtain

\[
\sup_{t>0} \{(\varphi_t \ast f)(x)\} \leq C \left( M_\gamma u \right)(x) \cdot \left( M_\beta (\nabla v) \right)(x).
\]

Since we can take \( \gamma \) and \( \beta \) so that

\[
1 < \gamma < p, \quad 1 < \beta < q < d,
\]

it follows from Lemma 1 that

\[
\|M_\gamma u\|_{L^p} \leq C \|u\|_{L^p}, \quad \|M_\beta (\nabla v)\|_{L^q} \leq C \|\nabla v\|_{L^q}.
\]

Lemma 2 now follows from Hölder’s inequality:

\[
\|f \cdot g\|_{L^r} \leq \|f\|_{L^p} \cdot \|g\|_{L^q} \quad \left( 0 < p < \infty, 0 < q < \infty, \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \right).
\]

This finishes the proof of the lemma. \( \square \)

We are now in a position to proof Theorem 2.

**Proof.** To prove this, we distinguish three cases.

**Case A.** Let us assume first that

\[
\nabla . u = 0.
\]

In this case we get

\[
div (vv) = (\nabla v) . u + v \, div \, u = u . \nabla v.
\]

Then we have \( u \in L^p(\mathbb{R}^d)^d, \ \nabla v \in L^q(\mathbb{R}^d) \) with \( \text{div} \, u = 0, \ \text{curl} \ (\nabla v) = 0 \) in the sense of distributions. It follows from Lemma 2 that

\[
u . \nabla v \in H^1(\mathbb{R}^d)
\]

and there exists an absolute constant \( C \) such that

\[
\|\text{div} (vu)\|_{H^1(\mathbb{R}^d)} \leq C \|u\|_{L^p} \|\nabla v\|_{L^q}.
\]
Case B. We may of course suppose under additional assumptions that $u$ and $v$ are supported on the ball $|x| \leq R_0$. In order to simplify the presentation, we take $p = q = 2$. We shall write $\Omega$ for the ball in $\mathbb{R}^d$ of radius $R_0$ centered at the origin. By $H_0^1(\Omega)$ we denote the closed subspace of $H^1(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in the $H^1$ norm. Let

$$g = \text{div} \ u \in L^2(\mathbb{R}^d).$$

By the classical result (see e.g. [20]) we know that

$$g = \partial_1 g_1 + \cdots + \partial_d g_d,$$

where $g_1, \ldots, g_d$ belong to $H_0^1(\Omega)$. Setting

$$G = (g_1, \ldots, g_d) \quad \text{and} \quad r = u - G,$$

Then it follows that

$$\text{div} \ r = 0 \quad \text{and} \quad r \in L^2(\Omega).$$

Using Lemma 2 we infer

$$\text{div} \ (ru) \in H^1(\mathbb{R}^d).$$

Further we set

$$f = \text{div} \ (Gv).$$

For this purpose we use Lemma 3 below, it follows that $f \in H^1(\mathbb{R}^d)$.

Case C. The general case. We call $\varphi$ a smooth bump function with compact support such that

$$1 = \sum_{k \in \mathbb{Z}^d} \varphi^2(x - k).$$

We have thus, if $f$ and $g$ are two functions,

$$f(x)g(x) = \sum_{k \in \mathbb{Z}^d} f(x)\varphi^2(x - k)g(x)$$

$$= \sum_{k \in \mathbb{Z}^d} f_k(x)g_k(x),$$

where

$$f_k(x) = \varphi(x - k)f(x) \quad \text{and} \quad g_k(x) = \varphi(x - k)g(x).$$

Now set

$$u_k(x) = \varphi(x - k)u(x) \quad \text{and} \quad v_k(x) = \varphi(x - k)v(x)$$

for $k \in \mathbb{Z}^d$. We then have

$$\text{div} \ (uv) = \sum_{k \in \mathbb{Z}^d} (u_kv_k) = \sum_{k \in \mathbb{Z}^d} w_k, \quad w_k = \text{div} \ (u_kv_k).$$

We are going to check that

$$\sum_{k \in \mathbb{Z}^d} \|w_k\|_{H^1(\mathbb{R}^d)} < \infty.$$
To do this, we apply the local version (Case A) and it follows
\[
\|w_k\|_{H^1(\mathbb{R}^d)} \leq C \left( \|u_k\|_{L^2} + \|\text{div } u_k\|_{L^2} \right) \left( \|v_k\|_{L^2} + \|\text{div } v_k\|_{L^2} \right)
\]
\[
= c_k \in l^1 \left( \mathbb{Z}^d \right).
\]

Up to now we have proved
\begin{equation}
(1.6) \quad \|\text{div } (uv)\|_{H^1(\mathbb{R}^d)} \leq C \left( \|u\|_{L^2} + \|\text{div } u\|_{L^2} \right) \left( \|v\|_{L^2} + \|\text{div } v\|_{L^2} \right).
\end{equation}
This automatically yields the estimate
\begin{equation}
(1.7) \quad \|\text{div } (uv)\|_{H^1(\mathbb{R}^d)} \leq C \left( \|u\|_{L^2} \|\nabla v\|_{L^2} + \|v\|_{L^2} \|\text{div } u\|_{L^2} \right).
\end{equation}
To see this, we may replace \( u \) in the inequality above by
\[
u = \delta^{\left(\frac{1}{2} - \frac{q}{d}\right)} u \left( \frac{x}{\delta} \right), \quad \text{whenever } 0 < \delta < \infty.
\]
and similarly \( v \) by
\[
v_\delta = \delta^{\left(\frac{1}{2} - \frac{q}{d}\right)} v \left( \frac{x}{\delta} \right), \quad \text{whenever } 0 < \delta < \infty.
\]
Thus the left-hand side of (1.6) fortunately does not change, while at right-hand we get rid of the undesirable terms by letting \( \delta \) either to 0, or to \( +\infty \). This completes the proof. \( \Box \)

Now we turn to the proof of Lemma 3. One can show that every function \( f \in L^p(\mathbb{R}^d), \ p \in (1, +\infty], \) with compact support and \( \int f \, dx = 0 \) belongs to \( \mathcal{H}^1(\mathbb{R}^d) \). In particular,

**Lemma 3.** If \( d^* = \frac{d}{d-1}, \ f \in L^{d^*}, \ \text{supp } f \subset \overline{\Omega} \) and

\[
\int f \, dx = 0,
\]
then \( f \in \mathcal{H}^1(\mathbb{R}^d) \).

**Proof.** We have
\[
f = \text{div } (G) v + G.\nabla v
\]
and we have to prove that the two terms belong to \( L^{d^*} \). We consider the first term on the right. Since \( \nabla v \in L^2 \), we have
\[
\text{div } (G) \in L^2 \quad \text{and} \quad v \in L^q, \quad \text{where} \quad \frac{1}{2} - \frac{1}{q} = \frac{1}{d^*}.
\]
Thus,
\[
v \text{div } (G) \in L^{d^*}.
\]
A similar argument works in the second term and this completes the proof of the lemma. \( \Box \)
1.2. Multipliers and Morrey-Campanato spaces

In this section, we give a description of the multiplier space \( \hat{X}_r \) introduced recently by P. G. Lemarié-Rieusset in his work [10] (see also [11]). The space \( \hat{X}_r \) of pointwise multipliers which map \( L^2 \) into \( \dot{H}^{-r} \) is defined in the following way

**Definition 2.** For \( 0 \leq r < \frac{d}{2} \), we define the homogeneous space \( \hat{X}_r \) by

\[
\hat{X}_r = \left\{ f \in L^2_{\text{loc}} : \forall g \in H^{-r} \ f g \in L^2 \right\},
\]

where we denote by \( H^{-r}(\mathbb{R}^d) \) the completion of the space \( \mathcal{D}(\mathbb{R}^d) \) with respect to the norm \( \|u\|_{H^{-r}} = \|(-\Delta)^{\frac{r}{2}} u\|_{L^2} \).

The norm of \( \hat{X}_r \) is given by the operator norm of pointwise multiplication

\[
\|f\|_{\hat{X}_r} = \sup_{\|g\|_{H^{-r}} \leq 1} \|fg\|_{L^2}.
\]

Similarly, we define the nonhomogeneous space \( X_r \) for \( 0 \leq r < \frac{d}{2} \) equipped with the norm

\[
\|f\|_{X_r} = \sup_{\|g\|_{H^{-r}} \leq 1} \|fg\|_{L^2}.
\]

We have the homogeneity properties: \( \forall x_0 \in \mathbb{R}^d \)

\[
\|f(x + x_0)\|_{X_r} = \|f\|_{X_r},
\]

\[
\|f(x + x_0)\|_{\hat{X}_r} = \|f\|_{\hat{X}_r},
\]

\[
\|f(\lambda x)\|_{X_r} \leq \frac{1}{\lambda^r} \|f\|_{X_r}, \quad 0 < \lambda \leq 1
\]

\[
\|f(\lambda x)\|_{\hat{X}_r} \leq \frac{1}{\lambda^r} \|f\|_{\hat{X}_r}, \quad \lambda > 0.
\]

The following imbedding

\[
L^\frac{d}{r} \subset X_r, \quad 0 \leq r < \frac{d}{2}, \quad 0 \leq r \leq r.
\]

\[
L^\frac{d}{r} \subset \hat{X}_r, \quad 0 \leq r < \frac{d}{2}
\]

holds.

**Example 1.** If \( u(x) \in \mathcal{D}(\mathbb{R}^d) \), \( \varphi(x) = \left( \sum_{k=1}^{d} |x_k|^{\gamma_k} \right)^{-1} \), \( \gamma_k > 0 \), \( d > 2 \), and

\[
\sum_{k=1}^{d} \gamma_k^{-1} = \frac{d}{2},
\]

then

\[
\int_{\mathbb{R}^d} \varphi(x) |u(x)|^2 \, dx \leq C \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx
\]
and \( \varphi \in X_1(\mathbb{R}^d) \).

Indeed, the inequality
\[
\int_{\lambda < |x| < 2\lambda} \varphi(x) |u(x)|^2 \, dx \\
\leq \left[ \int_{\lambda < |x| < 2\lambda} |u(x)|^{\frac{2d}{d-2}} \, dx \right]^{\frac{d-2}{2}} \cdot \left[ \int_{\lambda < |x| < 2\lambda} \varphi(x)^{\frac{d}{d-2}} \, dx \right]^{\frac{2}{d}}
\]
and the Sobolev theorem imply that for \( \lambda > 0 \)
\[
\int_{\lambda < |x| < 2\lambda} \varphi(x) |u(x)|^2 \, dx \\
\leq C \left[ \int_{\lambda < |x| < 2\lambda} |\nabla u(x)|^2 \, dx + \int_{\lambda < |x| < 2\lambda} \frac{|u(x)|^2}{|x|^2} \, dx \right] \cdot \left[ \int_{\lambda < |x| < 2\lambda} \varphi(x)^{\frac{d}{d-2}} \, dx \right]^{\frac{2}{d}},
\]
where \( C \) does not depend on \( \lambda \). Let us estimate the integral
\[
S(\lambda) = \int_{\lambda < |x| < 2\lambda} \varphi(x)^{\frac{d}{2}} \, dx.
\]
The domain \( \lambda < |x| < 2\lambda \) can be represented as a finite sum of domain \( \Omega_{j\lambda} \) such that \( |x_j| > \frac{\lambda}{2} \) if \( x \in \Omega_{j\lambda} \) for \( j = 1, \ldots, d \). Let for instance \( |x_1| > \frac{\lambda}{2} \). Then
\[
\int_{\Omega_{j\lambda}} \varphi(x)^{\frac{d}{2}} \, dx \leq \frac{3\lambda}{2} \int_{\lambda < |x| < 2\lambda} \frac{dx_1 \cdots dx_d}{\left( \left( \frac{\lambda}{2} \right)^{\gamma_1} + |x_2|^{\gamma_2} + \cdots + |x_d|^{\gamma_d} \right)^{\frac{d}{2}}}.
\]
The substitution \( x_j = t_j \left( \frac{\lambda}{2} \right)^{\frac{\gamma_j}{\gamma_1}} \) gives the relations
\[
S(\lambda) \leq C \int_{\mathbb{R}^{d-1}} \frac{dt_1 \cdots dt_d}{(1 + |t_2|^{\gamma_2} + \cdots + |t_d|^{\gamma_d})^{\frac{d}{2}}}
\leq C,
\]
since the integral is converging. To see this, set \( t_s = \frac{\lambda}{\gamma_s} \). Then
\[
\int_{\mathbb{R}^{d-1}} \frac{dt_1 \cdots dt_d}{(1 + |t_2|^{\gamma_2} + \cdots + |t_d|^{\gamma_d})^{\frac{d}{2}}}
\]
\[
\leq C \int_{\mathbb{R}^{d-1}} \frac{|\tau|^{\frac{1}{2} + \cdots + \frac{1}{d} - (d-1)} d\tau_1 \cdots d\tau_d}{(1 + |\tau|)^{\frac{d}{2}}} \\
\leq C \int_0^\infty \frac{d|\tau|}{(1 + |\tau|)^{\frac{d}{2} + 1}} < \infty.
\]

Therefore,
\[
\int_{\lambda < |x| < 2\lambda} \varphi(x) |u(x)|^2 dx \leq C_0 \left[ \int_{\lambda < |x| < 2\lambda} |\nabla u(x)|^2 dx + \int_{\lambda < |x| < 2\lambda} \frac{|u(x)|^2}{|x|^2} dx \right].
\]

Setting \( \lambda = 2^m, m \in \mathbb{Z} \) and assuming these inequalities over all \( m \), we obtain that
\[
\int_{\mathbb{R}^d} \varphi(x) |u(x)|^2 dx \leq C \left( \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx \right).
\]

By Hardy’s inequality in \( \mathbb{R}^d, d \geq 3 \)
\[
\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx, \quad u(x) \in \mathcal{D}(\mathbb{R}^d),
\]
and hence
\[
\int_{\mathbb{R}^d} \varphi(x) |u(x)|^2 dx \leq C \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx.
\]

Now we recall the definition of Morrey-Campanato spaces ([7], [19]):

**Definition 3.** For \( 1 < p \leq q \leq +\infty \), the Morrey-Campanato space \( \mathcal{M}_{p,q} \) is defined by:
\[
(1.8) \quad \mathcal{M}_{p,q} = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^d) : \|f\|_{\mathcal{M}_{p,q}} = \sup_{x \in \mathbb{R}^d} \sup_{0 < R \leq 1} R^{d/q - d/p} \|f(y)1_{B(x,R)}(y)\|_{L^p(dy)} < \infty \right\}.
\]

Let us define the homogeneous Morrey-Campanato spaces \( \tilde{\mathcal{M}}_{p,q} \) for \( 1 < p \leq q \leq +\infty \) by
\[
(1.9) \quad \|f\|_{\tilde{\mathcal{M}}_{p,q}} = \sup_{x \in \mathbb{R}^d} \sup_{R > 0} R^{d/q - d/p} \left( \int_{B(x,R)} |f(y)|^p dy \right)^{1/p}.
\]

It is easy to check the following properties:
\[
\|f(\lambda x)\|_{\tilde{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^d} \|f\|_{\tilde{\mathcal{M}}_{p,q}}, \quad 0 < \lambda \leq 1,
\]
\[
\|f(\lambda x)\|_{\tilde{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^d} \|f\|_{\tilde{\mathcal{M}}_{p,q}}, \quad \lambda > 0.
\]
We shall assume the following classical results [7].

a) For $1 \leq p \leq p', \ p \leq q \leq +\infty$ and for all function $f$ so that $f \in \mathcal{M}_{p,q} \cap L^\infty$:

$$
\|f\|_{\mathcal{M}_{p',q',p'}} \leq \|f\|_{L^p}^{1 - \frac{p}{p'}} \|f\|_{L^\infty}^{\frac{p}{p'}}.
$$

b) For $p, q, p', q'$ so that $\frac{1}{p} + \frac{1}{p'} \leq 1, \ \frac{1}{q} + \frac{1}{q'} \leq 1$, $f \in \mathcal{M}_{p,q}, \ g \in \mathcal{M}_{p',q'}$. Then

$$
f \ast g \in \mathcal{M}_{p'' ,q''} \text{ with } \frac{1}{p''} + \frac{1}{q''} = \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = \frac{1}{q''}.
$$

c) For $1 \leq p \leq d$, we have

$$
\forall \lambda > 0, \|\lambda f(\lambda x)\|_{\mathcal{M}_{p,d}} = \|f\|_{\mathcal{M}_{p,d}}.
$$

d) If $p' < p$,

$$
\mathcal{M}_{p,q} \subset \mathcal{M}_{p',q'},
$$

$$
\dot{\mathcal{M}}_{p,q} \subset \dot{\mathcal{M}}_{p',q'}.
$$

e) If $q_2 < q_1$, we have

$$
\mathcal{M}_{p,q_1} \subset \mathcal{M}_{p,q_2},
$$

$$
L^q = \dot{\mathcal{M}}_{q,q} \subset \dot{\mathcal{M}}_{p,q}, \ p \leq q.
$$

We have the following comparison between multipliers and Morrey-Campanato spaces:

**Proposition 1.** For $0 \leq r < \frac{d}{2}$, we have

$$
X_r \subseteq \mathcal{M}_{2, \frac{d}{r}},
$$

$$
\dot{X}_r \subseteq \dot{\mathcal{M}}_{2, \frac{d}{r}}.
$$

**Proof.** Let $f \in X_r$, $0 < R \leq 1$, $x_0 \in \mathbb{R}^d$ and $\phi \in \mathcal{D}$, $\phi \equiv 1$ on $B(\frac{x_0}{R}, 1)$. We have

$$
R^{r - \frac{d}{2}} \left( \int_{|x-x_0| \leq R} |f(x)|^2 \, dx \right)^{1/2} \leq R^r \left( \int_{|y-x_0| \leq 1} |f(Ry)|^2 \, dy \right)^{1/2}
$$

$$
\leq R^r \left( \int_{y \in \mathbb{R}^d} |f(Ry)\phi(y)|^2 \, dy \right)^{1/2}
$$

$$
\leq R^r \|f(Ry)\|_{X_r} \|\phi\|_{H^r}
$$

$$
\leq \|f(y)\|_{X_r} \|\phi\|_{H^r}
$$

$$
\leq C \|f(y)\|_{X_r}.
$$

We observe that the same proof is also valid for homogeneous spaces. □
Additionally, for $2 < p \leq \frac{d}{r}$ and $0 \leq r < \frac{d}{2}$, we have the following inclusion relations:

$$L^\frac{d}{r} (\mathbb{R}^d) \subset L^{\frac{d}{r}, \infty} (\mathbb{R}^d) \subset \mathcal{M}_{p, \frac{d}{r}} (\mathbb{R}^d) \subset \mathcal{M}_{2, \frac{d}{r}} (\mathbb{R}^d),$$

where $L^{p, \infty}$ denotes the usual Lorentz (weak $L^p$) space. For the definition and basic properties of Lorentz spaces $L^{p,q}$ we refer to [18].

2. Regularity theorem

In this section we give the regularity criterion by velocity to the Leray type weak solution of the Navier-Stokes equation (1.1). Before turning our attention to regularity issues, we start with some prerequisites for our main result. We use the notations

$$D_j = \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, d$$

means the $j^{th}$ partial derivative and

$$\nabla = (D_1, \ldots, D_d)$$

the gradient.

$$\nabla^2 = (D_j D_k)_{j,k=1}^d$$

means the matrix of the second order derivatives. Let

$$u : \mathbb{R}^d \to \mathbb{R}^d$$

$$x \mapsto u(x) = (u_1(x), \ldots, u_d(x))$$

be a vector field. Then we set

$$\text{div } u = \nabla \cdot u = D_1 u_1 + \cdots + D_d u_d,$$

$$\Delta u = \text{div} \nabla u = (D_1^2 + \cdots + D_d^2) u,$$

$$\nabla u = (D_1, \ldots, D_d) u = (D_j u_k)_{j,k=1}^d,$$

$$\nabla^2 u = (D_j D_k)_{j,k=1}^d u = (D_j D_k u_l)_{j,k,l=1}^d,$$

and

$$u.\nabla u = (u \cdot \nabla) u = (u_1 D_1 + \cdots + u_d D_d) u$$

$$= (u_1 D_1 u_k + \cdots + u_d D_d u_k)_{k=1}^d$$

whenever this is meaningful. Further we set

$$\text{div} (u \ u) = D_1 (u_1 u) + \cdots + D_d (u_d u)$$

$$= (D_1 (u_1 u_k) + \cdots + D_d (u_d u_k))_{k=1}^d,$$

where the matrix $u \ u = u \otimes u = (u_j u_k)_{j,k=1}^d$ means the usual tensor product. We prefer the simple notation $u \ u$. 
If $\text{div } u = 0$, we call $u$ is divergence free or solenoidal. In this case we get

$$
\begin{align*}
u. \nabla u &= D_1(u_1u) + \cdots + D_d(u_du) - (u_1D_1 + \cdots + u_dD_d)u \\
&= D_1(u_1u) + \cdots + D_d(u_du) \\
&= \text{div}(u\ u).
\end{align*}
$$

Let

$$
C^\infty_{0,\sigma}(\mathbb{R}^d) = \left\{ \varphi \in \left(C^\infty_0(\mathbb{R}^d)\right)^d : \text{div } \varphi = 0 \right\} \subseteq \left(C^\infty_0(\mathbb{R}^d)\right)^d.
$$

The subspace

$$
L^2(\mathbb{R}^d) = \overline{C^\infty_{0,\sigma}(\mathbb{R}^d)}_{||\cdot||_{L^2}} = \left\{ u \in L^2(\mathbb{R}^d)^d : \text{div } u = 0 \right\}
$$

obtained as the closure of $C^\infty_{0,\sigma}$ with respect to $L^2$-norm $||\cdot||_{L^2}$. $H^r_\sigma$ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$
||u||_{H^r} = ||u||_{L^2} + \left(1 - \Delta\right)^{\frac{r}{2}} ||u||_{L^2} \quad \text{for } r \geq 0.
$$

Our definition of Leray-Hopf weak solutions (see e.g., [9], [8]) now reads:

**Definition 4** (weak solutions). Let $a \in L^2_\sigma$ and $T > 0$. A measurable function $u$ is called a weak solution of (1.1) on $(0, T)$ if $u$ satisfies the following properties:

1. $u \in L^\infty((0, T); L^2_\sigma) \cap L^2((0, T); H^1_\sigma)$ for all $T > 0$;

2. $u(t)$ is continuous in time in the weak topology of $L^2_\sigma$ with

$$
\langle u(t), \phi \rangle \to \langle u, \phi \rangle \quad \text{as } t \to 0^+
$$

for all $\phi \in L^2_\sigma$;

3. for any $0 \leq s \leq t \leq T$, $u$ satisfies the identity

$$
(2.1) \int_s^t \left\{-\langle u, \partial_t \phi \rangle + \langle u. \nabla u, \phi \rangle + \langle \nabla u, \nabla \phi \rangle \right\} \, d\tau = -\langle u(t), \phi(t) \rangle + \langle u(s), \phi(s) \rangle
$$

for all $\phi \in H^1((s, t); H^1_\sigma)$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product and $||\cdot||_{L^2}$ denotes the norm in $L^2(\mathbb{R}^d)^d$.

**Remark 2.** For $u$ and $\phi$ as above, the integral

$$
\int_0^T \langle u. \nabla u, \phi \rangle \, d\tau
$$

is well defined since we have by the Sobolev inequality

$$
||u||_{L^2(\mathbb{R}^d)^d} \leq C ||\nabla u||_{L^2},
$$
that
\[
\left| \int_0^T \langle u, \nabla u, \phi \rangle \, dt \right| \leq \int_0^T \| u \|_{L^2}^2 \| \nabla u \|_{L^2} \| \phi \|_{L^2} \, dt \leq C \sup_{0 < t < T} \| \phi \|_{L^4} \int_0^T \| \nabla u \|_{L^2}^2 \, dt.
\]

Existence of weak solutions has been established by Leray in [12] for initial velocity in \( L^2_\sigma (\mathbb{R}^d) \). The result is the following

**Theorem 3** (Leray - Hopf). Let \( T > 0 \). Let \( a \in L^0_\sigma (\mathbb{R}^d) \) and
\[
u \in L^\infty \left((0, T) ; L^2_\sigma \right) \cap L^2 \left((0, T) ; H^1_\sigma \right)
\]
be a weak solution of the Navier-Stokes equation (1.1) satisfying the strong type energy inequality:
\[
\| u(t) \|_{L^2}^2 + 2 \int_0^t \| \nabla u(s) \|_{L^2}^2 \, ds \leq \| a \|_{L^2}^2 \quad \text{for a.a.} \quad 0 \leq t < T.
\]
We assume that the solution satisfies
\[
\| u(t) - a \|_{L^2} \to 0 \quad \text{as} \quad t \to +0.
\]

Let us introduced the class \( L^s \left((0, T) ; L^\gamma \right) \) with the norm \( \| \cdot \|_{L^s((0, T); L^\gamma)} \)
\[
\| u \|_{L^s((0, T); L^\gamma)} = \left( \int_0^T \| u(t) \|_{L^\gamma}^s \, dt \right)^{\frac{1}{s}}.
\]

The classical result on uniqueness and regularity of weak solutions in the class \( L^s \left((0, T) ; L^\gamma \right) \) was given by Foias, Serrin and Masuda [4], [16], [14].

**Theorem 4** (Foias-Serrin-Masuda). Let \( a \in L^2_\sigma (\mathbb{R}^d) \).
(i) Let \( u \) and \( v \) are two weak solutions of (1.1) on \( (0, T) \). Suppose that \( u \) satisfies
\[
\| u \|_{L^s((0, T); L^\gamma)} \quad \text{for} \quad \frac{2}{s} + \frac{d}{\gamma} = 1 \quad \text{with} \quad d < \gamma < \infty.
\]
Assume that \( v \) fulfills the energy inequality (2.2) for \( 0 \leq t < T \). Then we have \( u = v \) on \( [0, T) \).

(ii) Every weak solution \( u \) of (1.1) in the class (2.3) satisfies
\[
\frac{\partial u}{\partial t} + \alpha_1^{\alpha_1 + \cdots + \alpha_d} \in C \left((0, T) \times \mathbb{R}^d\right)
\]

for all multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_d) \) with \( |\alpha| = \alpha_1 + \cdots + \alpha_d \leq 2 \).

Kozono and Taniuchi [9] proved
Theorem 5 (Kozono-Taniuchi). Let \( a \in L^2_\sigma(\mathbb{R}^d) \).

(i) (uniqueness) Let \( u, v \) be two weak solutions of (1.1) on \((0, T)\). Suppose that

\[
    u \in L^2((0, T); \text{BMO})
\]

and that \( v \) satisfies the energy inequality (2.2). Then we have \( u = v \) on \([0, T]\).

(ii) (regularity) Suppose that \( u \) is a weak solution satisfying either of the following conditions

\[
    u \in L^2((0, T); \text{BMO}) \quad \text{or} \quad \text{rot } u \in L^1((0, T); \text{BMO}).
\]

Then \( u \) is a solution of (1.1) in the class

\[
    u \in C([\epsilon, T); H^s) \cap C^1([\epsilon, T); H^s) \cap C([\epsilon, T); H^{s+2}) \quad s > \frac{d}{2} - 1
\]

for all \( 0 < \epsilon < T \). Actually \( u \) is regular in \( \mathbb{R}^d \times (0, T) \).

Our aim result is to show a new regularity criterion for each of the problems to (1.1).

Theorem 6. Let \( u \) be a smooth solution to (1.1) in some interval \([0, T]\) with initial data \( a \in L^2_\sigma(\mathbb{R}^d)^d \). Suppose that the solution \( u \) satisfies

\[
    \int_0^T \| \nabla u(\tau) \|_{X_1(\mathbb{R}^d)}^2 \, d\tau < \infty.
\]

Then the solution

\[
    u \in C \left( (0, T) ; H^1_\sigma(\mathbb{R}^d)^d \right) \cap L^2 \left( (0, T) ; H^1_\sigma(\mathbb{R}^d)^d \cap H^2(\mathbb{R}^d)^d \right).
\]

Moreover,

\[
    \sup_{0 \leq \tau < T} \| \nabla u(t) \|_{L^2}^2 + \int_0^T \| \nabla^2 u(\tau) \|_{L^2}^2 \, d\tau
\]

\[
    \leq C \| \nabla u(0) \|_{L^2}^2 \left[ 1 + \exp \left( c \int_0^T \| \nabla u(\tau) \|_{X_1(\mathbb{R}^d)}^2 \, d\tau \right) \right].
\]

The same result holds when the assumption \( \nabla u \in L^2((0, T); X_1(\mathbb{R}^d)^d) \) is replaced by \( u \in L^2((0, T); \text{BMO}(\mathbb{R}^d)^d) \).

Remark 3. Theorem 6 covers the borderline case \( s = 2 \) and \( \gamma = d \). Our class \( L^2((0, T); X_1(\mathbb{R}^d)^d) \) is larger than \( L^2((0, T); L^d(\mathbb{R}^d)^d) \).

To clarify the main part of the result, we recall the known regularity criterion in the following.
Lemma 4 (Beirão da Veiga [1]). If we assume the following condition on the gradient of velocity for the Leray-Hopf weak solution \( u \):

\[
(2.6) \quad \int_0^T \| \nabla u(\tau) \|_{L^s}^s \, d\tau < \infty , \quad \frac{2}{s} + \frac{d}{\gamma} = 2 , \quad \frac{d}{2} < \gamma \leq \infty ,
\]

then the weak solution is smooth on \((0, T]\).

Corollary 1. If we assume the following condition on the gradient of velocity for the Leray-Hopf weak solution \( u \):

\[
(2.7) \quad \int_0^T \| \nabla u(\tau) \|_{X^1}^2 \, d\tau < \infty ,
\]

then the weak solution is smooth on \((0, T]\).

The marginale case \( q = \infty \) was considered by Kozono and Taniuchi in BMO frame work.

Lemma 5 (Kozono-Taniuchi [9]). Instead of the condition (2.6), if we assume the following condition on the vorticity of the weak solution \( u \):

\[
(2.8) \quad \int_0^T \| \mathbf{rot} u(\tau) \|_{BMO} \, d\tau < \infty ,
\]

then the weak solution is smooth on \((0, T]\).

The following lemmas play a fundamental role in estimating the nonlinear term.

Lemma 6. Let \( f \in H^1(\mathbb{R}^d) \), \( g(x) = (g_1(x))_{i=1}^d \) with \( \nabla \cdot g = 0 \) and \( g \in L^2(\mathbb{R}^d)^d \). Further we assume that \( \nabla h \in X^1(\mathbb{R}^d) \). Then there exists a constant \( C(d) > 0 \) independent of \( f, g \) and \( h \) such that

\[
(2.7) \quad \left| \int_{\mathbb{R}^d} f g \cdot \nabla h \, dx \right| \leq C \| \nabla f \|_{L^2(\mathbb{R}^d)} \| g \|_{L^2(\mathbb{R}^d)^d} \| \nabla h \|_{X^1(\mathbb{R}^d)}
\]

and

\[
(2.8) \quad \left| \int_{\mathbb{R}^d} \nabla f \cdot g h \, dx \right| \leq C \| \nabla f \|_{L^2(\mathbb{R}^d)} \| g \|_{L^2(\mathbb{R}^d)^d} \| \nabla h \|_{X^1(\mathbb{R}^d)}
\]
Proof. The proof is easy, due to definition of \( \dot{X}_1(\mathbb{R}^d) \). Suppose that \( \nabla h \in \dot{X}_1(\mathbb{R}^d) \) and using Cauchy-Schwarz inequality, we get

\[
\left| \int_{\mathbb{R}^d} f g \nabla h \, dx \right| \leq \left( \int_{\mathbb{R}^d} |f|^2 |\nabla h|^2 \, dx \right)^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^d)}^d \leq C \|\nabla h\|_{\dot{X}_1(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |\nabla f|^2 \, dx \right)^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^d)}^d,
\]

where the constant \( C \) is independent of \( f, g \) and \( h \). Thus the Lemma is proved in the case of (2.7). The proof is similar in the case of (2.8). \( \square \)

The same result holds when we replace the assumption \( \nabla h \in \dot{X}_1(\mathbb{R}^d) \) by the assumption \( h \in H^1(\mathbb{R}^d) \cap BMO(\mathbb{R}^d) \). Indeed, we known that

\[
h(x) = \log |x| \in BMO
\]

and

\[
|\nabla h|^2 \leq \frac{1}{|x|^2},
\]

then by Hardy’s inequality in \( \mathbb{R}^d \) \( (d \geq 3) \), we have

\[
\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx \leq C(d) \int_{\mathbb{R}^d} |\nabla f|^2 \, dx, \quad \forall f \in H^1(\mathbb{R}^d).
\]

This remark suggest that the lemma will also be holds when we replace the \( \dot{X}_1(\mathbb{R}^d) \)--norm of \( \nabla h \) by \( BMO \)--norm of \( h \). In fact, the following is a combination of the compensated compactness results of Coifman, Lions, Meyer and Semmes [2] and the duality of the space \( BMO \), we have:

Lemma 7. Let \( f \in H^1(\mathbb{R}^d) \), \( g = (g_i(x))_{i=1}^d \) with \( \nabla \cdot g = 0 \) and \( g \in L^2(\mathbb{R}^d)^d \) and a function \( h \in H^1(\mathbb{R}^d) \cap BMO(\mathbb{R}^d) \). Then there exists a constant \( C(d) > 0 \) independent of \( f, g \) and \( h \) such that

\[
\|g \nabla f, h\| \leq C \|\nabla f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)^d} \|h\|_{BMO(\mathbb{R}^d)}. \tag{2.9}
\]

Proof. It is an immediate consequence of Lemma 2 and the duality inequality (1.4)

\[
\|g \nabla f, h\| \leq C \|g \nabla f\|_{H^1(\mathbb{R}^d)} \|h\|_{BMO(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)^d} \|h\|_{BMO(\mathbb{R}^d)}. \tag{1.4}
\]

Next we recall the following well-known result:
Lemma 8 (Poincaré inequality). Suppose $Q$ is a unit cube in $\mathbb{R}^d$ of side length $\rho$ and $f$ is $C^2$ on $Q$ with $\nabla f \in L^2(Q)$. There exists $c$ not depending on $f$ such that

$$
\int_Q |f - m_Q f|^2 \, dy \leq c \rho^2 \int_Q |\nabla f(y)|^2 \, dy,
$$

where $m_Q f = \frac{1}{|Q|} \int_Q f(y) \, dy$ is the integral mean of $f$ on $Q$.

Combining this result with Proposition 1 gives:

Proposition 2. If $f \in H^1(\mathbb{R}^d)$ and $\nabla f \in X_1(\mathbb{R}^d)$, then $f \in BMO(\mathbb{R}^d)$.

Proof. Since $X_1(\mathbb{R}^d) \subset M_{2,d}(\mathbb{R}^d)$, it follows that $\nabla f \in M_{2,d}(\mathbb{R}^d)$.

By the classical Poincaré inequality (2.10), we have

$$
\int_{B(x,R)} |f(y) - m_{B(x,R)} f(y)|^2 \, dy \leq C R^2 \int_{B(x,R)} |\nabla f(y)|^2 \, dy
$$

$$
\leq C R^d \|\nabla f\|_{M_{2,d}}^2
$$

for every ball $B(x,R)$ of any radius $R$ and there holds

$$
\|f\|_{BMO} = \sup_{x \in \mathbb{R}^d} \sup_{R > 0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y) - m_{B(x,R)} f(y)|^2 \, dy
$$

$$
\leq C \|\nabla f\|_{M_{2,d}}^2
$$

$$
\leq C \|\nabla f\|_{X_1(\mathbb{R}^d)}^2.
$$

Now we turn into the proof of our Theorem 2.

Proof. Let $u$ be a smooth solution to (1.1) on $[0,T)$. By operating the Laplacian to the equation and then taking a $L^2$ inner product of the equation with $(-\Delta u)$, we have

$$
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \, dT = \langle u, \nabla u, \Delta u \rangle - \langle p, \Delta u \rangle
$$

$$
= \sum_{j,d=1}^d \int_{\mathbb{R}^d} u_j D_j u_t \Delta u_t \, dx + \langle p, \text{div} \Delta u \rangle
$$

$$
= \sum_{j,d=1}^d \int_{\mathbb{R}^d} u_j D_j u_t \Delta u_t \, dx,
$$

where we have used $\text{div} u = 0 = \text{div} \Delta u$. 

\]
Now, we use integration by parts to have
\[
\sum_{j,l=1}^{d} \int_{\mathbb{R}^d} u_j D_j u_l \Delta u_l \, dx
\]
\[
= - \sum_{j,k,l=1}^{d} \int_{\mathbb{R}^d} D_k u_j D_j u_k u_l \, dx - \sum_{j,k,l=1}^{d} \int_{\mathbb{R}^d} u_j D_j D_k u_l D_k u_l \, dx,
\]
or
\[
\sum_{j,k,l=1}^{d} \int_{\mathbb{R}^d} u_j D_j (D_k u_l D_k u_l) \, dx = \frac{1}{2} \sum_{j=1}^{d} \int_{\mathbb{R}^d} u_j D_j |\nabla u|^2 \, dx
\]
\[
= -\frac{1}{2} \int_{\mathbb{R}^d} \text{div} u |\nabla u|^2 \, dx = 0.
\]
Then
\[
\sum_{j,l=1}^{d} \int_{\mathbb{R}^d} u_j D_j u_l \Delta u_l \, dx = \sum_{j,k,l=1}^{d} \int_{\mathbb{R}^d} (D_k u_j) (D_j D_k u_l) u_l \, dx
\]
\[
= \langle u, \nabla u, \nabla^2 u \rangle.
\]
From Lemma 6 with
\[
g = \nabla u, \quad \nabla f = \nabla^2 u \quad \text{and} \quad h = u
\]
yields directly
\[
|\langle u, \nabla u, \nabla^2 u, \rangle| \leq C \|\nabla u\|_{L^2(\mathbb{R}^d)} \|\nabla^2 u\|_{L^2(\mathbb{R}^d)^d} \|\nabla u\|_{X_1(\mathbb{R}^d)}.
\]
By the Young inequality, we have
\[
\left| \int_{0}^{t} \langle \nabla u, \Delta u, u \rangle \, d\tau \right|
\]
(2.11) \[
\leq \frac{1}{2} \int_{0}^{t} \|\nabla^2 u\|_{L^2(\mathbb{R}^d)}^2 \, d\tau + \frac{C}{2} \int_{0}^{t} \|\nabla u\|_{L^2(\mathbb{R}^d)^d}^2 \|\nabla u\|_{X_1(\mathbb{R}^d)}^2 \, d\tau.
\]
Hence
\[
(2.12) \quad \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \, d\tau \leq C \int_{0}^{t} \|\nabla u\|_{L^2(\mathbb{R}^d)^d}^2 \|\nabla u\|_{X_1(\mathbb{R}^d)}^2 \, d\tau
for all $t > 0$. Since $\nabla u \in L^2 \left( (0, T) ; \dot{X}_1(\mathbb{R}^d)^d \right)$, it follows from the Gronwall inequality that
\[
\sup_{0 \leq t < T} \|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 \left( 1 + \exp \left\{ C \int_0^t \|\nabla u\|_{\dot{X}_1(\mathbb{R}^d)}^2 \, d\tau \right\} \right)
\]
from which we get the desired result. $\square$

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