IFP RINGS AND NEAR-IFP RINGS

KYUNG-YUEN HAM, YOUNG CHEOL JEON, JINWOO KANG, NAM KYUN KIM, WONJAE LEE, YANG LEE, SUNG JU RYU, AND HAE-HUN YANG

ABSTRACT. A ring $R$ is called IFP, due to Bell, if $ab = 0$ implies $aRh = 0$ for $a, b \in R$. Hub et al. showed that the IFP condition need not be preserved by polynomial ring extensions. But it is shown that $\sum_{i=0}^{n} E_{i} E$ is a nonzero nilpotent ideal of $E$ whenever $R$ is an IFP ring and $0 \neq f \in F$ is nilpotent, where $E$ is a polynomial ring over $R$, $F$ is a polynomial ring over $E$, and $a_i's$ are the coefficients of $f$. We shall use the term near-IFP to denote such a ring as having place near at the IFPness. In the present note the structures of IFP rings and near-IFP rings are observed, extending the classes of them. IFP rings are NI (i.e., nilpotent elements form an ideal). It is shown that the near-IFPness and the N1ness are distinct each other, and the relations among them and related conditions are examined.

1. Near-IFP rings

Throughout every ring is associative with identity unless otherwise stated. $X$ denotes a nonempty set of commuting indeterminates over rings. Let $R$ be a ring. The polynomial ring over $R$ with $X$ is denoted by $R[X]$, and if $X$ is a singleton, say $X = \{x\}$, then we write $R[x]$ in place of $R[\{x\}]$. Every polynomial in $R[X]$ is written by $a_0 + \sum_{j=1}^{n} a_j X^j$, with $X^j$, a finite product of indeterminates over $R$, according to the notations in the proof of [10, Theorem 1.1]. The $n \times n$ matrix ring over a ring $R$ is denoted by $\text{Mat}_n(R)$, and $E_{ij}$ denotes the $n \times n$ matrix with $(i, j)$-entry 1 and zero elsewhere. The $n \times n$ upper and lower triangular matrix rings over $R$ are denoted by $\text{UTM}_n(R)$ and $\text{LTM}_n(R)$, respectively.

An element $a$ of a ring is called nilpotent if $a^m = 0$ for some positive integer $m$. A subset $S$ of a ring is called nilpotent if $S^n = 0$ for some positive integer $n$. A subset $T$ of a ring is called nil if each element of $T$ is nilpotent. Given a ring $R$, $N^*(R)$ and $N(R)$ denote the nilradical (i.e., the sum of all nil ideals) of $R$ and the set of all nilpotent elements in $R$, respectively. Note $N^*(R) \subseteq N(R)$.

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The $r_R(-)$ (resp. $\ell_R(-)$) is used for the right (resp. left) annihilator in a ring $R$. An element $a \in R$ is said to be right (resp. left) regular if $r_R(a) = 0$ (resp. $\ell_R(a) = 0$). A $\ell_R(a)$ is called a left (resp. right) zero-divisor if $r_R(a) \neq 0$ (resp. $\ell_R(a) \neq 0$). A zero-divisor means an element that is neither right nor left regular. A domain means a ring whose nonzero elements are two-sided regular.

A ring $R$ is called reduced if $N(R) = 0$. Marks [15] called a ring $R$ NI when $N^*(R) = N(R)$ (equivalently, $N(R)$ forms an ideal in $R$). Reduced rings are clearly NI and it is obvious that a ring $R$ is NI if and only if $R/N^*(R)$ is reduced. A prime ideal $P$ of a ring $R$ is called completely prime if $R/P$ is a domain. Hong et al. showed that a ring $R$ is NI if and only if every minimal strongly prime ideal of $R$ is completely prime [8, Corollary 13].

A well-known property between “commutative” and “NI” is the insertion-of-factors-property (simply IFP) due to Bell [1]; a right (or left) ideal $I$ of a ring $R$ is said to have the IFP if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. So a ring $R$ is called IFP if the zero ideal of $R$ has the IFP. Shin [17] used the term SI for the IFP; while IFP rings are also known as semicommutative in Narbonne’s paper [16]. IFP rings are NI by [17, Theorem 1.5], and reduced rings are IFP by a simple computation. A ring is called abelian if each idempotent is central. IFP rings are abelian by a simple computation.

Huh et al. showed that the IFP condition need not be preserved by polynomial ring extensions [11, Example 2]. But IFP rings have the following useful facts.

**Lemma 1.1.** (1) A ring $R$ is IFP if and only if $r_R(S)$ is an ideal of $R$ for any $S \subseteq R$ if and only if $\ell_R(S)$ is an ideal of $R$ for any $S \subseteq R$.

(2) IFP rings are NI.

(3) If $R$ is an NI ring and $a_0 + \sum_{j=1}^{n} a_j X^j \in N(R[X])$ then $\sum_{j=0}^{n} Ra_j R$ is nil.

(4) Let $R$ be an IFP ring. Then $\sum_{j=0}^{n} Ra_j R$ is nilpotent whenever $a_0 + \sum_{j=1}^{n} a_j X^j \in R[X]$ is nilpotent.

**Proof.** (1) and (2) are proved by [17, Lemma 1.2] and [17, Theorem 1.5], respectively.

(3) Let $R$ be an NI ring and $a_0 + \sum_{j=1}^{n} a_j X^j \in N(R[X])$. Then $R/N^*(R)$ is reduced with $N^*(R) = N(R)$ by the definition, and so from

$$\frac{R[X]}{N^*(R)[X]} \cong \frac{R}{N^*(R)[X]}$$

we have $N(R[X]) \subseteq N^*(R)[X]$, entailing $a_j \in N^*(R)$ for all $j$. Then $\sum_{j=0}^{n} Ra_j R$ is nil since $N^*(R)$ is an ideal of $R$.

(4) Let $R$ be an IFP ring and $a_0 + \sum_{j=1}^{n} a_j X^j \in N(R[X])$. Then by (2, 3) all $a_j$'s are in $N(R)$. Say $a_j^{k_j} = 0$ for some positive integer $k_j$, then $(Ra_j R)^{k_j} = 0$ for all $j$.
since \( R \) is IFP. Thus we obtain
\[
\left( \sum_{j=0}^{n} Ra_j R \right)^k = 0 \text{ with } k = \sum_{j=0}^{n} k_j.
\]

Here we consider the following condition that is weaker than the result in Lemma 1.1(4): (*) \( \sum_{j=0}^{n} Ra_j R \) contains a nonzero nilpotent ideal whenever a nonzero polynomial \( \sum_{i=0}^{n} a_i x^i \) over a ring \( R \) is nilpotent. Then the condition (*) is placed near at the IFPness by Lemma 1.1(4); hence we call a ring near-IFP if it satisfies the condition (*). However the near-IFPness is distinct from the NIness as we see below. IFP rings are near-IFP by Lemma 1.1(4).

**Proposition 1.2.** For a ring \( R \) the following conditions are equivalent:

1. \( R \) is near-IFP;
2. \( RaR \) contains a nonzero nilpotent ideal of \( R \) for any \( 0 \neq a \in N(R) \);
3. \( \sum_{j=0}^{n} Ra_j R \) contains a nonzero nilpotent ideal of \( R \) whenever \( 0 \neq a_0 + \sum_{j=1}^{n} a_j X^j \in R[X] \) is nilpotent.

**Proof.** It suffices to obtain (3) from (2). Let \( 0 \neq f(X) = a_0 + \sum_{j=1}^{n} a_j X^j \in N(R[X]) \) with \( I_j < I_{j+1} \) for all \( j \geq 1 \). Without loss of generality, we can put \( a_1 \neq 0 \) when \( a_0 = 0 \). Then by the proof of [10, Theorem 1.1], we get \( a_0 \in N(R) \) (when \( a_0 \neq 0 \)) or \( a_1 \in N(R) \) (when \( a_0 = 0 \)). By the condition (2), there exists a nonzero nilpotent ideal of \( R \) contained in \( \sum_{k=0}^{n} Ra_k R \subseteq \sum_{k=0}^{n} Ra_k R \), completing the proof. \( \square \)

We will use Proposition 1.2 freely. In the following we confirm that there are no containing relations between the classes of near-IFP rings and NI rings, and that NI rings and near-IFP rings need not be IFP.

**Example 1.3.**

1. Let \( R = UTM_2(S) \) over a reduced ring \( S \). Note that \( N^*(R) = (0 \ 0 \ \ 0 \ 0) = N(R) \) (hence \( R \) is NI) and \( N^*(R)^2 = 0 \). Since \( N(R) \neq 0 \) we can take \( 0 \neq a_0 \in N(R) \). But \( (Ra_0 R)^2 = 0 \) and thus \( R \) is near-IFP. However \( R \) is not IFP since \( R \) is non-abelian.

2. There is an NI ring but not near-IFP. Let \( T \) be a reduced ring, \( n \) be a positive integer and \( R_n \) be the \( 2^n \) by \( 2^n \) upper triangular matrix ring over \( T \). Define a map \( \sigma : R_n \rightarrow R_{n+1} \) by \( A \mapsto \left( \begin{array}{cc} A & \ 0 \\ \ 0 & A \end{array} \right) \), then \( R_n \) can be considered as a subring of \( R_{n+1} \) via \( \sigma \) (i.e., \( A = \sigma(A) \) for \( A \in R_n \)). Let \( R \) be the direct limit of the direct system \( (R_n, \sigma_{ij}) \), where \( \sigma_{ij} = \sigma^j-1 \). Then \( R \) is NI by [12, Proposition 1.1], and semiprime by [7, Corollary 1.3]. Note that \( N(R) \) is an infinite subset of \( R \), but \( RaR \) cannot contain any nonzero nilpotent ideal for each \( 0 \neq a \in N(R) \) since \( R \) is semiprime. Thus \( R \) is not near-IFP.

3. There is a near-IFP ring but not NI. Let \( S = \mathbb{Z}_4 \), the ring of integers modulo 4, and \( R = \text{Mat}_n(S) \) for \( n \geq 2 \). Note \( (\text{Mat}_n(2\mathbb{Z}_4))^2 = 0 \). Let \( 0 \neq A \in N(R) \). Since \( \text{Mat}_n(2\mathbb{Z}_4) \) is the only nonzero proper ideal of \( R \), \( RAR \) is either
Mat\(_n(2\mathbb{Z}_4)\) (when \(A \in \text{Mat}_n(2\mathbb{Z}_4)\)) or \(R\) (when \(A \in R\setminus\text{Mat}_n(2\mathbb{Z}_4)\)). Thus \(RAR\) must contain \(\text{Mat}_n(2\mathbb{Z}_4)\) and so \(R\) is near-IFP. However \(R\) is not NI as can be seen by \(E_{12} + E_{21} \notin N(R)\).

If given rings are semiprime then near-IFP rings are reduced as follows.

**Proposition 1.4.** Let \(R\) be a semiprime ring. Then the following conditions are equivalent:

1. \(R\) is near-IFP;
2. \(R\) is IFP;
3. \(R\) is reduced.

**Proof.** It suffices to show \((1) \Rightarrow (3)\). Let \(R\) be near-IFP and \(a^2 = 0\) for \(a \in R\). If \(a \neq 0\) then \(RaR\) contains a nonzero nilpotent ideal \(I\) of \(R\) since \(R\) is near-IFP; but \(R\) is semiprime by hypothesis and so \(I = 0\), a contradiction. Thus \(R\) is reduced.

When \(R\) is a semiprime ring we may conjecture that a ring \(R\) is NI if and only if \(R\) is reduced, based on Proposition 1.4. However there is a semiprime NI ring but not reduced as we see in Example 1.3(2).

The *index of nilpotency* of a subset \(I\) of a ring is the supremum of the indices of nilpotency of all nilpotent elements in \(I\). If such a supremum is finite, then \(I\) is said to be of bounded index of nilpotency.

**Proposition 1.5.** Let \(R\) be a semiprime ring of bounded index of nilpotency. Then the following conditions are equivalent:

1. \(R\) is near-IFP;
2. \(R\) is IFP;
3. \(R\) is reduced;
4. \(R\) is NI.

**Proof.** With the help of Proposition 1.4, it suffices to show \((4) \Rightarrow (3)\) since reduced rings are clearly NI. Let \(R\) be NI and assume \(N(R) \neq 0\). Take \(0 \neq a \in N(R)\). Then \(RaR\) is a nonzero nil ideal of \(R\). Since \(R\) is of bounded index of nilpotency, \(RaR\) contains a nonzero nilpotent ideal, say \(J\), by Levitzki [6, Lemma 1.1] or Klein [14, Lemma 5]. But \(R\) is semiprime and so \(J = 0\), a contradiction. Thus \(N(R) = 0\).

The condition "of bounded index of nilpotency" in Proposition 1.5 is not superfluous by Example 1.3(2) (this ring is semiprime but not of bounded index of nilpotency); while, the condition "semiprime" in Proposition 1.5 is also not superfluous by Example 1.3(3) (this ring is of bounded index of nilpotency but not semiprime).

A ring \(R\) is called *von Neumann regular* if for each \(a \in R\) there exists \(x \in R\) such that \(a = axa\). A ring is called *right (resp. left) duo* if every right (resp. left) ideal is two-sided. Right or left duo rings are IFP by Lemma 1.1(1). Von Neumann regular rings need not be near-IFP in spite of being
semiprime. \( \text{Mat}_n(R) \) is von Neumann regular by [4, Lemma 1.6] over a von Neumann regular ring \( R \), but it is not near-IFP by Proposition 1.10(2) below when \( n \geq 2 \). In the following we see some conditions under which von Neumann regular rings can be near-IFP.

**Proposition 1.6.** Let \( R \) be a von Neumann regular ring. Then the following conditions are equivalent:

1. \( R \) is right (left) duo;
2. \( R \) is reduced;
3. \( R \) is abelian;
4. \( R \) is IFP;
5. \( R \) is near-IFP;
6. \( R \) is NI.

**Proof.** The equivalences of the conditions (1), (2), and (3) are proved by [4, Theorem 3.2]. The equivalences of the conditions (2), (4), and (5) are proved by Proposition 1.4 since von Neumann regular rings are semiprime. (2)\( \Rightarrow \) (6) is obvious.

(6)\( \Rightarrow \) (2): Let \( R \) be NI and assume \( N(R) \neq 0 \). Take \( 0 \neq a \in N(R) \). Since \( R \) is von Neumann regular, there exists \( b \in R \) such that \( a = aba \). Then we get \( a = aba = ababa = \cdots \). But \( N(R) \) is an ideal of \( R \) and so \( ab \in N(R) \); hence \( (ab)^n = 0 \) for some positive integer \( n \). This entails \( 0 \neq a = aba = \cdots = (ab)^n a = 0 \), a contradiction. Thus \( N(R) = 0 \). \( \square \)

A ring \( R \) is called **strongly regular** if for each \( a \in R \) there exists \( x \in R \) such that \( a = a^2x \). A ring is strongly regular if and only if it is abelian and von Neumann regular [4, Theorem 3.5]. From Proposition 1.6 we obtain a similar result to [4, Theorem 3.5].

**Corollary 1.7.** A ring is strongly regular if and only if it is near-IFP and von Neumann regular.

A ring \( R \) is called **directly finite** if \( ab = 1 \) implies \( ba = 1 \) for \( a, b \in R \). NI rings are directly finite by [12, Proposition 2.7(1)]. Abelian rings are also directly finite (hence so are IFP rings). For, if \( R \) is an abelian ring and \( a, b \in R \) with \( ab = 1 \), then \( bab = ba \) and so \( ba = baab = abab = 1 \). Thus from Proposition 1.6 one may conjecture that near-IFP rings are directly finite. However the answer is negative by the following.

**Example 1.8.** Let \( F \) be a field and \( \mathcal{V} \) be a vector space over \( F \) with \( \dim_F(\mathcal{V}) = \aleph_0 \). Set \( S = \text{Hom}_F(\mathcal{V}, \mathcal{V}) \). Take \( a = \sum_{i=1}^{\infty} E_{(i+1)j} \) and \( b = \sum_{j=1}^{\infty} E_{(j+1)j} \) in \( S \). Then \( ab = 1 \) but \( ba \neq 1 \) in \( S \). Next let \( R = \text{UTM}_n(S) \) for \( n \geq 2 \). Then \( R \) is near-IFP by Proposition 1.10(1) below, but \( (aE)(bE) = E \) but \( (bE)(aE) \neq E \) in \( R \), where \( E \) is the identity matrix in \( R \). Thus \( R \) is not directly finite.

A ring \( R \) is called **\( \pi \)-regular** if for each \( a \in R \) there exist a positive integer \( n \), depending on \( a \), and \( b \in R \) such that \( a^n = a^n ba^n \). Von Neumann regular
rings are obviously $\pi$-regular, and so one may ask if a $\pi$-regular ring is near-IFP when it is abelian and semiprime, based on Proposition 1.6. However the answer is negative by the following.

Example 1.9. Let $S$ be a division ring. Consider the ring extension of $S$, that is a subring of $\text{UTM}_{2^n}(S)$,

$$D_n = \{M \in \text{UTM}_{2^n}(S) \mid \text{the diagonal entries of } M \text{ are equal}\}.$$ 

Define a map $\sigma : D_n \to D_{n+1}$ by $A \mapsto \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A} \end{pmatrix}$, then $D_n$ can be considered as a subring of $D_{n+1}$ via $\sigma$ (i.e., $A = \sigma(A)$ for $A \in D_n$). Set $R$ be the direct limit of the direct system $(D_n, \sigma_{ij})$ with $\sigma_{ij} = \sigma^{j-i}$. Then $R$ is semiprime by [5, Theorem 2.2(2)]. Every $D_n$ is abelian by [9, Lemma 2] such that every idempotent in $D_n$ is of the form

$$\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$$

with $f^2 = f \in S$. Thus $R$ is also abelian. Every element of $D_n$ is either invertible or nilpotent and therefore $D_n$ is $\pi$-regular; hence $R$ is also $\pi$-regular. However $R$ is not near-IFP by Proposition 1.4.

In the following proposition we see some criteria by which we examine the near-IFPness of given rings.

Proposition 1.10. (1) For any ring $S$, $\text{UTM}_{n}(S)$ and $\text{LTM}_{n}(S)$ are near-IFP for $n \geq 2$.

(2) $\text{Mat}_{n}(S)$ over a semiprime ring $S$ cannot be near-IFP for $n \geq 2$.

(3) Suppose that a ring $S$ contains a nonzero nilpotent ideal $I$ such that every element in $S \setminus I$ is invertible. Then $\text{Mat}_{n}(S)$ is near-IFP for any $n$.

Proof. (1) Let $R = \text{UTM}_{n}(S)$ for $n \geq 2$ and $0 \neq A = (a_{ij}) \in R$ with $a_{st} \neq 0$. Then $RE_{1s}AE_{tn}R = S\alpha_{st}SE_{1n}$ is a nonzero nilpotent ideal of $R$ that is contained in $RAR$. Thus $R$ is near-IFP. The proof of the case $\text{LTM}_{n}(S)$ is similar.

(2) Let $R = \text{Mat}_{n}(S)$. Since $n \geq 2$ we have $N(R) \neq 0$. Thus $R$ is not near-IFP by Proposition 1.4.

(3) Let $R = \text{Mat}_{n}(S)$ and $0 \neq A = (a_{ij}) \in N(R)$. If every $a_{ij}$ is in $I$ then $RAR$ is nilpotent because $I$ is nilpotent. Otherwise we get $RAR = R$ since some $a_{ij}$ is invertible by hypothesis, hence $RAR$ contains the nonzero nilpotent ideal $R\text{Mat}_{n}(I)R$. \hfill \Box

We can apply Proposition 1.10(3) to Example 1.3(3). $Z_4$ contains a nonzero nilpotent ideal $2Z_4$ such that $Z_4 \setminus 2Z_4$ is the subset of invertible elements in $Z_4$. Thus $\text{Mat}_{n}(Z_4)$ is near-IFP by Proposition 1.10(3).
2. Structure and examples of (near-)IFP rings

Huh et al. showed that $R[x]$ need not be IFP when $R$ is an IFP ring [11, Example 2]. But $R[X]$ can be near-IFP over an IFP ring $R$ as in the following. The prime radical of a ring $R$ is denoted by $N_*(R)$. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$ for any ring $R$.

**Proposition 2.1.** Let $R$ be an IFP ring. Then $\sum_{i=0}^{n} f_i R[X]y^i$ is nilpotent whenever $\sum_{i=0}^{n} f_i y^i \in R[X][y]$ is nilpotent, where $f_i \in R[X]$ and $y$ is an indeterminate over $R[X]$: especially $R[X]$ is near-IFP.

**Proof.** Let $R$ be an IFP ring. Suppose that $0 \neq g(y) = \sum_{i=0}^{n} f_i y^i \in N(R[X][y])$ with $f_i = a(i)_0 + \sum_{j=1}^{m_i} a(i)_j X(i)_j^I \in R[X]$ for $i = 0, 1, \ldots, n$, where $y$ is an indeterminate over $R[X]$. We can say $f_0 \neq 0$ after dividing $g(y)$ by powers of $y$ if necessary. Note that there is a finite subset $X_0$ of $X$ such that $f_i \in R[X_0]$ for all $i$, say $X_0 = \{x_1, \ldots, x_r\}$. Then $g(y)$ is nilpotent in $R[x_1, \ldots, x_r, y]$. Write $g(y) = g(x_1, \ldots, x_r, y)$.

By [17, Theorem 1.5] we have $N_*(R) = N^*(R) = N(R)$ since $R$ is IFP, and so by [2, Proposition 2.6] we have $N_*(R[X]) = N^*(R[X]) = N(R[X])$. It is well-known that $N_*(A[X]) = N_*(A)[X]$ for any ring $A$. Consequently we get $N(R)[X] = N_*(R[X]) = N^*(R[X]) = N(R[X])$. Applying this result we obtain that every coefficient $f_i$ is nilpotent in $R[x_1, \ldots, x_r, y]$, from $g(x_1, \ldots, x_r, y) \in N(R[x_1, \ldots, x_r, y])$. Then by Lemma 1.1(4), we obtain that

$$\sum_{i=0}^{n} \sum_{j=0}^{m_i} Ra(i)_j R$$

is nilpotent.

It then follows that

$$\sum_{i=0}^{n} R[X]f_i R[X] \subseteq \sum_{i=0}^{n} R[X](K(i)_0 + \sum_{j=1}^{m_i} K(i)_j X(i)_j^I)R[X],$$

where $K(i)_j = Ra(i)_j R$. But $\sum_{i=0}^{n} R[X](K(i)_0 + \sum_{j=1}^{m_i} K(i)_j X(i)_j^I)R[X]$ is nilpotent and therefore $\sum_{i=0}^{n} R[X]f_i R[X]$ is also nilpotent. It is immediate that $R[X]$ is near-IFP. $\square$

GF$(p^n)$ means the Galois field of order $p^n$. In the following an infinite direct sum is considered as a ring without identity.

**Proposition 2.2.** (1) Every minimal noncommutative near-IFP ring is isomorphic to UTm$_2$(GF(2)).

(2) The class of near-IFP rings is closed under direct sums and direct products.

**Proof.** (1) By [3, Proposition] a finite noncommutative ring $R$ is isomorphic to UTm$_2$(GF(p)) when the order of $R$ is $p^2$, $p$ a prime. Next by [3, Theorem] a finite ring $R$ of order $m$ is commutative when $m$ has a cube free factorization. Thus every minimal noncommutative ring is isomorphic to UTm$_2$(GF(2)). But
UTM$_2(GF(2))$ is near-IFP by Proposition 1.10(1), and hence every minimal noncommutative near-IFP ring is isomorphic to UTM$_2(GF(2))$.

(2) Let $R_i$ be a near-IFP ring for $i \in I$ and $R = \oplus_{i \in I} R_i$. Take $0 \neq (a_i) \in N(R)$. There are $j = 1, \ldots, n$ such that $a_{ij} \neq 0$. Since each $R_i$ is near-IFP, $R_{ij} a_{ij} R_{ij}$ contains a nonzero nilpotent ideal of $R_{ij}$, say $N_{ij}$. Put $N = \oplus_{i \in I} N_i$ such that $N_i = 0$ for $i \neq i_j$. Then clearly $N$ is nilpotent and is contained in $R(a_i)R$.

Next let $R = \prod_{i \in I} R_i$ and $0 \neq (a_i) \in N(R)$. We can take a finite number of $a_i$'s, say $a_{ij} \neq 0$ for $j = 1, \ldots, n$. Then as above $N$ is a nilpotent ideal of $R$ contained in $R(a_i)R$.

From Proposition 2.2(2) and the relation between direct sums and direct products, one may suspect that the class of near-IFP rings may be closed under subrings. However there exists a near-IFP ring whose subrings are non-near-IFP as follows.

**Example 2.3.** Let $S$ be a semiprime ring and $T = \text{Mat}_n(S)$ for $n \geq 2$. Then $T$ is not near-IFP by Proposition 1.10(2). Next let $R = UTM_n(T)$ for $n \geq 2$. Then $R$ is near-IFP by Proposition 1.10(1) but the subring $T$ of $R$ is not near-IFP.

**Proposition 2.4.** For a ring $R$ suppose that $R/I$ is a near-IFP ring for some ideal $I$ of $R$. If $I$ is nilpotent then $R$ is near-IFP.

**Proof.** Let $0 \neq a \in N(R)$. If $a \in I$ then $RaR$ is nilpotent and so we assume $a \notin I$. Write $\hat{R} = R/I$ and $\hat{r} = r + I$ for $r \in R$. Since $\hat{R}$ is near-IFP, $\hat{R}a\hat{R}$ contains a nonzero nilpotent ideal of $\hat{R}$, say $J/I$ with $J^k \subseteq I$ for some positive integer $k$. But since $I$ is nilpotent, say $I^n = 0$ for some positive integer $n$, we get $J^{kn} = 0$. Take $0 \neq b \in J$. Then there exists $0 \neq c \in RaR$ with $c - b \in I$, and hence $0 \neq c \in J$. Since $J$ is nilpotent, $RcR$ is a nonzero nilpotent ideal of $R$ contained in $RaR$. Thus $R$ is near-IFP.

Instead of the condition "$I$ is nilpotent" in Proposition 2.4 we may consider a weaker one "$I$ is nil". However this one cannot guarantee the near-IFPness of $R$ as we see in the following.

**Example 2.5.** Consider the ring $R$ in Example 1.3(2). Let

$$I = \{ M \in R \mid \text{each diagonal entry of } M \text{ is zero} \}.$$ 

Then $I$ is a nil ideal of $R$ such that $R/I$ is reduced (hence near-IFP). But $R$ is not near-IFP.

Given rings $A$ and $B$, suppose that $A_U B$, $B V_A$ are bimodules and

$$\theta : U \otimes_B V \to A, \quad \psi : V \otimes_A U \to B$$

are bimodule homomorphisms (called pairings). The array $T = (\theta_U \psi_U)$ can be given the formal operations of 2 by 2 matrices, using $\theta$ and $\psi$ in defining multiplication. If $\theta$, $\psi$ satisfy the associativity conditions required to make $T$
a ring then the collection \((A, B, U, V, \theta, \psi)\) is called a Morita context, and \(T\) is called the ring of the Morita context. If \(\theta, \psi\) are zero then they are called zero pairings.

**Proposition 2.6.** Suppose that \(T\) is the ring of a Morita context
\[
(A, B, U, V, \theta, \psi)
\]
with zero pairings.

1. If \(A\) and \(B\) are near-IFP then so is \(T\).
2. \(T\) is NI if and only if so are \(A\) and \(B\).

**Proof.**

1. Let \(I = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\). Then \(I\) is a nilpotent ideal of \(T\) by the zero pairings. Since \(\frac{B}{I} \cong \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \cong A \oplus B\), \(T\) is near-IFP by Propositions 2.2(2) and 2.4.

2. Since \(\left(\begin{smallmatrix} 0 & U \\ V & 0 \end{smallmatrix}\right)\) is a nilpotent ideal of \(T\), we have \(\mathcal{N}(T) = \left(\begin{smallmatrix} \mathcal{N}(A) & U \\ V & \mathcal{N}(B) \end{smallmatrix}\right)\) because \(\theta, \psi\) are zero pairings by the hypothesis. Note that \(\mathcal{N}(T)\) is an ideal of \(T\) if and only if \(\mathcal{N}(A)\) and \(\mathcal{N}(B)\) are ideals of \(A\) and \(B\), respectively. The proof is then complete. \(\square\)

In Proposition 2.6(2) we have the isomorphisms
\[
\frac{T}{\mathcal{N}(T)} \cong \begin{pmatrix} A & 0 \\ \mathcal{N}(A) & B \end{pmatrix} \cong \frac{A}{\mathcal{N}(A)} \oplus \frac{B}{\mathcal{N}(B)} \subseteq \mathcal{N}(T).
\]

Proposition 2.6(1) can be applied to prove Proposition 1.10(1) when \(n = 2\), letting \(U = S\) and \(V = 0\). But the converse of Proposition 2.6(1) need not hold by the following.

**Example 2.7.** Let \(A = B = \text{Mat}_2(\mathbb{Z}_6)\), where \(\mathbb{Z}_6\) is the ring of integers modulo 6, and \(U = \text{Mat}_2(2\mathbb{Z}_6), V = \text{Mat}_2(3\mathbb{Z}_6)\). Define \(\theta(U \otimes_B V) = UV\) and \(\psi(V \otimes_A U) = VU\), then they are zero pairings. Let \(T\) be the ring of a Morita context \((A, B, U, V, \theta, \psi)\). Take \(0 \neq a = \left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{N}(T)\). Then \(x, y \in \mathcal{N}(A)\). If \(x = 0 = y\) then \(\alpha \neq 0\) or \(\beta \neq 0\) and so \(TaT\) itself is a nonzero nilpotent ideal of \(T\). Assume \(x \neq 0\). If \(x \in A\setminus V\), then \(TaT\) contains a nonzero nilpotent ideal
\[
Ta \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} T = \begin{pmatrix} 0 & A(2x)B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}.
\]

If \(x \in A\setminus U\), then \(TaT\) contains a nonzero nilpotent ideal
\[
T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \alpha T = \begin{pmatrix} 0 & 0 \\ B(3x)A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}.
\]

The computation for the case of \(y \neq 0\) is similar. However \(A, B\) are both not near-IFP by Proposition 1.10(2) since \(\mathbb{Z}_6\) is semiprime.
Over a reduced ring $R$ the subring of $\text{Mat}_3(R)$

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is an IFP ring by [13, Proposition 1.2]. In the following we see an IFP ring of similar structure.

**Proposition 2.8.** Let $R$ be a ring and suppose that $I, J$ are ideals of $R$ satisfying $IJ = 0 = JI$. If $R$ is reduced then

$$\left\{ \begin{pmatrix} a & b & c \\ e & a & d \\ f & g & a \end{pmatrix} \mid a \in R \text{ and } b, c, d \in I, e, f, g \in J \right\}$$

is an IFP ring.

**Proof.** Put

$$S = \left\{ \begin{pmatrix} a & b & c \\ e & a & d \\ f & g & a \end{pmatrix} \mid a \in R \text{ and } b, c, d \in I, e, f, g \in J \right\}.$$

The addition and multiplication of

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ e_1 & a_1 & d_1 \\ f_1 & g_1 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 & c_2 \\ e_2 & a_2 & d_2 \\ f_2 & g_2 & a_2 \end{pmatrix}$$

in $S$ can be rewritten by

$$(a_1; b_1, c_1, d_1; e_1, f_1, g_1) + (a_2; b_2, c_2, d_2; e_2, f_2, g_2)$$

$$= (a_1 + a_2; b_1 + b_2, c_1 + c_2, d_1 + d_2; e_1 + e_2, f_1 + f_2, g_1 + g_2)$$

and

$$(a_1; b_1, c_1, d_1; e_1, f_1, g_1)(a_2; b_2, c_2, d_2; e_2, f_2, g_2)$$

$$= (a_1 a_2; a_1 b_2 + a_1 a_2, a_1 c_2 + b_1 d_2 + c_1 a_2, a_1 d_2 + d_1 a_2;$$

$$e_1 a_2 + a_1 e_2, f_1 a_2 + g_1 e_2 + a_1 f_2, g_1 a_2 + a_1 g_2)$$

respectively. Now let $AB = 0$, then

$$(a_1; b_1, c_1, d_1; e_1, f_1, g_1)(a_2; b_2, c_2, d_2; e_2, f_2, g_2) = 0$$

and hence we have the following system of equations:

$$a_1 a_2 = 0$$

$$a_1 b_2 + b_1 a_2 = 0, a_1 c_2 + b_1 d_2 + c_1 a_2 = 0, a_1 d_2 + d_1 a_2 = 0$$

$$e_1 a_2 + a_1 e_2 = 0, f_1 a_2 + g_1 e_2 + a_1 f_2 = 0, g_1 a_2 + a_1 g_2 = 0.$$

Reduced rings are IFP. So $a_1 R a_2 = 0$ and by the proof of [13, Proposition 1.2], we have

$$a_1 R b_2 = b_1 R a_2 = a_1 R c_2 = b_1 R d_2 = c_1 R a_2 = a_1 R d_2 = d_1 R a_2 = 0.$$
In a similar way to the case of the upper triangular part, we also get
\[ e_1Ra_2 = a_1Re_2 = f_1Ra_2 = g_1Re_2 = a_1Rf_2 = g_1Ra_2 = a_1Rg_2 = 0. \]

It is an immediate consequence that for any \((r; s, t, u; x, y, z) \in S\) we have
\[(a_1; b_1, c_1; d_1; e_1, f_1, g_1)(r; s, t, u; x, y, z)(a_2; b_2, c_2; d_2; e_2, f_2, g_2) = (a_1ra_2, a_1rb_2 + a_1sa_2 + b_1ra_2, a_1rc_2 + a_1sd_2 + b_1rd_2 + a_1ta_2 + b_1ua_2 + c_1ra_2, a_1rd_2 + a_1ua_2 + d_1ra_2) = 0.\]

Thus we get
\[
\begin{pmatrix}
a_1 & b_1 & c_1 \\
e_1 & a_1 & d_1 \\
f_1 & g_1 & a_1
\end{pmatrix}
\begin{pmatrix}
r & s & t \\
x & r & u \\
y & z & r
\end{pmatrix}
\begin{pmatrix}
a_2 & b_2 & c_2 \\
e_2 & a_2 & d_2 \\
f_2 & g_2 & a_2
\end{pmatrix} = 0
\]

for any
\[
\begin{pmatrix}
r & s & t \\
x & r & u \\
y & z & r
\end{pmatrix} \in S;
\]

hence \(S\) is IFP. \[\square\]

Let \(R = D_1 \oplus D_2\) for domains \(D_i\) and \(I = D_1 \oplus 0, J = 0 \oplus D_2\). Then \(IJ = 0 = JI\) and so by Proposition 2.8 we get an IFP ring
\[
\left\{ \begin{pmatrix} a & b & c \\ e & a & d \\ f & g & a \end{pmatrix} \mid a \in R \text{ and } b, c, d \in I, e, f, g \in J \right\}.
\]

The converse of Proposition 2.8 need not hold as we see below. For that we define a kind of subring of \(\text{UTM}_n(S)\)
\[
D_n(S) = \{ M \in \text{UTM}_n(S) \mid \text{the diagonal entries of } M \text{ are equal}\},
\]
where \(S\) is a given ring.

**Example 2.9.** Let \(S\) be a commutative domain and
\[
R = D_2(S) \oplus S \oplus S.
\]

Take the ideals \(I = 0 \oplus S \oplus 0, J = 0 \oplus 0 \oplus S\) of \(R\), then clearly \(IJ = 0 = JI\). Let
\[
S = \left\{ \begin{pmatrix} a & b & c \\ e & a & d \\ f & g & a \end{pmatrix} \mid a \in R \text{ and } b, c, d \in I, e, f, g \in J \right\}
\]
and \(AB = 0\), where
\[
A = \begin{pmatrix} a_1 & b_1 & c_1 \\ e_1 & a_1 & d_1 \\ f_1 & g_1 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 & c_2 \\ e_2 & a_2 & d_2 \\ f_2 & g_2 & a_2 \end{pmatrix}
\]
are in \(S\). Since \(R\) is commutative, \(R\) is IFP and hence we have \(a_1Ra_2 = 0\). Notice that remaining computations are actually done in a reduced \(0 \oplus S \oplus S\), from the structure of \(I\) and \(J\). Thus we can obtain the same result as in the
proof of Proposition 2.8, i.e., \( ASB = 0 \), concluding that \( S \) is IFP. But \( R \) is not reduced.

But, in Proposition 2.8, if \((I = R, J = 0)\) or \((I = 0, J = R)\) then \( R \) is reduced by [5, Proposition 2.8] when \( S \) is IFP.

For a reduced ring \( R \), \( D_3(R) \) is an IFP ring by [13, Proposition 1.2], but \( D_n(R) \) \((n \geq 4)\) is not IFP by [13, Example 1.3]. For the near-IFP case we have a different situation as in the following.

**Proposition 2.10.** \( D_n(S) \) is a near-IFP ring for any ring \( S \) when \( n \geq 2 \).

**Proof.** Let \( R = D_n(S) \) for \( n \geq 2 \) and \( 0 \neq A = (a_{ij}) \in R \) with \( a_{st} \neq 0 \). When the diagonal of \( A \) is nonzero we have the nonzero nilpotent ideal \( RAE_{1n}R = Sa_{11}SE_{1n} \subseteq RAR \) of \( R \). So we assume that the diagonal of \( A \) is zero. Then \( RAR \) itself is a nonzero nilpotent ideal of \( R \). Thus \( R \) is near-IFP. \( \Box \)

A ring \( R \) is called right Ore if it has a classical right quotient ring. It is well-known that semiprime right Goldie rings and right Noetherian domains are both right Ore. But not every domain has a classical right quotient ring (e.g., the free algebra in two indeterminates over a field). We denote the set of all regular elements in a ring \( R \) by \( C(0) \).

**Proposition 2.11.** Let \( R \) be a right Ore ring and \( Q \) be the classical right quotient ring of \( R \).

1. Let \( R \) be an IFP ring. If \( a_1b_1^{-1} \cdots a_nb_n^{-1} = 0 \) for \( a_ib_i^{-1} \in Q \) then \( a_1 \cdots a_n = 0 \).

2. Let \( R \) be an IFP ring. If \( 0 \neq ab^{-1} \in N(Q) \) then \( Qab^{-1}Q \) contains a nonzero nilpotent ideal of \( R \).

**Proof.** (1) Put \( a_1b_1^{-1} \cdots a_ib_i^{-1} = 0 \) for \( a_ib_i^{-1} \in Q \). Since \( R \) is right Ore, we have the following computation: There exist \( c_1 \in R, d_1 \in C(0) \) with \( a_2d_1 = b_1c_1 \); there exist \( c_2 \in R, d_2 \in C(0) \) with \( a_3d_2 = (b_2d_1)c_2 \); inductively there exist \( c_{n-1} \in R, d_{n-1} \in C(0) \) with \( a_nd_{n-1} = (b_{n-1}d_{n-2})c_{n-1} \); consequently we have

\[
0 = a_1b_1^{-1} \cdots a_ib_i^{-1} = a_1c_1 \cdots c_{n-1}d_{n-1}^{-1}b_n^{-1}
\]

and so \( a_1c_1 \cdots c_{n-1} = 0 \). But

\[
0 = a_1c_1 \cdots c_{n-1} = a_1b_1c_1 \cdots c_{n-1} = a_1a_2d_1 \cdots c_{n-1} = a_1a_2(b_2d_1)c_2 \cdots c_{n-1} = a_1a_2a_3d_2 \cdots c_{n-1} = \cdots = a_1a_2 \cdots a_{n-1}(b_{n-1}d_{n-2})c_{n-1} = a_1a_2 \cdots a_{n-1}a_nb_{n-1}.
\]

Since \( d_{n-1} \) is regular, we have \( a_1a_2 \cdots a_{n-1}a_n = 0 \).

(2) Let \( 0 \neq ab^{-1} \in N(Q) \), say \((ab^{-1})^n = 0 \). Then \( a^n = 0 \) by (1). Since \( R \) is IFP, \( RaR \) is nilpotent by Lemma 1.1(4) such that

\[
RaR \subseteq QaQ = Qab^{-1}bQ \subseteq Qab^{-1}Q.
\]
References


KYUNG-YUN HAM
DEPARTMENT OF MATHEMATICS
KOREA SCIENCE ACADEMY
BUSAN 614 103, KOREA

YOUNG CHEOL JEON
DEPARTMENT OF MATHEMATICS
KOREA SCIENCE ACADEMY
BUSAN 614 103, KOREA
E-mail address: jachun@chol.com
Jinwoo Kang  
Department of Mathematics  
Korea Science Academy  
Busan 614-103, Korea

Nam Kyun Kim  
College of Liberal Arts  
Hanbat National University  
Daejeon 305-719, Korea  
E-mail address: nkkim@hanbat.ac.kr

Wonjae Lee  
Department of Mathematics  
Korea Science Academy  
Busan 614-103, Korea

Yang Lee  
Department of Mathematics Education  
Busan National University  
Busan 609-735, Korea  
E-mail address: ylee@pusan.ac.kr

Sung Ju Ryu  
Department of Mathematics  
Busan National University  
Busan 609-735, Korea  
E-mail address: sung1530@dreamwiz.com

Hae-Hun Yang  
Department of Mathematics  
Korea Science Academy  
Busan 614-103, Korea