HOMOLOGY OF THE GAUGE GROUP OF EXCEPTIONAL LIE GROUP $G_2$

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Abstract. We study homology of the gauge group associated with the principal $G_2$ bundle over the four-sphere using the Eilenberg–Moore spectral sequence and the Serre spectral sequence with the aid of homology and cohomology operations.

1. Introduction

Let $G$ be a compact, connected simple Lie group. The fact that $\pi_3(G) = \pi_4(BG) = Z$ leads to the classification of principal $G$ bundles $P_k$ over $S^4$ by the integer $k$ in $Z$. The gauge group $G_k(G)$ acts freely on the space $Map(P_k, EG)$ of all $G$ equivariant maps from $P_k$ to $EG$ and its orbit space is given by the $k$–component of the space $Map_k(S^4, BG)$ of maps from $S^4$ to $BG$. Since $Map(P_k, EG)$ is contractible, the classifying space of $G_k(G)$ is homotopy equivalent to $Map_k(S^4, BG)$. Then the number of homotopy types of $G_k(G)$ is finite [7]. Similarly, if $G_k^p(G)$ is the based gauge group which consists of base point preserving automorphisms on $P_k$, $BG_k^p(G)$ is homotopy equivalent to $\Omega_k^p G$ [1].

In this paper we study the mod $p$ homology of the gauge group associated with principal bundle of the exceptional Lie group $G_2$ by computing the Serre spectral sequence for the following fibration:

$$G_k^p(G_2) \rightarrow G_k(G_2) \rightarrow G_2.$$ 

The main result is that the Serre spectral sequence converging to

$$H_n(G_k(G_2); \mathbb{F}_p)$$

collapses at the $E_2$-term except for $p = 3, 7$.

2. Preliminaries

Let $E(x)$ be the exterior algebra on $x$ and $\Gamma(x)$ be the divided power Hopf algebra on $x$ which is free over $\gamma_i(x)$ with product $\gamma_i(x)\gamma_j(x) = (i+j)^{i+j}(x)$
and with coproduct $\Delta(\gamma_n(x)) = \sum_{i=0}^{n} \gamma_{n-i}(x) \otimes \gamma_i(x)$. Throughout this paper, the subscript of an element always means the degree of an element; for example the degree of $a_i$ is $i$.

There are only four division algebras over $R$, that is, the real numbers $R$, complex numbers $C$, the quaternions $H$ and the Cayley numbers $K$. $K$ is $R^8$ as a vector space and it is the non-associative algebra. The exceptional Lie group $G_2$ is the group of automorphisms of $K$. By the Cartan-Killing classification we call $G_2$ the exceptional Lie group of type $(3,11)$. The following theorem is well-known [14].

**Theorem 2.1.** The cohomology of $G_2$ are given by

$$H^*(G_2; \mathbb{F}_2) = \mathbb{F}_2[x_3]/(x_3^2) \otimes E(Sq^2 x_3),$$

$$H^*(G_2; \mathbb{F}_3) = E(x_3, x_{11}) \text{ for } p > 2$$

with all $x_i$ primitive, where $P^1 x_3 = x_{11}$ for $p = 5$.

We have homology operations $Q_i(p-1)$ on the $n$-fold loop space $\Omega^n X$

$$Q_i(p-1) : H_q(\Omega^n X; \mathbb{F}_p) \to H_{pq+i(p-1)}(\Omega^n X; \mathbb{F}_p)$$

for $0 \leq i \leq n-1$ when $p = 2$, and for $0 \leq i \leq n-1$ and $i + q$ even when $p > 2$. They are natural with respect to $n$-fold loop maps. In particular, we have $Q_0x = x^p$. The iterated power $Q_i^a$ denotes the composition of $Q_i$'s $a$ times. If $G$ is a Lie group, $G$ is homotopy equivalent to $\Omega BG$. Hence $Q_3(p-1)$ is defined in $H_*(\Omega^3 G; \mathbb{F}_p)$ and $Q_4(p-1)$ is defined in $H_*(\Omega^4 G; \mathbb{F}_p)$. These operations satisfy the following properties [6].

**Theorem 2.2.** In the path-loop fibration $\Omega^{n+1} X \to P \Omega^n X \to \Omega^n X$, we have the following.

(a) If $x \in H_*(\Omega^n X; \mathbb{F}_p)$ is transgressive in the Serre spectral sequence, so is $Q_i x$ and $\tau \circ Q_i(p-1)x = Q(Q_i+1)(p-1) \circ \tau x$ for each $i$, $0 \leq i \leq n-1$, where $\tau$ is the transgression.

(b) For $p > 2$ and $n > 1$, $d^2q(p-1)(x^{p-1} \otimes \tau(x)) = -\beta Q_i(p-1)\tau(x)$ if $x \in H_2q(\Omega^n X; \mathbb{F}_p)$.

(c) For $p = 2$, $Sq^1 Q_i x = Q_i-1x$ if $x \in H_q(\Omega^n X; \mathbb{F}_2)$ and $q + i$ is even.

From now on, we will simplify the notation as follows: We denote $\mathbb{F}_p[Q_i^a u]$ by $\mathbb{F}_p[Q_i^a u]$ and $\mathbb{F}_p[Q_i^{a+1} u] : a \geq 0$ by $\mathbb{F}_p[Q_i^{a+1} u]$ and so on. Now we recall the mod $p$ homology of the four fold loop space of a sphere.

$$H_*(\Omega^4 S^{n+4}; \mathbb{F}_2) = \mathbb{F}_2[Q_i^a Q_j^b Q_k^c Q_l^d], n \geq 1,$$

$$H_*(\Omega^4 S^{n+4}; \mathbb{F}_p) = E(Q_i^a Q_j^{b(p-1)} l_n) \otimes \mathbb{F}_p[\beta Q_i^{a+1} Q_j^{b(p-1)} l_n]$$

$$\otimes \mathbb{F}_p[Q_2(p-1) \beta Q_5^{b+1} l_n] \otimes E(Q_i^{(p-1)} \beta Q_2^{b+1} l_n) \otimes \mathbb{F}_p[\beta Q_4^{a+1} \beta Q_2^{b+1} Q_5^{c+1} l_n], n \geq 1.$$
Theorem 2.3 ([3, Theorem 5.14]). Let $X$ be a path connected $H$-space. Then the following are true.

(a) The Eilenberg Moore spectral sequence collapses at $E_2$ if and only if $\ker \sigma = 0$.

(b) The suspension $\sigma : QH^k(X; \mathbb{F}_p) \to PH^{k-1}(\Omega X; \mathbb{F}_p)$ is injective if $k \not\equiv 2 \mod 2p$.

(c) The suspension $\sigma : QH^k(X; \mathbb{F}_p) \to PH^{k-1}(\Omega X; \mathbb{F}_p)$ is surjective if $k - 1 \not\equiv -2 \mod 2p$.

From now on we denote $H_*(\Omega^i S^n; \mathbb{F}_p)$ by $\Omega_i(n)$, $\otimes_{k=1}^r H_*(\Omega^i S^{n_k}; \mathbb{F}_p)$ by $\Omega_i(n_1, \ldots, n_r)$, and $\otimes_{k=1}^r H^*(\Omega^i S^{n_k}; \mathbb{F}_p)$ by $\Omega^i(n_1, \ldots, n_r)$ for each $i, n \geq 1$.

Theorem 2.4. The cohomology of the loop space of $G_2$ are

$$H^*(\Omega G_2; \mathbb{F}_2) = \mathbb{F}_2[y_2]/(y_2^4) \otimes \Gamma(y_8, y_{10}),$$

$$H^*(\Omega G_2; \mathbb{F}_5) = \bigotimes_{i \geq 0} \mathbb{F}_5[\gamma_i(y_2)]/(\gamma_5(y_2)^{25}),$$

$$H^*(\Omega G_2; \mathbb{F}_p) = \Omega^1(3, 11) \text{ for odd primes } p \neq 5.$$

Proof. By the Eilenberg Moore spectral sequence for the path loop fibration converging to $H^*(\Omega G_2; \mathbb{F}_2)$, as a Hopf algebra we have

$$E_2 = \text{Tor}_{H^*(G_2; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)$$

$$= \text{Tor}_{\mathbb{F}_2[x_3]/(x_3^3) \otimes E(x_5)}(\mathbb{F}_2, \mathbb{F}_2)$$

$$= \text{Tor}_{\mathbb{F}_2[x_3]/(x_3^3)}(\mathbb{F}_2, \mathbb{F}_2) \otimes \text{Tor}_{E(x_5)}(\mathbb{F}_2, \mathbb{F}_2)$$

$$= E(y_2) \otimes \Gamma(y_{10}) \otimes \Gamma(y_4).$$

Since the $E_2$-term concentrates on even dimensions, the spectral sequence collapses at the $E_2$-term. Hence $E_2 = E_\infty$. Since the Eilenberg-Moore spectral sequence preserves the Steenrod actions, from $Sq^2 x_3 = x_5$, we get $Sq^2 y_2 = y_4$, that is, $y_2^2 = y_4$. From this, we can solve the algebra extension problem and we have

$$H^*(\Omega G_2; \mathbb{F}_2) = \mathbb{F}_2[y_2]/(y_2^4) \otimes \Gamma(y_8) \otimes \Gamma(y_{10}).$$

Now we turn to the odd prime cases. Like the mod 2 case, we use the Eilenberg-Moore spectral sequence converging to $H^*(\Omega G_2; \mathbb{F}_p)$ with

$$E_2 = \text{Tor}_{H^*(G_2; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

$$= \text{Tor}_{E(x_3) \otimes E(x_{10})}(\mathbb{F}_p, \mathbb{F}_p)$$

$$= \Gamma(y_2) \otimes \Gamma(y_{10}).$$

Since all elements in $E_2$ are even dimensional, the spectral sequence collapses at the $E_2$-term, so $E_2 = E_\infty$. For $p \neq 5$, there is no extension problem and we get

$$H^*(\Omega G_2; \mathbb{F}_p) = \Gamma(y_2) \otimes \Gamma(y_{10}) = \Omega^1(3, 11).$$
as a Hopf algebra. Now we consider the case \( p = 5 \). In the bar construction, \( y_2 \) is represented by \([x_3] \). Since \( P^1 x_3 = x_{11} \) in \( H^*(G_2; \mathbb{F}_5) \), we have the following algebra extension:

\[ y_2^5 = P^1 y_2 = P^1 [x_3] = [P^1 x_3] = [x_{11}] = y_{10}. \]

The element \( \gamma_5(y_2) \) is represented by \([x_3] \cdots [x_3] \) (5 factors) in the bar construction. By the Cartan formula, we also have the following algebra extension:

\[
(\gamma_5, y_2)^5 = P^5 \gamma_5(y_2) = P^5 [x_3] \cdots [x_3]
= [P^1 x_3] \cdots [P^1 x_3] = [x_{11}] \cdots [x_{11}]
= \gamma_5(y_{10}).
\]

Since \( \Gamma(y_2) = \bigotimes_{i \geq 0} \gamma_5^i(y_2) \) and \( \Gamma(y_{10}) = \bigotimes_{i \geq 0} \gamma_5^i(y_{10}) \) as an algebra, it follows that \( \Gamma(y_2) \otimes \Gamma(y_{10}) \) in \( E_\infty \) produces \( \bigotimes_{i \geq 0} \mathbb{F}_5 [\gamma_5(y_2)]/(\gamma_5(y_2))^{25} \) in \( H^*(G_2; \mathbb{F}_5) \) and we get the conclusion. \( \square \)

To get \( H_*(\Omega_0^5 G; \mathbb{F}_p) \), where \( \Omega_0^5 G \) is the zero component of \( \Omega^5 G \), we need the following result [10].

**Theorem 2.5.** The Eilenberg–Moore spectral sequences for the path loop fibrations converging to the mod \( p \) (co)homology of the double and the triple loop spaces of any simply connected finite \( H \)-space collapse at the \( E_2 \)-term.

By Theorem 2.5 and the formal computations of Tor, Ext and Cotor, we get the following two theorems.

**Theorem 2.6.** The homology of the double loop space of \( G_2 \) are

\[
H_*(\Omega^2 G_2; \mathbb{F}_2) = E(z_1) \otimes \mathbb{F}_2 [\beta z_7] \otimes \mathbb{F}_2 [Q_1^4 z_7] \otimes \Omega_2(11),
H_*(\Omega^2 G_2; \mathbb{F}_5) = E(Q_4^2 z_1) \otimes \mathbb{F}_5 [Q_8^6 z_4],
H_*(\Omega^2 G_2; \mathbb{F}_p) = \Omega_2(3, 11) \text{ for odd primes } p \neq 5.
\]

**Theorem 2.7.** The homology of the triple loop space of \( G_2 \) are

\[
H_*(\Omega_0^5 G_2; \mathbb{F}_2) = \mathbb{F}_2 [Q_1^4 \beta w_6] \otimes \mathbb{F}_2 [Q_1^4 Q_5^8 w_6] \otimes \Omega_3(11),
H_*(\Omega_0^5 G_2; \mathbb{F}_5) = \mathbb{F}_5 [Q_8^6 (Q_8[1] * [-5])] \otimes E(Q_4^2 Q_1^{12} w_{47}) \otimes \mathbb{F}_5 [\beta Q_8^6 Q_1^{12} w_{47}],
H_*(\Omega^3 G_2; \mathbb{F}_p) = \Omega_3(3, 11) \text{ for odd primes } p \neq 5.
\]

### 3. Based gauge group

For the exceptional Lie group \( G_2 \) we have

\[
\pi_3(G_2) = \mathbb{Z},
\pi_4(G_2) = 0.
\]
Exploiting the fibration leading to the 3-connected cover \( G_2(3) \), for which the base is \( G_2 \) and the fiber is \( K(Z, 2) \)[13], we obtain

\[
\begin{align*}
H^*(G_2(3); \mathbb{F}_2) &= \mathbb{F}_2[y_8] \otimes E(\beta y_8, S^q_2 S^q_1 y_8), \\
H^*(G_2(3); \mathbb{F}_3) &= \mathbb{F}_3[y_{50}] \otimes E(y_{11}, \beta y_{50}), \\
H^*(G_2(3); \mathbb{F}_p) &= \mathbb{F}_p[y_{2p}] \otimes E(y_{11}, \beta y_{2p}) \quad \text{for primes } p \neq 2, 5.
\end{align*}
\]

Hence \( H_i(G_2(3); \mathbb{F}_p) = 0 \) for \( 1 \leq i \leq 7 \) for all primes \( p \) and \( H_i(\Omega^4 G_2(3); \mathbb{F}_p) = H_i(\Omega^4 G_2; \mathbb{F}_p) = 0 \) for \( 1 \leq i \leq 3 \). Note that \( \Omega^4 G_2 \cong \Omega^4 G_2(3) \). Since \( BQ_k^b(G_2) \cong \Omega^3_k G_2 \), we have \( Q_k^b(G_2) \cong \Omega(\Omega^3_k G_2) \). Note \( \Omega^3 G_2 \cong \Omega^3 G_2 \times Z \) and \( \Omega(\Omega^3_k G_2) \cong \Omega^4 G_2 \) for any \( k \in Z \).

**Theorem 3.1.** The mod 2 homology of the four fold loop space of \( G_2 \) is

\[
H_*(\Omega^4 G_2; \mathbb{F}_2) = \mathbb{F}_2[Q^b_1 Q^b_2 \beta u_5] \otimes \mathbb{F}_2[Q^a_1 Q^b_2 Q^b_3 u_1 : i = 5, 7].
\]

**Proof.** We compute the \( H_*(\Omega^4 G_2; \mathbb{F}_2) \) by the almost same method computing the triple loop space in [4]. The group \( G_2 \), as a subgroup of \( O(7) \), acts on \( S^7 \). The action is transitive and the isotropy group is \( SU(3) \). So we have the following fibration

\[
SU(3) \longrightarrow G_2 \longrightarrow S^6.
\]

Consider the Serre spectral sequence for the following fibration:

\[
\Omega^4 SU(3) \longrightarrow \Omega^4 G_2 \longrightarrow \Omega^4 S^6.
\]

We have

\[
\begin{align*}
H_*(\Omega^4 S^6; \mathbb{F}_2) &= \mathbb{F}_2[Q^a_1 Q^b_2 Q^b_3 u_2], \\
H_*(\Omega^4 SU(3); \mathbb{F}_2) &= \mathbb{F}_2[Q^a_1 Q^b_3 u_1] \otimes \mathbb{F}_2[Q^a_1 Q^b_2 Q^b_3 u_4], \\
d^2 Q^a_3 + u_1 &= Q^a_1 Q^b_3 u_4.
\end{align*}
\]

Since \( H_i(\Omega^4 G_2; \mathbb{F}_2) = H_i(\Omega^4 G_2(3); \mathbb{F}_2) = 0 \) for \( 1 \leq i \leq 3 \), we have

\[
\tau(Q^a_1 Q^b_3 u_2) = Q^a_1 Q^b_3 u_1, \; a, b \geq 0.
\]

Note that \( \tau(Q^a_1 Q^b_2 Q^b_3 u_2) = 0 \) since \( Q^a_1 Q^b_2 Q^b_3 u_1 = 0 \). Then the \( E_\infty \) term is

\[
\mathbb{F}_2[Q^a_1 Q^b_2 Q^b_3 u_2] \otimes \mathbb{F}_2[Q^a_1 Q^b_2 Q^b_3 u_4] \otimes \mathbb{F}_2[Q^a_1 Q^b_2 Q^b_3 u_1 : i = 5, 7].
\]

On the other hand, \( S^q_1 Q^a_1 Q^a_2 Q^a_3 u_2 = Q^a_0 Q^a_1 Q^a_2 u_4 \), \( Q^a_0 Q^a_1 Q^a_2 u_4 \), \( a \geq 0 \) by the Nishida relation. Since \( d^2 Q^a_3 + u_1 = Q^a_1 Q^b_3 u_4 \), we have \( S^q_1 Q^a_1 Q^a_2 Q^a_3 u_2 = Q^a_1 Q^b_3 u_4 \), \( a \geq 0 \) by Theorem 2.2 in [4]. By the Nishida relation again, \( Q^a_0 Q^a_1 Q^a_2 u_4 \). So if we put \( Q^a_1 Q^b_3 u_4 \), then \( Q^a_0 Q^a_1 Q^a_2 Q^a_3 u_5 \) can be expressed as \( Q^a_0 Q^a_1 Q^a_2 Q^a_3 u_5 \). We also put \( u_7 = Q^a_3 u_2 \). Then we get \( H_*(\Omega^4 G_2; \mathbb{F}_2) = \mathbb{F}_2[Q^a_1 Q^b_3 u_4] \otimes \mathbb{F}_2[Q^a_1 Q^b_2 Q^b_3 u_1 : i = 5, 7] \). This result implies that the Eilenberg-Moore spectral sequence converging to \( H^*(\Omega^4 G_2; \mathbb{F}_2) \) for the path loop fibration collapses at \( E_2 \). Since \( H_*(\Omega^4 G_2(3); \mathbb{F}_2) = H_*(\Omega^1 G_2; \mathbb{F}_2) \), we have \( \beta u_5 = u_4 \) from the information of \( H^*(G_2(3); \mathbb{F}_2) \).
Theorem 3.2. For odd primes $p$, the mod $p$ homology of the four fold loop space of $G_2$ are as follows:

$$H_*(\Omega^4 G_2; \mathbb{F}_p) = E(Q^a_4 Q^b_4 u_7) \otimes \mathbb{F}_5[\beta Q^a_4 Q^b_4 u_7] \otimes \mathbb{F}_5[Q^a_8 Q^b_8 u_4 6]$$

$$\otimes E(Q^a_4 Q^b_8 Q^c_8 u_4 6) \otimes \mathbb{F}_5[\beta Q^a_4 Q^b_8 Q^c_8 u_4 6],$$

$$H_*(\Omega^4 G_2; \mathbb{F}_p) = \Omega_4(3, 11) \quad \text{for odd primes} \quad p \neq 5.$$  

Proof. Consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega^4 G_2; \mathbb{F}_p)$

with

$$E_2 \cong \text{Tor}_{H^*}(\Omega^5 G_2; \mathbb{F}_p)(\mathbb{F}_p, \mathbb{F}_p).$$

Then by Theorem 2.3, the collapse at $E_2$ depends on whether

$$\sigma : Q H^{2kp+2}(\Omega^5 G_2; \mathbb{F}_p) \to P H^{2kp+1}(\Omega^4 G_2; \mathbb{F}_p)$$

is injective or not. By the exact sequence of Milnor–Moore [12] and Theorem 2.3, we have that

$$Q H^{2kp+2}(\Omega^5 G_2; \mathbb{F}_p) \cong P H^{2kp+2}(\Omega^5 G_2; \mathbb{F}_p)$$

$$\cong Q H^{2kp+3}(\Omega^5 G_2; \mathbb{F}_p).$$

For odd primes $p$, every primitive element in $H_*(\Omega^2 G_2; \mathbb{F}_p)$ is one of the following types:

$$Q^a_{(p-1)} z_{2i+1}, \quad (\beta Q^a_{(p-1)} z_{2i+1})^p, \quad (\beta^2 Q^a_{(p-1)} z_{2i+1})^p.$$ 

Since $|Q^a_{(p-1)} z_{2i-1}| = 2p^a i - 1$ and $|\beta Q^a_{(p-1)} z_{2i-1}| = |\beta^2 Q^a_{(p-1)} z_{2i-1}| = 2p^a i - 2$, there is no primitive element with degree $2kp + 3$. By duality, there is no indecomposable element with degree $2kp + 3$ in $H^*(\Omega^5 G_2; \mathbb{F}_p)$. Hence the Eilenberg–Moore spectral sequence collapses at $E_2$ and there is no coalgebra extension problem in such a case [8]. Hence by duality, the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^4 G_2; \mathbb{F}_p)$ with

$$E^2 \cong \text{Cotor}_{H_*}(\Omega^5 G_2; \mathbb{F}_p)(\mathbb{F}_p, \mathbb{F}_p)$$

also collapses at $E^2$. Then we get the conclusion for $H_*(\Omega^4 G_2; \mathbb{F}_p)$ by the formal cotor calculation since there is no algebra extension problem by the duality. \qed

4. Full gauge group

Let $P_k$ be a principal $G_2$ bundle over $S^4$ classified by the integer $k$ in $Z$ and $G_k(G_2)$ be the gauge group of the principal $G_2$ bundle $P_k$. From [1, Prop. 2.4] we can get

$$BG_k(G_2) \simeq \text{Map}_{P_k}(S^4, BG_2),$$

where the subscript $P_k$ denotes the component of a map of $M$ into $BG_2$ which induces $P_k$. By the natural evaluation map, we have

$$\Omega^5 G_2 \to \text{Map}_{P_k}(S^4, BG_2) \to BG_2.$$
Looping again, we get
\[ \Omega^4 G_2 \cong \Omega(\Omega^3 G_2) \rightarrow G_k(G_2) \rightarrow 2. \]
So we have the following fiber sequence
\[ \cdots \rightarrow \Omega G_2 \xrightarrow{\Omega h_k} \Omega(\Omega^3 G_2) \rightarrow G_k(G_2) \rightarrow G_2 \rightarrow h_k \rightarrow \Omega^3 G_2 \rightarrow \cdots. \]
Main idea to compute \( H(G_k(G_2); \mathbb{F}_p) \) is to exploit the property of the map
\( (\Omega h_k)_* : H(\Omega G_2; \mathbb{F}_p) \rightarrow H(\Omega(\Omega^3 G_2); \mathbb{F}_p). \)
Recall the following \( p \)-primary component of homotopy groups of odd spheres [15, p. 176].

**Proposition 4.1.** Let \( p \) be an odd prime. Then we have the following.
\[ \pi_{2m+1+2i(p-1)-2}(S^{2m+1}; p) = \mathbb{Z}/(p) \text{ for } 1 \leq m < i, \text{ and } i = 2, \ldots, p-1. \]
\[ \pi_{2m+1+2i(p-1)-1}(S^{2m+1}; p) = \mathbb{Z}/(p) \text{ for } 1 \leq m, \text{ and } i = 1, 2, \ldots, p-1. \]
\[ \pi_{2m+1+k}(S^{2m+1}; p) = 0 \text{ otherwise for } k < 2p(p-1)-2. \]

From this, we have
\[ \pi_i(S^{4j-1}; p) = \begin{cases} \mathbb{Z}/(p), & \text{if } i = 2p \text{ and } j = 1, \\ 0, & \text{otherwise for } j > 1 \text{ and } 0 \leq i \leq 2p. \end{cases} \]

When localized at \( p \geq 7 \), \( G_2 \) splits as \( G_2 \cong_S S^3 \times S^1 \) [14]. So when localized at \( p > 7 \), the map \( h : (G_2)_{(p)} \rightarrow (\Omega^3 G_2)_{(p)} \) is null homotopic since \( h : (S^3 \times S^1)_{(p)} \rightarrow (\Omega^3 S^3 \times \Omega^3 S^1)_{(p)} \) is null homotopic. Here \( X_{(p)} \) denote \( X \) localized at the prime \( p \). Note that \( \pi_6(S^3; p) = \pi_{14}(S^3; p) = 0 \) for \( p > 7 \), but \( \pi_6(S^3; 3) = \mathbb{Z}/(3) \) and \( \pi_{14}(S^3; 7) = \mathbb{Z}/(7) \).

**Theorem 4.2.** For any primes \( p \neq 3, 7 \), we have
\[ H_*(G_k(G_2); \mathbb{F}_p) = H_*(\Omega^4 G_2; \mathbb{F}_p) \cong H_*(G_2; \mathbb{F}_2) \]
as an algebra for any \( k \in \mathbb{Z} \).

**Proof.** Consider the Serre spectral sequence converging to \( H_*(G_k(G_2); \mathbb{F}_p) \) for \( \Omega(\Omega^3 G) \rightarrow G_k(G_2) \rightarrow G_2 \). For \( p = 2 \), by the degree reason the first possible nontrivial differentials are \( \tau((S^2 q^2 x_3)^*) = \beta u_5 \) and \( \tau((x_3^2)^*) = u_5 \) where \((S^2 q^2 x_3)^* \) and \((x_3^2)^* \) are dual homology elements in \( H_*(G_2; \mathbb{F}_2) \). Note that \( \beta(x_3^2)^* = (S^2 q^2 x_3)^* \). Consider the following homotopy commutative morphisms of fibrations:
\[
\begin{array}{ccc}
\Omega G_2 & \longrightarrow & * \\
\Omega h_k & \downarrow & \downarrow \\
\Omega(\Omega^3 G_2) & \longrightarrow & \Omega BG_k(G_2) \longrightarrow G_2.
\end{array}
\]
Assume that \( \tau((S^2 q^2 x_3)^*) = \beta u_5 \). Then \( \tau(Q_0(S^2 q^2 x_3)^*) = Q_1 \beta u_5 \). But \( Q_0(S^2 q^2 x_3)^* = 0 \)
in $H_*(G_2; \mathbb{F}_2)$, while $Q_1 \beta u_5$ is not trivial in $H_*(\Omega^4 G_2; \mathbb{F}_2)$. This is a contradiction. Hence the Serre spectral sequence collapses at $E^2$ and there is no algebra extension problem by the degree reason.

Similarly for $p = 5$, the degree of target primitive of the first possible nontrivial differential should be 2 or 10. But there is no primitive of such degrees in $H_*(\Omega^4 G_2; \mathbb{F}_5)$. So the Serre spectral sequence collapses at $E_2$ and there is no algebra extension problem by the degree reason. Localized at $p > 7$, the map $h_k : (G_2)_{(p)} \to (\Omega^3 G_2)_{(p)}$ is null homotopic by the above argument following Proposition 4.1. So $\Omega h_k : (\Omega G_2)_{(p)} \to (\Omega(\Omega^3 G_2))_{(p)} \simeq (\Omega^4 G_2)_{(p)}$ is null homotopic for $p > 7$. Hence the Serre spectral sequence collapses at $E^2$ for $p > 7$. □

**Theorem 4.3.** For $k \not\equiv 0 \mod 3$, the mod 3 homology of $G_k(G_2)$ is as follows:

$$H_*(G_k(G_2); \mathbb{F}_3) = E(Q_3^k Q_6^k u_3) \otimes \mathbb{F}_3[\beta Q_2^k Q_6^k u_3] \otimes \mathbb{F}_3[Q_4^k \beta Q_6^k u_3]$$

$$\otimes \mathbb{F}_3[Q_4^k (Q_4 \beta u_3)] \otimes E(Q_3^k \beta Q_4^k \beta Q_6^k u_3)$$

$$\otimes \mathbb{F}_3[\beta Q_2^k \beta Q_4^k \beta Q_6^k u_3] \otimes \Omega_4(11) \otimes E(e_{11}).$$

**Proof.** We have the following homotopy commutative morphism of fibrations:

$$\begin{align*}
\Omega_2 S^3 & \xrightarrow{f} \Omega_2 G_2 \\
\downarrow & \\
\text{Map}_k(S^4, BS^3) & \longrightarrow \text{Map}_k(S^4, BG_2) \\
\downarrow & \\
BS^3 & \longrightarrow BG_2.
\end{align*}$$

Then $H_*(f; \mathbb{F}_3) = f_*$ is onto by Theorem 2.7. Consider the following homotopy commutative morphism of fibrations:

$$\begin{align*}
\Omega G_2 & \longrightarrow * \longrightarrow G_2 \\
\Omega h_k & \downarrow \\
\Omega(\Omega^3_k G_2) & \longrightarrow * \longrightarrow \Omega^3_k G_2.
\end{align*}$$

From the case for $S^3$ [9, 11], $(h_k)_*(x_3) = k \beta w_4$ up to a choice of a generator. So we have

$$(\Omega h_k)_*(x_2) = (\Omega h_k)_*(\tau(x_3)) = \tau((h_k)_*(x_3)) = \tau(\beta w_4) = \beta u_3.$$ 

There is the following homotopy commutative morphism of fibrations:

$$\begin{align*}
\Omega G_2 & \longrightarrow * \longrightarrow G_2 \\
\Omega h_k & \downarrow \\
\Omega(\Omega^3_k G_2) & \longrightarrow G_k(G_2) \longrightarrow G_2.
\end{align*}$$
Then we have $\tau(e_3) = \beta u_3$ in the Serre spectral sequence for the bottom row fibration converging to $H_*(G_k(G_2);F_3)$ where $H_*(G_2;F_3) = E(e_3,e_{11})$. Differentials from $e_{11}$ are trivial because of degree reason. Hence we get the conclusion.

For the case of $p = 7$, we follow the procedure used in [5]. Localized at $p = 7$, there are the following $p$-equivalences [14]:

$$Sp(3) \cong_7 S^3 \times S^7 \times S^{11} \cong_7 G_2 \times S^7.$$ 

Now we consider the exact sequence of homotopy groups:

$$\cdots \rightarrow \pi_{11}(Sp(3)) \xrightarrow{(\partial_k)_#} \pi_{11}(\Omega^3_k Sp(3)) \rightarrow \pi_{11}(\text{Map}_k(S^4, BSp(3))) \rightarrow \cdots$$

Then the boundary map $(\partial_k)_#$ can be expressed in terms of the Samelson product $(\cdot)$ as follows [2, 15]. For $\alpha$ in $\pi_{11}(Sp(n))$, we have

$$(\partial_k)_# \alpha = \pm k(\alpha, \beta),$$

where $\beta$ generates $\pi_3(Sp(3))$ [15, Proposition 2.1]. We recall the following fact.

**Proposition 4.4** ([2, Theorem 2]). The kernel of the homomorphism

$$\pi_{4n-1}(Sp(n)) \otimes \pi_{4m-1}(Sp(m)) \rightarrow \pi_{4n+4m-2}(Sp(n+m-1))$$

$$(\alpha \otimes \beta \mapsto \langle \alpha, \beta \rangle)$$

induced by the Samelson product $(\cdot)$ is precisely divisible by $k_{n+m}/k_nk_m$, where

$$k_r = \begin{cases} (2r-1)!2 & \text{for even } r, \\ (2r-1)! & \text{for odd } r. \end{cases}$$

**Theorem 4.5.** For $k \not\equiv 0$ mod 7, the mod 7 homology of $G_k$ is as follow:

$$H_*(G_k(G_2); F_7) = E(Q^0_6 Q^1_8 u_{11}) \otimes F_7[Q^0_6 \beta Q^b_7 Q^1_8 u_{11}] \otimes F_7[Q^0_6 \beta Q^b_7 Q^1_8 u_{11}]$$

$$\times F_7[Q^0_6 \beta Q^b_7 Q^1_8 u_{11}] \bigtimes E(Q^0_6 \beta Q^b_7 Q^1_8 u_{11})$$

$$\times F_7[Q^0_6 \beta Q^b_7 Q^1_8 u_{11}] \bigtimes \Omega_4(11) \otimes E(e_3).$$

**Proof.** By Proposition 4.4, for $\alpha \in \pi_{11}(Sp(3))$ and $\beta \in \pi_3(Sp(3))$ we get

the order of $\langle \alpha, \beta \rangle = 84$.

So if $p = 7$, then $\partial_k : (Sp(3))_{(7)} \rightarrow (\Omega^3_k Sp(3))_{(7)}$ is not null homotopic for $k \not\equiv 0$ mod 7. Localized at $p = 7$, $Sp(3) \cong_7 S^3 \times S^7 \times S^{11}$. So $\partial_k : (S^3 \times S^7 \times S^{11})_{(7)} \rightarrow (\Omega^3_k S^3 \times \Omega^3 S^7 \times \Omega^3 S^{11})_{(7)}$ is not null homotopic for $k \not\equiv 0$ mod 7. Note that

$$\pi_i(S^3; 7) = 0 \text{ for } i = 6, 10, \quad \pi_i(S^7; 7) = Z/(7),$$

$$\pi_i(S^{11}; 7) = 0 \text{ for } i = 6, 10, 14.$$ 

This implies that $\partial_k : (S^3 \times S^{11})_{(7)} \rightarrow (\Omega^3_k S^3 \times \Omega^3 S^{11})_{(7)}$ is not null homotopic for $k \not\equiv 0$ mod 7. Since $G_2 \cong_7 S^3 \times S^{11}$, $\partial_k : (G_2)_{(7)} \rightarrow (\Omega^3_k G_2)_{(7)}$ is not null homotopic for $k \not\equiv 0$ mod 7.
homotopic for $k \not\equiv 0 \pmod{7}$. Consider the following:

$$S^{11} \xrightarrow{\iota} S^3 \times S^{11} \xrightarrow{\partial_k} \Omega^3_k S^3 \times \Omega^3 S^{11} = \Omega^3_k G_2$$

$$p \downarrow$$

$$\Omega^3_k S^3.$$ 

Let $\partial_k' = p \circ \partial_k$ and $\partial_k'' = p \circ \partial_k \circ \iota$. Since $(\partial_k)'$ is nonzero, $(\partial_k'')$ is nonzero.

We have $h_\# : \pi_i(S^{11}; 7) \to \pi_i(\Omega^3_k S^3; 7)$ is an isomorphism for $i \leq 11$. Note that $\pi_i(\Omega^3_k S^3; 7) = 0$ for $1 \leq i \leq 10$. By J. H. C. Whitehead Theorem, $(\partial_k'')_* : H_{11}(S^{11}; \mathbb{F}_7) \to H_{11}(\Omega^3_k S^3; \mathbb{F}_7)$ is also an isomorphism. Hence we have

$$(\partial_k)_*(e_{11}) = \begin{cases} 0, & k \equiv 0 \pmod{7} \\ \neq 0, & k \not\equiv 0 \pmod{7}. \end{cases}$$

where $H_*(G_2; \mathbb{F}_7) = E(e_3, e_{11})$. There is the following homotopy commutative morphism of fibrations:

$$\text{Homotopy commutative morphism}$$

$$\Omega G_2 \longrightarrow * \longrightarrow G_2$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\Omega G_2 \longrightarrow * \longrightarrow G_2.$$

We have $(\partial_k)_*(e_{11}) = k\beta w_{12}$ for $\beta w_{12} \in H_*(\Omega^3_k G_2; \mathbb{F}_7)$ up to a choice of a generator. Let $\tau(e_{11}) = v_{10}$ for $v_{10} \in H_*(\Omega G_2; \mathbb{F}_7)$. Then we have that

$(\Omega \partial_k)_*(v_{10}) = (\Omega \partial_k)_*(\tau(e_{11})) = \tau((\partial_k)_*(e_{11})) = \tau(k\beta w_{12}) = k\beta u_{11}.$

Consider the following homotopy commutative morphism of fibrations:

$$\text{Homotopy commutative morphism}$$

$$\Omega G_2 \longrightarrow * \longrightarrow G_2$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\Omega G_2 \longrightarrow G_k(G_2) \longrightarrow G_2.$$

Then we have $\tau(e_{11}) = k\beta u_{11}$ in the Serre spectral sequence converging to $H_*(G_k(G_2); \mathbb{F}_7)$ for the bottom row fibration. From this we get the conclusion.

□

References


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