HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. In this paper we consider the hyponormality of Toeplitz operators $T_\varphi$ on the Bergman space $L^2_a(\mathbb{D})$ in the cases, where $\varphi := f + \overline{g}$ ($f$ and $g$ are polynomials). We present some necessary or sufficient conditions for the hyponormality of $T_\varphi$ under certain assumptions about the coefficients of $\varphi$.

1. Introduction

The purpose of this paper is to study the hyponormality of Toeplitz operators acting on the Bergman space $L^2_a(\mathbb{D})$. Our interest is with Toeplitz operators with trigonometric polynomial symbols.

A bounded linear operator $A$ on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^* A - AA^*$ is positive semidefinite. Let $\mathbb{D}$ denote the open unit disk in the complex plane, $dA$ the area measure on the plane. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

The Bergman space $L^2_a(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of functions analytic on $\mathbb{D}$. Let $L^\infty(\mathbb{D})$ be the space of bounded area measurable function on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator $M_\varphi$ on the Bergman space are defined by $M_\varphi(f) = \varphi \cdot f$, where $f$ is in $L^2_a$. If $P$ denotes the orthogonal projection of $L^2(\mathbb{D})$ onto the Bergman space $L^2_a$, the Toeplitz operator $T_\varphi$ on the Bergman space is defined by

$$T_\varphi(f) = P(\varphi \cdot f),$$

where $\varphi$ is measurable and $f$ is in $L^2_a$. It is clear that those operators are bounded if $\varphi$ is in $L^\infty(\mathbb{D})$. The Hankel operator $H_\varphi : L^2_a \rightarrow L^2_a$ is defined by

$$H_\varphi(f) = (I - P)(\varphi \cdot f).$$

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Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T} = \partial \mathbb{D}$. Recall that given $\psi \in L^\infty(\mathbb{T})$, the Toeplitz operator on the Hardy space is the operator $T_\psi$ on $H^2(\mathbb{T})$ defined by $T_\psi f = P_+(\psi \cdot f)$, where $f$ is in $H^2(\mathbb{T})$ and $P_+$ denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Basic properties of the Bergman space and the Hardy space can be found in [1, 4, 5]. In [2], Cowen characterized the hyponormality of Toeplitz operator $T_\psi$ on $H^2(\mathbb{T})$ by properties of the symbol $\psi \in L^\infty(\mathbb{T})$. Cowen’s theorem states that if $\psi \in L^\infty(\mathbb{T})$, then the Toeplitz operator $T_\psi$ is hyponormal if and only if the following ‘Cowen’ set $\mathcal{E}(\psi)$ is nonempty:

$$\mathcal{E}(\psi) = \{ k \in H^\infty(\mathbb{T}) : ||k||_\infty \leq 1 \text{ and } \psi - k\overline{\psi} \in H^\infty(\mathbb{T}) \}.$$ 

We record here some results on the hyponormality of Toeplitz operators on the Hardy space, which have been recently developed in [3, 6, 8, 9, 10].

**Proposition 1.1.** Suppose that $\psi$ is a trigonometric polynomial of the form $\psi(z) = \sum_{n=-m}^{N} a_n z^n$, where $a_{-m}$ and $a_N$ are nonzero.

(i) If $T_\psi$ is a hyponormal operator, then $m \leq N$ and $|a_{-m}| \leq |a_N|$.

(ii) If $T_\psi$ is a hyponormal operator, then $N - m \leq \text{rank} [T_\psi^*, T_\psi] \leq N$.

(iii) The hyponormality of $T_\psi$ is independent of the particular values of the Fourier coefficients $a_0, a_1, \ldots, a_{N-m}$ of $\psi$. Moreover the rank of the selfcommutator $[T_\psi^*, T_\psi]$ is also independent of those coefficients.

(iv) If $|a_{-m}| = |a_N| \neq 0$, then $T_\psi$ is hyponormal if and only if the following equation holds:

$$\begin{pmatrix}
  a_{-1} \\
a_{-2} \\
  \vdots \\
a_{-m}
\end{pmatrix}
\begin{pmatrix}
  \overline{a_N} \\
  a_{N-m+1} \\
  \vdots \\
  a_{N-m+2}
\end{pmatrix} = 0.$$ 

In this case, the rank of $[T_\psi^*, T_\psi]$ is $N - m$.

(v) $T_\psi$ is normal if and only if $m = N$, $|a_{-m}| = |a_N|$, and (1) holds with $m = N$.

The solution (Cowen’s theorem) of the hyponormality of $T_\psi$ on the Hardy space is based on a dilation theorem of Sarason. It also exploited the fact that functions in $H^2_{-1}$ are conjugates of functions in $zH^2$. For the Bergman space, $L^2_{-1}$ is much larger than the conjugates of functions in $zL^2_a$, and no dilation theorem (similar to Sarason’s theorem) is available. So we cannot get a similar version of Cowen’s theorem for $T_\psi$ on the Bergman space. Therefore, at present, it seems to be quite difficult to determine the hyponormality of $T_\psi$.

We will now consider the hyponormality of Toeplitz operators on the Bergman space with a symbol in the class of functions $g + f$, where $f$ and $g$ are polynomials. Since the hyponormality of operators is translation invariant we may assume that $f(0) = g(0) = 0$. We shall list the well-known properties of Toeplitz operators $T_\phi$ on the Bergman space.
If \( f, g \) are in \( L^\infty(\mathbb{D}) \), then we can easily check that

\[
\begin{align*}
\text{a) } T_{f+g} &= T_f + T_g \\
\text{b) } T_f^* &= T_{\overline{f}} \\
\text{c) } T_f T_g &= T_{\overline{f} g} \text{ if } f \text{ or } g \text{ is analytic.}
\end{align*}
\]

These properties enable us to establish several consequences of hyponormality.

**Proposition 1.2** ([11]). Let \( f, g \) be bounded and analytic. Then the followings are equivalent.

1. \( \overline{T}_{\varphi + f} \) is hyponormal.
2. \( H_\varphi^* H_\varphi \leq H_{\overline{\varphi}}^* H_{\overline{\varphi}} \).
3. \( H_\varphi = CH_{\overline{\varphi}} \), where \( C \) is of norm less than or equal to one.

Very recently, in [7], it was shown that if \( \varphi(z) = a_{-m} \overline{z}^m + a_{-N} \overline{z}^N + a_m z^m + a_N z^N \) (\( 0 < m < N \)) and \( a_m \overline{a}_N = \overline{a}_{-m} a_{-N} \), then

\( T_{\varphi} \) is hyponormal.

\[
\begin{align*}
\frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) &\leq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) \quad \text{if } |a_{-N}| \leq |a_N| \\
N^2(|a_{-N}|^2 - |a_N|^2) &\leq m^2(|a_m|^2 - |a_{-m}|^2) \quad \text{if } |a_N| \leq |a_{-N}|.
\end{align*}
\]

In this paper we continue to examine the hyponormality of \( T_{\varphi} \) in the cases, where \( \varphi \) is a trigonometric polynomial.

### 2. Some necessary conditions for hyponormality of \( T_{\varphi} \)

In this section we present some necessary conditions for hyponormality of \( T_{\varphi} \). First of all, observe that for any \( s, t \) nonnegative integers,

\[
P(\overline{z}^t z^s) = \begin{cases} 
\frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\
0 & \text{if } s < t.
\end{cases}
\]

Let \( \varphi = \overline{g} + f \), where

\[
f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.
\]

For \( m, n = 1, 2, \ldots, N \), define

\[
A_{m,n} := \det \begin{pmatrix} a_m & a_{-m} \\ \overline{a}_n & \overline{a}_{-n} \end{pmatrix}
\]

and we abbreviate \( A_{m,n} \) to \( A_{m,n} \).

The following lemma was shown in [7].
Lemma 2.1 ([7]). Let \( \varphi = \bar{g} + f \), where
\[
f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.
\]
Suppose \( T_\varphi \) is hyponormal. Then

(i) For each \( i = 0, 1, 2, \ldots, N - 1 \),
\[
\sum_{n=1}^{i} \frac{n^2 A_n}{(n+i+1)(n+1)^2} + \sum_{n=i+1}^{N} \frac{A_n}{n+i+1} \geq 0.
\]

(ii) For each \( i \geq N \),
\[
\sum_{n=1}^{N} \frac{n^2 A_n}{(n+i+1)(n+1)^2} \geq 0.
\]

Our main result treats the extremal cases in view of Lemma 2.1:

Theorem 2.2. Let \( \varphi = \bar{g} + f \), where \( f(z) = \sum_{n=1}^{N} a_n z^n \) and \( g(z) = \sum_{n=1}^{N} a_{-n} z^n \). Suppose that \( T_\varphi \) is hyponormal, and that for some \( 0 \leq i_0 \leq N - 1 \),
\[
(3) \quad \sum_{n=0}^{i_0} \frac{n^2 A_n}{(i_0+n+1)(i_0+1)^2} + \sum_{n=i_0+1}^{N} \frac{A_n}{i_0+n+1} = 0.
\]
Then the following conditions hold

(i) \( AB = C \), where
\[
A = [a_{ij}]_{i_0 \times (N-i_0-1)} \quad \text{with} \quad a_{ij} = \begin{cases} 0 & \text{if } i > j \text{ or } j > N-i_0+i-1 \\ A_{i_0+i-j+1} & \text{if } i \leq j, \end{cases}
\]
\[
B = [b_{ij}]_{(N-i_0-1) \times 1} \quad \text{with} \quad b_{j} = \frac{1}{i_0+j+1},
\]
\[
C = [c_{ij}]_{1 \times 1} \quad \text{with} \quad c_{1} = 0 \quad \text{and} \quad c_{j} = -\sum_{n=1}^{j-1} \frac{n(i_0-j+1+n)}{j(i_0+(n+1))} A_{n,i_0-j+1+n}.
\]

(ii) \( AB = D \), where
\[
A = [a_{ij}]_{(N-i_0-1) \times (N-i_0-1)} \quad \text{with} \quad a_{ij} = \begin{cases} 0 & \text{if } i > j \\ A_{i_0+j-i+1,i_0+j+1} & \text{if } i \leq j, \end{cases}
\]
\[
B = [b_{ij}]_{(N-i_0-1) \times 1} \quad \text{with} \quad b_{j} = \frac{1}{2(i_0+1)+j},
\]
\[
D = [d_{ij}]_{(N-i_0-1) \times 1} \quad \text{with} \quad d_{j} := -\sum_{n=1}^{i_0} \frac{n(j+n)}{(i_0+1)(i_0+j+1)(i_0+j+n+1)} A_{n,j+n+1}.
\]

(iii)
\[
\sum_{n=1}^{i_0+j-1} \frac{n(N+j-i_0+n)}{(i_0+1)(N+j+1)(N+j+n+1)} A_{n,N+j-i_0+n} = 0
\]
for each \( 0 \leq j \leq i_0 - 1 \).
Proof. Let $T_\varphi$ be a hyponormal operator and suppose (3) holds for some $0 \leq i_0 \leq N - 1$. Then it follows from Proposition 1.2 that for each non-negative integer $m \neq i_0$ and $c_{i_0}, c_m \in \mathbb{C}$, we have
\[
\left\langle (H_f^* H_f - H_g^* H_g) (c_{i_0} z^{i_0} + c_m z^m), c_{i_0} z^{i_0} + c_m z^m \right\rangle \geq 0,
\]
or equivalently
\[
|c_{i_0}|^2 \left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{i_0} \right\rangle + |c_m|^2 \left\langle (H_f^* H_f - H_g^* H_g) z^m, z^m \right\rangle \\
+ 2 \text{Re} \left( c_{i_0} \overline{c_m} \left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^m \right\rangle \right) \geq 0.
\]
(4)

Observe that for $0 \leq i_0 \leq N - 1$,
\[
\left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{i_0} \right\rangle \\
= \sum_{n=1}^{N} \frac{1}{i_0 + n + 1} (|a_n|^2 - |a_{-n}|^2) - \sum_{n=1}^{i_0} \frac{i_0 - n + 1}{(i_0 + 1)^2} (|a_n|^2 - |a_{-n}|^2) \\
= \sum_{n=1}^{i_0} \frac{n^2 A_n}{(i_0 + n + 1)(i_0 + 1)^2} + \sum_{n=i_0+1}^{N} \frac{A_n}{i_0 + n + 1}.
\]

Hence by the assumption,
\[
\left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{i_0} \right\rangle = 0.
\]
(5)

Since $c_{i_0}$ and $c_m$ are arbitrary, it follows from (4) and (5) that
\[
\left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{m} \right\rangle = 0.
\]
(6)

If $i_0 < m$ ($i_0 + 1 \leq m \leq N + i_0 - 1$), then we have
\[
\left\langle M_f z^{i_0}, M_f z^{m} \right\rangle = \sum_{n=1}^{N+i_0-m} \frac{1}{m + n + 1} a_{m+n-i_0} \overline{a_n}.
\]
(7)

If instead $i_0 < m < N$, then
\[
\left\langle T_f z^{i_0}, T_f z^{m} \right\rangle = \sum_{n=1}^{i_0} \frac{i_0 + 1 - n}{(i_0 + 1)(m + 1)} a_{m+n-i_0} \overline{a_n}.
\]
(8)

Also if $N \leq m \leq N + i_0 - 1$, then
\[
\left\langle T_f z^{i_0}, T_f z^{m} \right\rangle = \sum_{n=1}^{N-m+i_0} \frac{i_0 + 1 - n}{(i_0 + 1)(m + 1)} a_{m+n-i_0} \overline{a_n}.
\]
(9)
Therefore (7), (8) and (9) give that for $i_0 < m$ ($i_0 + 1 \leq m \leq N + i_0 - 1$),

$$
\langle H_f^z H_f^z z^{i_0}, z^m \rangle = \begin{cases}
\sum_{n=1}^{i_0} \frac{n(m - i_0 + n)}{(i_0 + 1)(m + 1)(m + n + 1)} a_{m+n-i_0} \bar{a}_n & \text{if } i_0 < m < N \\
+ \sum_{n=i_0+1}^{N+i_0-m} \frac{1}{m + n + 1} a_{m+n-i_0} \bar{a}_n & \text{if } N \leq m \leq N + i_0 - 1.
\end{cases}
$$

Similarly, we have

$$
\langle H_f^z H_g^z z^{i_0}, z^m \rangle = \begin{cases}
\sum_{n=1}^{i_0} \frac{n(m - i_0 + n)}{(i_0 + 1)(m + 1)(m + n + 1)} a_{-(m+n-i_0)} \bar{a}_n & \text{if } i_0 < m < N \\
+ \sum_{n=i_0+1}^{N+i_0-m} \frac{1}{m + n + 1} a_{-(m+n-i_0)} \bar{a}_n & \text{if } N \leq m \leq N + i_0 - 1.
\end{cases}
$$

Thus by (10) and (11) we have that for $i_0 < m$ ($i_0 + 1 \leq m \leq N + i_0 - 1$)

$$
\langle (H_f^z H_f^z - H_g^z H_g^z) z^{i_0}, z^m \rangle = \begin{cases}
\sum_{n=1}^{i_0} \frac{n(m - i_0 + n)}{(i_0 + 1)(m + 1)(m + n + 1)} \overline{A_{n,m-i_0+n}} & \text{if } i_0 < m < N \\
+ \sum_{n=i_0+1}^{N+i_0-m} \frac{1}{m + n + 1} \overline{A_{n,m-i_0+n}} & \text{if } N \leq m \leq N + i_0 - 1.
\end{cases}
$$

If $0 \leq m < i_0$, then we get

$$
\langle M_f^z z^{i_0}, M_f^z z^m \rangle = \sum_{n=1}^{N+i_0-m} \frac{1}{i_0 + n + 1} a_n a_{i_0-m+n}
$$

and

$$
\langle T_f^z z^{i_0}, T_f^z z^m \rangle = \sum_{n=1}^{m} \frac{m + 1 - n}{(i_0 + 1)(m + 1)} a_n a_{i_0-m+n}.
$$
Thus we have, for $0 \leq m < i_0$,

$$
\langle H_{i_0}^* H_{i_0} z^{i_0}, z^m \rangle = \sum_{n=1}^{m} \frac{n(i_0 - m + n)}{(i_0 + 1)(m + 1)(i_0 + n + 1)} a_n \overline{a_{i_0-m+n}}
+ \sum_{n=m+1}^{N+m-i_0} \frac{1}{i_0 + n + 1} a_n \overline{a_{i_0-m+n}}.
$$

(13)

Similarly, we have that for $0 \leq m < i_0$,

$$
\langle H_{i_0}^* H_{i_0} z^{i_0}, z^m \rangle = \sum_{n=1}^{m} \frac{n(i_0 - m + n)}{(i_0 + 1)(m + 1)(i_0 + n + 1)} a_n \overline{a_{i_0-m+n}}
+ \sum_{n=m+1}^{N+m-i_0} \frac{1}{i_0 + n + 1} a_n \overline{a_{i_0-m+n}}.
$$

(14)

Thus by (13) and (14) we also have, for $0 \leq m < i_0$,

$$
\langle (H_{i_0}^* H_{i_0} - H_{i_0}^* H_{i_0}) z^{i_0}, z^m \rangle = \sum_{n=1}^{m} \frac{n(i_0 - m + n)}{(i_0 + 1)(m + 1)(i_0 + n + 1)} A_{n,i_0-m+n}
+ \sum_{n=m+1}^{N+m-i_0} \frac{1}{i_0 + n + 1} A_{n,i_0-m+n}.
$$

(15)

It follows from (5), (12), and (15) that for $0 \leq i_0 \leq N - 1$,

$$
\begin{align*}
\sum_{n=m+1}^{N+m-i_0} \frac{1}{i_0 + n + 1} A_{n,i_0-m+n} &= -\sum_{n=1}^{m} \frac{n(i_0 - m + n)}{(i_0 + 1)(m + 1)(i_0 + n + 1)} A_{n,i_0-m+n} & \text{if } 0 \leq m < i_0, \\
\sum_{n=i_0+1}^{N+i_0-m} \frac{1}{m + n + 1} A_{n,i_0+n} &= -\sum_{n=1}^{i_0} \frac{n(m - i_0 + n)}{(i_0 + 1)(m + 1)(m + n + 1)} A_{n,i_0+n} & \text{if } i_0 < m < N,
\end{align*}
$$

and

$$
\begin{align*}
\sum_{n=1}^{N+i_0-m} \frac{n(m - i_0 + n)}{(i_0 + 1)(m + 1)(m + n + 1)} A_{n,i_0+n} &= 0 & \text{if } N \leq m \leq N + i_0 - 1.
\end{align*}
$$

This proves (i), (ii), and (iii).  \qed
Theorem 2.3. Let \( \varphi = \varphi + f \), where \( f(z) = \sum_{n=1}^{N} a_n z^n \) and \( g(z) = \sum_{n=1}^{N} a_{-n} z^n \). Suppose that \( T_\varphi \) is hyponormal, and that for some \( i_0 \geq N \),

\[
(16) \quad \sum_{n=1}^{N} \frac{n^2 A_n}{(i_0 + n + 1)(i_0 + 1)^2} = 0.
\]

Then we have

\[
(17) \quad \sum_{n=1}^{N-j} \frac{n(n+j)}{(i_0 + j + 1)(i_0 + j + n + 1)} A_{n,n+j} = 0 \quad \text{for} \quad 1 \leq j \leq N - 1;
\]

\[
(18) \quad \sum_{n=1}^{N-j} \frac{n(n+j)}{(i_0 - j + 1)(i_0 + n + 1)} A_{n,n+j} = 0 \quad \text{for} \quad 1 \leq j \leq N - 1.
\]

Proof. Let \( T_\varphi \) be a hyponormal operator and suppose (16) holds for some \( i_0 \geq N \). Then by assumption we have that for \( i_0 \geq N \),

\[
\left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{i_0} \right\rangle = \sum_{n=1}^{N} \frac{n^2 A_n}{(i_0 + n + 1)(i_0 + 1)^2} = 0.
\]

Thus it follows from (4) that for each non-negative integer \( m \neq i_0 \), we have

\[
(19) \quad \left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{m} \right\rangle = 0.
\]

If \( i_0 < m \leq N + i_0 - 1 \), then

\[
\left\langle M_f z^{i_0}, M_f z^{m} \right\rangle = \sum_{n=1}^{N-m+i_0} \frac{1}{m+n+1} \overline{a_n} a_{m-i_0+n}
\]

and

\[
\left\langle T_f z^{i_0}, T_f z^{m} \right\rangle = \sum_{n=1}^{N-m+i_0} \frac{i_0 + 1 - n}{(i_0 + 1)(m + 1)} \overline{a_n} a_{m-i_0+n}.
\]

Thus for \( i_0 < m \leq N + i_0 - 1 \) \( i_0 \geq N \), we get

\[
(20) \quad \left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{m} \right\rangle = \sum_{n=1}^{N+i_0-m} \frac{n(n+m-i_0)}{(i_0 + 1)(m+1)(m+n+1)} \overline{A_{n,m-i_0+n}}.
\]

Similarly, for \( i_0 - N + 1 \leq m < i_0 \) \( i_0 \geq N \) we have

\[
(21) \quad \left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{m} \right\rangle = \sum_{n=1}^{N+m-i_0} \frac{n(n+m-i_0)}{(i_0 + 1)(m+1)(i_0 + n + 1)} \overline{A_{n,i_0-m+n}}.
\]

By (19), (20), and (21), we see that for \( i_0 \geq N \),

\[
(22) \quad \sum_{n=1}^{N+i_0-m} \frac{n(n+m-i_0)}{(i_0 + 1)(m+1)(m+n+1)} \overline{A_{n,m-i_0+n}} = 0 \quad \text{if} \quad i_0 < m \leq N + i_0 - 1;
\]
Putting \( j = m - i_0 \) and \( j = i_0 - m \), respectively, in (22) and (23) gives the result.

From Theorems 2.2 and 2.3 we get the following corollaries.

**Corollary 2.4.** Let \( \varphi = \bar{g} + f \), where \( f(z) = \sum_{n=1}^{N} a_n z^n \) and \( g(z) = \sum_{n=1}^{N} a_{-n} z^n \). If \( T_\varphi \) is hyponormal and (3) holds for some \( 0 \leq i_0 \leq N - 1 \), then

\[
\sum_{n=1}^{N-i_0} \frac{1}{n+i_0+1} A_{n,n+i_0} = 0.
\]

**Corollary 2.5 ([7]).** Let \( \varphi = \bar{g} + f \), where \( f(z) = \sum_{n=1}^{N} a_n z^n \) and \( g(z) = \sum_{n=1}^{N} a_{-n} z^n \). If \( T_\varphi \) is hyponormal and \( ||f|| = ||g|| \), then we have

\[
\begin{pmatrix}
A_{1,1} & A_{2,2} & \cdots & \cdots & A_{N,N} \\
0 & A_{1,2} & A_{2,3} & \cdots & A_{N-1,N} \\
0 & 0 & A_{1,3} & \cdots & A_{N-2,N} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & A_{1,N}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{1} \\
\frac{1}{2} \\
\frac{1}{3} \\
\vdots \\
\frac{1}{N+1}
\end{pmatrix}
= 0.
\]

**Proof.** We have the result by putting \( i_0 = 0 \) in Theorem 2.2 (ii). \( \square \)

**Corollary 2.6.** Let \( \varphi = \bar{g} + f \), where \( f(z) = \sum_{n=1}^{N} a_n z^n \) and \( g(z) = \sum_{n=1}^{N} a_{-n} z^n \) \((N \geq 3)\). If \( T_\varphi \) is hyponormal and (16) holds for some \( i_0 \geq N \), then we have

\[
A_{1,N} = A_{1,N-1} = A_{2,N} = 0.
\]

**Proof.** Putting \( j = N - 1 \) in (17) gives \( A_{1,N} = 0 \) and putting \( j = N - 2 \) in (17) and (18) gives that

\[
\begin{pmatrix}
N-1 \\
N-1 \\
N-1 \\
N-1 \\
N-1
\end{pmatrix}
\begin{pmatrix}
\frac{2N}{(i_0+N-1)(i_0+1)} \\
\frac{2N}{(i_0+N-1)(i_0+1)} \\
\frac{2N}{(i_0+N-1)(i_0+1)} \\
\frac{2N}{(i_0+N-1)(i_0+1)} \\
\frac{2N}{i_0+3)(i_0+2)}
\end{pmatrix}
\begin{pmatrix}
A_{1,N-1} \\
A_{2,N}
\end{pmatrix}
= \begin{pmatrix} 0 \\
0 \end{pmatrix}.
\]

Observe that

\[
\det \begin{pmatrix}
N-1 \\
N-1 \\
N-1 \\
N-1 \\
N-1
\end{pmatrix}
\begin{pmatrix}
\frac{2N}{(i_0+N-1)(i_0+1)} \\
\frac{2N}{(i_0+N-1)(i_0+1)} \\
\frac{2N}{(i_0+N-1)(i_0+1)} \\
\frac{2N}{i_0+3)(i_0+2)}
\end{pmatrix}
= 0 \quad \text{if and only if} \quad N = 2.
\]

Thus we have that \( A_{1,N-1} = A_{2,N} = 0 \). \( \square \)

**Corollary 2.7.** Let \( \varphi = \bar{g} + f \), where \( f(z) = \sum_{n=1}^{3} a_n z^n \) and \( g(z) = \sum_{n=1}^{3} a_{-n} z^n \). If (3) or (16) holds for some \( i_0 \geq 1 \), then \( T_\varphi \) is hyponormal if and only if \( \varphi(z) \) satisfies one of the following two conditions:
(i) \( f(z) = \alpha g(z) \) for some \( |\alpha| = 1 \) (in this case \( T_\varphi \) is normal);
(ii) \( f(z) = a_m z^m + a_N z^N, \ g(z) = a_{-m} z^m + a_{-N} z^N, \ A_{m,N} = 0 \) (1 \( \leq m \leq N \leq 3 \)) and (2) holds.

**Proof.** Suppose \( T_\varphi \) is hyponormal. We will show that \( A_{1,2} = A_{2,3} = A_{1,3} = 0 \). If \( i_0 \geq 3 \), this follows from Corollary 2.6. If \( i_0 = 1, \) then putting \( j = 0 \) in Theorem 2.2 (iii) gives \( A_{1,3} = 0 \) and by Theorem 2.2 (i) and (ii) we have

\[
\begin{pmatrix}
\frac{1}{3} \\
\frac{1}{12}
\end{pmatrix}
\begin{pmatrix}
A_{1,2} \\
A_{2,3}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

Therefore \( A_{1,2} = A_{2,3} = A_{1,3} = 0 \). If \( i_0 = 2 \) then by Theorem 2.2 (i) we get \( A_{1,3} = 0 \) and \( \frac{1}{12} A_{1,2} + \frac{1}{5} A_{2,3} = 0 \). Putting \( j = 0 \) in Theorem 2.2 (iii) we see that \( 2 A_{1,2} + 5 A_{2,3} = 0 \) and therefore \( A_{1,2} = A_{2,3} = 0 \). Thus by (2), (i) or (ii) holds. The converse follows from Proposition 1.2 and (2). This completes the proof. \( \Box \)

**Example 2.8.** Consider the polynomial

\[ \varphi(z) = 4\overline{z}^3 + 2z^2 + \bar{z} + z + 2z^2 + \beta z^3 \quad (|\beta| = 4). \]

Then \( \varphi(z) \) satisfies the equality (3). Thus by Corollary 2.7, \( T_\varphi \) is hyponormal if and only if \( \beta = 4 \).

**Example 2.9.** Consider the polynomial

\[ \varphi(z) = 8\overline{z}^3 + \overline{z}^2 + \beta \overline{z} + \gamma z + 7z^2 + 2z^3 \quad (|\beta| = |\gamma|). \]

Then \( \varphi(z) \) satisfies the equality (3). Thus Corollary 2.7 shows that \( T_\varphi \) is not hyponormal.

### 3. Some sufficient conditions for hyponormality of \( T_\varphi \)

If \( f(z) = \sum_{n=2}^{N} a_n z^n \) (\( N \geq 2 \)) and \( h(z) = az + f(z) \), then the Toeplitz operator \( T_{f+h} \) on the Hardy space is hyponormal if and only if \( a = 0 \) (Proposition 1.1(iv)). On the contrary, the following theorem shows that the Toeplitz operator \( T_{f+h} \) on the Bergman space is hyponormal if \( |a| \) is sufficiently large.

**Theorem 3.1.** If \( f(z) = \sum_{n=2}^{N} a_n z^n \) (\( N \geq 2 \)), \( h(z) = az + f(z) \), and \( A := \max\{|a_i| : 2 \leq i \leq N\} \), then \( T_{f+h} \) is hyponormal when \( |a| \geq 2N^2 A \).

**Proof.** Let \( K_i := \{k_i(z) \in L^2_\alpha : k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \} \) for \( i = 0, 1, 2, \ldots, N-1 \). Then by Proposition 1.2, we have that \( T_{f+h} \) is hyponormal if and only if \( \langle (H_{f+h} - H_{f+h}) \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \rangle \geq 0 \) for all \( k_i \in K_i \) (\( i = 0, 1, 2, \ldots, N-1 \)).
0, 1, 2, \ldots, N - 1), or equivalently

\[
\sum_{i=0}^{N-1} \left( 2\text{Re}\left( H_{\bar{z}} k_i(z), \bar{a}H_{\bar{z}} k_i(z) \right) + |a|^2 \left\langle H_{\bar{z}} k_i(z), H_{\bar{z}} k_i(z) \right\rangle \right) \\
+ \sum_{i \neq j, \ i, j \geq 0} \left( 2\text{Re}\left( H_{\bar{z}} k_i(z), \bar{a}H_{\bar{z}} k_j(z) \right) + |a|^2 \left\langle H_{\bar{z}} k_i(z), H_{\bar{z}} k_j(z) \right\rangle \right) \geq 0.
\]

(24)

But we have

\[
\left\langle H_{\bar{z}} k_i(z), \bar{a}H_{\bar{z}} k_i(z) \right\rangle = 0
\]

and for \( i \neq j \) (\( i, j = 0, 1, 2, \ldots, N - 1 \)),

\[
\left\langle H_{\bar{z}} k_i(z), H_{\bar{z}} k_j(z) \right\rangle = 0.
\]

(26)

Putting (25) and (26) in (24) we have that \( T_{\bar{z}+n} \) is hyponormal if and only if

\[
\sum_{i=0}^{N-1} |a|^2 \left\langle H_{\bar{z}} k_i(z), H_{\bar{z}} k_i(z) \right\rangle + \sum_{i \neq j, \ i, j \geq 0} 2\text{Re}\left( a\left\langle H_{\bar{z}} k_i(z), H_{\bar{z}} k_j(z) \right\rangle \right) \geq 0.
\]

(27)

Observe that

\[
\sum_{i=0}^{N-1} \left\langle H_{\bar{z}} k_i(z), H_{\bar{z}} k_i(z) \right\rangle = \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)^2} |c_n|^2
\]

and

\[
\sum_{i \neq j, \ i, j \geq 0} \left\langle H_{\bar{z}} k_i(z), H_{\bar{z}} k_j(z) \right\rangle = \sum_{m=2}^{N} \bar{a}_m \sum_{i \neq j, \ i, j \geq 0} \left\langle H_{\bar{z}}^{m} k_i(z), H_{\bar{z}} k_j(z) \right\rangle.
\]

(29)

For \( m = 2, 3, \ldots, N \), we have

\[
\sum_{i \neq j, \ i, j \geq 0} \left\langle M_{\bar{z}}^{m} k_i(z), M_{\bar{z}} k_j(z) \right\rangle = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{Nn+j+m+1} \frac{c_{Nn+j+m-1}}{c_{Nn+j}}
\]

and

\[
\sum_{i \neq j, \ i, j \geq 0} \left\langle T_{\bar{z}}^{m} k_i(z), T_{\bar{z}} k_j(z) \right\rangle
\]

\[
= \sum_{j=0}^{N-1} \sum_{n=0}^{\infty} \frac{Nn+j}{(Nn+j+m)(Nn+j+1)} c_{Nn+j+m-1} c_{Nn+j}.
\]

(31)

Combining (30) and (31) gives that

\[
\sum_{i \neq j, \ i, j \geq 0} \left\langle H_{\bar{z}}^{m} k_i(z), H_{\bar{z}} k_j(z) \right\rangle = \sum_{n=0}^{\infty} \frac{m}{(n+m+1)(n+m)(n+1)} \frac{c_{n} c_{n+m-1}}{c_{n} c_{n+m-1}}.
\]

(32)
Putting (32) in (29) and putting (28) and (29) in (27) we see that $T_{f+h}$ is hyponormal if and only if

$$
|a|^2 \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)^2} |c_n|^2 \\
+ 2 \text{Re} \left( a \sum_{m=2}^{N} \frac{1}{a_m} \sum_{n=0}^{\infty} \frac{m}{(n+m+1)(n+m)(n+1)} c_n c_{n+m-1} \right) \geq 0.
$$

(33)

Note that the inequality (33) holds if the following inequality holds for each $m = 2, 3, \ldots, N$,

$$
\sum_{n=0}^{\infty} \frac{|c_n|^2}{(n+2)(n+1)^2} \geq \frac{2(N-1)|a_m|}{|a|} \sum_{n=0}^{\infty} \frac{m}{(n+m+1)(n+m)(n+1)} |c_n||c_{n+m-1}|.
$$

(34)

So it follows from (34) that $T_{f+h}$ is hyponormal if for all $n \geq 0$, $m = 2, 3, \ldots, N$,

$$
\frac{\alpha_m}{(n+m+1)(n+m)(n+1)} |c_n||c_{n+m-1}| \leq \frac{1}{(n+2)(n+1)^2} |c_n|^2 + \frac{1}{(n+m+1)(n+m)(n+1)^2} |c_{n+m-1}|^2,
$$

(35)

where $\alpha_m = \frac{4(N-1)|a_m|}{|a|}$. Observe that the inequality (35) holds if

$$
\alpha_m^2 \leq \frac{4(n+m+1)}{n+2}.
$$

Let $A := \max\{|a_i| : i = 2, 3, \ldots, N\}$. Then $T_{f+h}$ is hyponormal when $|a| \geq 2N^2 A$. This completes the proof. \qed

**Corollary 3.2.** Let $f(z) = \sum_{n=2}^{N} a_n z^n$ ($N \geq 2$), $g \in H^\infty$ and $T_{\overline{g}+f}$ be a hyponormal operator. If $h(z) = az + f(z)$ and $|a| \geq 2(N-1)A$, where $A := \max\{|a_i| : 2 \leq i \leq N\}$, then $T_{\overline{g}+h}$ is hyponormal.

**Proof.** This follows from Proposition 1.2 and Theorem 3.1. \qed

Let $f(z) = \sum_{n=1}^{N-1} (N \geq 2)$ and $h(z) = f(z) + az^N$. Then the Toeplitz operator $T_{f+h}$ on the Hardy space is hyponormal if $|a|$ is sufficiently large ([6]). The following theorem shows that the Toeplitz operator $T_{f+h}$ on the Bergman space has the same property.

**Theorem 3.3.** Let $f(z) = \sum_{n=1}^{N-1} a_n z^n$ ($N \geq 2$), $h(z) = f(z) + az^N$ and $A := \max\{|a_i| : 1 \leq i \leq N-1\}$. If $|a| \geq 2\sqrt{2(N-1)A}$, then $T_{f+h}$ is hyponormal.
Proof. Let \( K_i := \{ k_i(z) \in L_a^2 : k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \} \) for \( i = 0, 1, 2, \ldots, N - 1 \). Then Proposition 1.2 gives that \( T_{f+h}^* \) is hyponormal if and only if
\[
\langle (H_h^* H_h^* - H_f^* H_f^*) \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \rangle \geq 0 \text{ for all } k_i \in K_i \ (i = 0, 1, 2, \ldots, N - 1),
\]
or equivalently
\[
\begin{align*}
&\sum_{i=0}^{N-1} |a|^2 \langle H_{\frac{1}{z}} k_i(z), H_{\frac{1}{z}} k_i(z) \rangle \\
&+ \sum_{i\neq j, i,j \geq 0} N \text{Re} \left( a \sum_{m=1}^{N-1} \overline{a}_m \langle H_{\frac{1}{z}}^m k_i(z), H_{\frac{1}{z}}^m k_j(z) \rangle \right) \geq 0.
\end{align*}
\]
On the other hand, we have
\[
\sum_{i=0}^{N-1} \langle H_{\frac{1}{z}}^N k_i(z), H_{\frac{1}{z}}^N k_i(z) \rangle
\]
\[
= \sum_{n=0}^{N-1} \frac{1}{n + N + 1} |c_n|^2 + \sum_{n=N}^{\infty} \frac{N^2}{(n + N + 1)(n + 1)^2} |c_n|^2,
\]
and for each \( m = 1, 2, \ldots, N - 1 \),
\[
\sum_{i\neq j, i,j \geq 0} \langle M_{\frac{1}{z}}^m k_i(z), M_{\frac{1}{z}}^m k_j(z) \rangle = \sum_{i=0}^{N-1} \sum_{n=0}^{\infty} \frac{1}{N(n+1) + i + 1} c_{Nn+i} \overline{c}_{N(n-1)-m+i}
\]
and
\[
\sum_{i\neq j, i,j \geq 0} \langle T_{\frac{1}{z}}^m k_i(z), T_{\frac{1}{z}}^m k_j(z) \rangle
\]
\[
= \sum_{m=1}^{N-1} \sum_{n=1}^{\infty} \frac{Nn - m + i + 1}{(Nn + i + 1)(N(n + 1) - m + i + 1)} c_{Nn+i} \overline{c}_{N(n-1)-m+i}
\]
\[
+ \sum_{i=m}^{N-1} \sum_{n=0}^{\infty} \frac{Nn - m + i + 1}{(Nn + i + 1)(N(n + 1) - m + i + 1)} c_{Nn+i} \overline{c}_{N(n-1)-m+i}.
\]
Combining (38) and (39) we see that
\[
\sum_{i\neq j, i,j \geq 0} \langle H_{\frac{1}{z}}^m k_i(z), H_{\frac{1}{z}}^m k_j(z) \rangle
\]
\[
= \sum_{n=0}^{m-1} \frac{1}{n + N + 1} c_n \overline{c}_{n+N-m}
\]
\[
+ \sum_{n=m}^{\infty} \frac{MN}{(n+1)(n+N-m+1)(n+N+1)} c_n \overline{c}_{n+N-m}.
\]
Putting (37) and (40) in (36) we have that $T_{f+h}$ is hyponormal if and only if

$$
|a|^2 \left( \sum_{n=0}^{N-1} \frac{1}{n + N + 1} |c_n|^2 + \sum_{n=N}^{\infty} \frac{N^2}{(n + N + 1)(n + 1)^2} |c_n|^2 \right) \\
+ 2 \text{Re} \left\{ a \sum_{m=1}^{N-1} a_m \left( \sum_{n=0}^{m-1} \frac{1}{n + N + 1} c_n \overline{c}_{n+N-m} \right) \right. \\
\left. + \sum_{n=m}^{\infty} \frac{mN}{(n + 1)(n + N - m + 1)(n + N + 1)} c_n \overline{c}_{n+N-m} \right\} \geq 0.
$$

(41)

The inequality (41) holds if for each $m = 1, 2, 3, \ldots, N - 1$,

$$
\sum_{n=0}^{N-1} \frac{1}{n + N + 1} |c_n|^2 + \sum_{n=N}^{\infty} \frac{N^2}{(n + N + 1)(n + 1)^2} |c_n|^2 \\
\geq \alpha_m \left( \sum_{n=0}^{m-1} \frac{1}{n + N + 1} |c_n||c_{n+N-m}| \right) \\
+ \sum_{n=m}^{\infty} \frac{mN}{(n + 1)(n + N - m + 1)(n + N + 1)} |c_n||c_{n+N-m}|,
$$

(42)

where $\alpha_m = \frac{2(N-1)|a_m|}{|a|}$. Note that (42) holds if for $m = 1, 2, \ldots, N - 1$,

$$
|a|^2 \geq \frac{4(N-1)^2|a_m|^2(n + 2N - m + 1)}{N + n + 1} ||c_{n+2N-m+1}| |c_{n+2N-m+1}| \quad \text{if } n = 0, 1, 2, \ldots, m - 1,
$$

(43)

$$
|a|^2 \geq \frac{4(N-1)^2|a_m|^2m^2(n + 2N - m + 1)}{(n + 1)^2(n + N + 1)} ||c_{n+2N-m+1}| |c_{n+2N-m+1}| \quad \text{if } n = m, m + 1, \ldots, N - 1,
$$

$$
|a|^2 \geq \frac{4(N-1)^2|a_m|^2m^2(n + 2N - m + 1)}{(n + N + 1)N^2} ||c_{n+2N-m+1}| |c_{n+2N-m+1}| \quad \text{if } n \geq N.
$$

Observe that (43) holds if $|a| \geq 2\sqrt{2}(N-1)|a_m|$ for all $m = 1, 2, \ldots, N - 1$. This completes the proof.

□

**Corollary 3.4.** Let $f(z) = \sum_{n=1}^{N-1} a_n z^n$ ($n \geq 2$), $g \in H^\infty$ and $T_{g+f}$ be a hyponormal operator. If $|a| \geq 2\sqrt{2}(N-1)|A|$, where $A := \max\{|a_i| : 1 \leq i \leq N - 1\}$ and $h(z) = f(z) + az^N$, then $T_{g+h}$ is hyponormal.

**Proof.** This follows from Proposition 1.2 and Theorem 3.3. □

**Example 3.5.** Consider the polynomial

$$
\varphi(z) = 2z^2 + 2\overline{z} + 4z + z^2.
$$

Then (2) shows that $T_{\varphi}$ is hyponormal. Put $\psi(z) = 2z^2 + 2\overline{z} + 4z + z^2 + 32z^3$. Then Corollary 3.4 shows that $T_{\psi}$ is hyponormal.

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