HYPERCYCLICITY ON INVARIANT SUBSPACES

HENRIK PETERSSON

ABSTRACT. A continuous linear operator $T : \mathcal{X} \to \mathcal{X}$ is called hypercyclic if there exists an $x \in \mathcal{X}$ such that the orbit $\{T^n x\}_{n \geq 0}$ is dense. We consider the problem: given an operator $T : \mathcal{X} \to \mathcal{X}$, hypercyclic or not, is the restriction $T|_{\mathcal{Y}}$ to some closed invariant subspace $\mathcal{Y} \subseteq \mathcal{X}$ hypercyclic? In particular, it is well-known that any non-constant partial differential operator $p(D)$ on $H(\mathbb{C}^d)$ (entire functions) is hypercyclic. Now, if $q(D)$ is another such operator, $p(D)$ maps $\ker q(D)$ invariantly (by commutativity), and we obtain a necessary and sufficient condition on $p$ and $q$ in order that the restriction $p(D) : \ker q(D) \to \ker q(D)$ is hypercyclic. We also study hypercyclicity for other types of operators on subspaces of $H(\mathbb{C}^d)$.

1. Introduction

In all that follows, $\mathcal{X}$ denotes a real or complex separated locally convex space, and $L(\mathcal{X})$ the algebra of continuous linear operators $T : \mathcal{X} \to \mathcal{X}$ on $\mathcal{X}$. An operator $T \in L(\mathcal{X})$ is said to be hypercyclic if for some vector $x \in \mathcal{X}$, called hypercyclic for $T$, the orbit $\{T^n x\}_{n \geq 0}$ is dense. Thus, the existence of a hypercyclic operator in $L(\mathcal{X})$ requires that $\mathcal{X}$ is separable and, see [15], infinite dimensional (unless $\mathcal{X} = \{0\}$). For Fréchet spaces we have the following well-known Hypercyclicity Criterion to establish hypercyclicity, see [1]:

**Proposition 1.** Let $\mathcal{X}$ be a separable Fréchet space and assume $T \in L(\mathcal{X})$ satisfies the Hypercyclicity Criterion (HC): There exist dense subspaces $\mathcal{Z}, \mathcal{Y} \subseteq \mathcal{X}$, and sequences $(S_k)$ and $(n_k)$ of linear maps $S_k : \mathcal{Y} \to \mathcal{X}$ and of natural numbers $n_k$, such that:

1. $T^{n_k} z \to 0$ for all $z \in \mathcal{Z},$
2. $S_{k} y \to 0$ for all $y \in \mathcal{Y},$
3. $T^{n_k} S_k y \to y$ for all $y \in \mathcal{Y}.$

Then $T$ is hypercyclic.

We say that $T$ satisfies the HC with respect to a given sequence $(n_k) \subseteq \mathbb{N}$, if this sequence can be used in the criterion (i.e., in (1) and (3)). It is convenient

Received April 19, 2006.

2000 Mathematics Subject Classification. 47A15, 47A16, 47B38, 32A70, 35A35.

Key words and phrases. hypercyclic, restriction, extension, invariant subspace.

Supported by the The Royal Swedish Academy of Sciences.
to note that if $T$ satisfies the HC with respect to $(n_k)$, then $T$ satisfies the HC for any subsequence of $(n_k)$.

In this work we highlight the following two problems:

**Problem 1.** Given a hypercyclic operator $T \in L(\mathcal{X})$, can we find a closed invariant subspace $\mathcal{Y} \subset \mathcal{X}$ for which the restriction $T|_{\mathcal{Y}}$ of $T$ to $\mathcal{Y}$ is also hypercyclic?

Another problem is to go in the other direction:

**Problem 2.** Given a hypercyclic operator $T : \mathcal{Y} \to \mathcal{Y}$, where $\mathcal{Y}$ is a proper closed subspace of $\mathcal{X}$, can we extend $T$ to a hypercyclic operator $\hat{T} \in L(\mathcal{X})$?

Note that, in the first problem, every hypercyclic vector $x$ for $T : \mathcal{X} \to \mathcal{X}$ must necessarily be outside of $\mathcal{Y}$ (since $\mathcal{Y}$ is closed). Thus the hypercyclic vectors for $T$ and $T|_{\mathcal{Y}}$ (and thus those for $T : \mathcal{Y} \to \mathcal{Y}$ and $\hat{T}$ in Problem 2) are distinct. In other words, $T|_{\mathcal{Y}}$ does not by no means inherit the hypercyclic property of $T$. Let us recall that every infinite dimensional separable Fréchet space $\mathcal{X}$ supports a hypercyclic operator [2], and hence so does any infinite dimensional closed subspace $\mathcal{Y} \subset \mathcal{X}$.

The following simple observation was one of the factors that motivated us to pose Problems 1 and 2. Recall that the differentiation operators $D_i \equiv \partial / \partial z_i$ are hypercyclic on the Fréchet space $H(\mathbb{C}^d)$ of $d$-variable entire functions provided with the compact-open topology (in fact, they satisfy the HC with respect to the full sequence $(n_k = k)$). (See Proposition 3 for a more general result.) Consider now the space $H(\mathbb{C}^2)$ and the operators $D_1$ and $D_2$. We know that $D_1$ is hypercyclic on $H(\mathbb{C}^2)$, and it is clear that $\mathcal{Y} \equiv \ker D_2 = H(\mathbb{C}_1) \equiv \{ f(z_1) : f \in H(\mathbb{C}) \}$ and $D_1$ maps $\mathcal{Y}$ invariantly. It follows now that $D_1$ is hypercyclic on $\mathcal{Y}$, since $D$ is hypercyclic on $H(\mathbb{C})$. Even more can be said. $\mathcal{Y}$ is complemented in $H(\mathbb{C}^2)$. Indeed, $H(\mathbb{C}^2) = \mathcal{Z} \oplus \mathcal{Y}$ is a topological decomposition where $\mathcal{Z} \equiv z_2 H(\mathbb{C}^2)$. (Recall that a subspace $\mathcal{Y} \subset \mathcal{X}$ is said to be complemented in $\mathcal{X}$ if there exists a subspace $\mathcal{Z} \subset \mathcal{X}$ such that $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$, and where the corresponding projecdc $\mathcal{X} \to \mathcal{Y}$ (or equivalently $\mathcal{X} \to \mathcal{Z}$) is continuous. In this case we say that $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$ is a topological decomposition of $\mathcal{X}$.) We notice that $D_1$ maps $\mathcal{Z}$ invariantly, and if $f = f(z_1, z_2)$ is a hypercyclic vector for $D_1$ acting on $H(\mathbb{C}^2)$, then $z_2 f \in \mathcal{Z}$ is evidently hypercyclic for $D_1 : \mathcal{Z} \to \mathcal{Z}$. Thus, we can decompose $H(\mathbb{C}^2)$ into a sum of two closed invariant subspaces $\mathcal{Z}, \mathcal{Y}$ for $D_1$, and for which the corresponding restrictions of $D_1$ are hypercyclic. At this point the following question arises; if $z \in \mathcal{Z}$ is hypercyclic for $D_1 : \mathcal{Z} \to \mathcal{Z}$ and $y \in \mathcal{Y}$ for $D_1 : \mathcal{Y} \to \mathcal{Y}$, does $h \equiv z + y$ form a hypercyclic vector for $D_1$ acting on the full space $H(\mathbb{C}^2)$? What we know is that for any given $f \in H(\mathbb{C}^2)$ and neighborhood $U_f$ of $f$, there exist $n, m \in \mathbb{N}$ such that $D_1^n z + D_2^m y \in U_f$. Our question remains thus open, since we do not know if, for a general choice of $z$ and $y$, we always can choose $n = m$. However, we shall see (by Proposition 7) that there exist vectors $z, y$ for which this is always possible:
Proposition 2. \( \mathcal{Y} \equiv H(\mathbb{C}_1) \) and \( Z \equiv \pi_2 H(\mathbb{C}_2) \) are closed invariant subspaces for \( D_1 : H(\mathbb{C}_2) \to H(\mathbb{C}_2) \) with \( H(\mathbb{C}_2) = Z \oplus \mathcal{Y} \) (topological decomposition), and the restrictions \( D_1 : \mathcal{Y} \to \mathcal{Y} \) and \( D_1 : Z \to Z \) are hypercyclic. Further, for any hypercyclic vector \( y \in \mathcal{Y} \) for \( D_1 : \mathcal{Y} \to \mathcal{Y} \), there exists a hypercyclic vector \( z \in Z \) for \( D_1 : Z \to Z \) such that \( z + y \) forms a hypercyclic vector for \( D_1 : H(\mathbb{C}_2) \to H(\mathbb{C}_2) \), and the analogue holds if we start with a hypercyclic vector \( z \in Z \) for \( D_1 : Z \to Z \).

The idea of this paper is to consider Problems 1 and 2 in the complex analysis setting. In particular we approximate kernels to PDE:s in the following way. If \( p(D) \) and \( q(D) \) are partial differential operators with constant coefficients acting on \( H(\mathbb{C}_d) \), then \( p(D) \) and \( q(D) \) commute. Consequently, \( p(D) \) maps ker \( q(D) \) invariantly, and in Section 2 we establish a necessary and sufficient condition on the polynomials \( p \) and \( q \) in order that \( p(D) : \ker q(D) \to \ker q(D) \) is hypercyclic (Theorem 1). For any such pair \( p, q \), there exists thus a solution \( f \in H(\mathbb{C}_d) \) to \( q(D)f = 0 \) such that any other homogeneous solution lies arbitrarily close to some \( p(D)^nf = p^n(D)f \). In Section 3, we construct certain closed subspaces of \( H(\mathbb{C}^{2d}) \) and prove that for a suitable choice \( E \subset H(\mathbb{C}^{2d}) \), we can to every operator \( T \) on \( H(\mathbb{C}_d) \) that satisfies the HC, associate a hypercyclic operator \( T_E \) on \( E \) (Theorem 2). Further, when \( d = 1, T_E \) extends to a hypercyclic operator \( \tilde{T}_E \) on the full space \( H(\mathbb{C}_2) \). In the last section, Section 4, we discuss extensions of our obtained results, especially extensions to other spaces than \( H(\mathbb{C}_d) \).

2. On Problem 1

Recall that a convolution operator \( T \) on \( H(\mathbb{C}_d) \) is a continuous linear operator that commutes with all translations \( \tau_a : f \mapsto f(z + a), a \in \mathbb{C}_d \), and \( T \) is called trivial if it is a scalar multiple of the identity. The following well-known result of Godefroy and Shapiro [5, Section 5] is some sort of basis for our investigation in this section. Recall first that the algebra \( \text{Exp}(\mathbb{C}_d) \) of exponential type functions is formed by all \( \phi \in H(\mathbb{C}_d) \) such that, for some \( M, r > 0 \), \( |\phi(z)| \leq M e^{r||z||} \) where \( ||\cdot|| \) denotes the Euclidian norm on \( \mathbb{C}_d \). In view of our purposes, it is also convenient to recall that \( \text{Exp}(\mathbb{C}_d) \) can be identified with \( H'(\mathbb{C}_d) \) via the bilinear form \( (f, \phi) \equiv \sum_{\alpha \in \mathbb{N}^d} f^{(\alpha)}(0)\phi^{(\alpha)}(0)/\alpha! \) (\( \alpha! \equiv \prod \alpha_i! \)) on \( H \times \text{Exp} \).

Proposition 3 (Godefroy, Shapiro [5]). The map \( \phi \mapsto \phi(D) \equiv \sum_{\alpha \in \mathbb{N}^d} \phi_D \alpha D^\alpha \), where \( \phi(z) = \sum_{\alpha \in \mathbb{N}^d} \phi_D \alpha z^\alpha \) and \( D^\alpha \equiv D_1^{\alpha_1} \cdots D_d^{\alpha_d} \), defines an algebra isomorphism between \( \text{Exp}(\mathbb{C}_d) \) and the set \( C \) of convolution operators on \( H(\mathbb{C}_d) \). Any nontrivial convolution operator \( \phi(D) \), i.e., \( \phi \) is not a constant mapping, satisfies the HC (with respect to the full sequence) and is thus hypercyclic on \( H(\mathbb{C}_d) \).
Note that $\tau_a = e_a(D)$, where $e_a(z) \equiv \exp \sum z_i a_i$, and $\varphi(D)f(z) = \langle f, \varphi e_z \rangle$. In particular Proposition 3 yields that the operators in $C$ commute and, accordingly, any $\varphi(D) \in C$ maps any kernel $\ker \psi(D)$, $\psi \in \text{Exp}(C^d)$, invariantly. This suggests, in view of Problem 1, the following definition:

**Definition 1.** Let $\varphi, \psi \in \text{Exp}(C^d)$. We say that $\varphi(D)$ is $\psi$-hypercyclic if the restriction $\varphi(D) : \ker \psi(D) \to \ker \psi(D)$ is hypercyclic.

In the Introduction (Proposition 2) we noticed that $D_1$ is $z_2$-hypercyclic ($d = 2$) and, in general, a $\psi$-hypercyclic operator $T = \varphi(D)$ solves Problem 1 with $\mathcal{V} = \ker \psi(D)$. Let us consider some trivial cases. If $\psi = 0$, then $\ker \psi(D) = H$ and Proposition 3 gives the corresponding $\psi$-hypercyclic operators. Next, if $\psi$ is a unit in $\text{Exp}$, i.e., $\psi = Ae_a$ where $A \neq 0$, then $\psi(D) = A\tau_a$ and $\ker \psi(D) = \{0\}$. Thus every convolution operator is $\psi$-hypercyclic in this case, and especially this holds if $\psi$ is a constant $\neq 0$. Any $\varphi(D)$ fails of course to be $\psi$-hypercyclic, unless $\varphi$ is a unit. Further, when $d = 1$ we know that no nonconstant polynomial $\psi$ supports any $\psi$-hypercyclic operator, for $\ker \psi(D)$ is then finite dimensional and $\neq \{0\}$. We shall improve this by proving that, when $d = 1$, $\psi$ admits a $\psi$-hypercyclic operator if and only if $\psi = Ae_a$ for some $A, a \in \mathbb{C}$.

Noteworthy is that for arbitrary $d$, any kernel $\ker \psi(D)$ is complemented in $H(C^d)$. (This is a consequence of Taylor and Meise's result from [9], saying that any convolution operator has a continuous linear right inverse.)

After these preliminary observations, the objective is to give a necessary and sufficient condition for $\varphi(D)$ to be $\psi$-hypercyclic in the case when $\psi$ is a polynomial $p$. By the discussion above, we may assume $p$ is nonconstant.

Let us recall some terminology from analytic geometry (for an excellent exposition of this theory we refer to [3]). A regular point of an analytic set $A$ in $\mathbb{C}^d$, e.g., $A = Z(f) \equiv \{ z : f(z) = 0 \}$ where $f \in H$, is a point $a \in A$ for which there exists a neighborhood $U$ of $a$ in $\mathbb{C}^d$ such that $A \cap U$ forms a complex manifold. The set of regular points, $\text{reg} A$, forms an open and dense subset of $A$, and its connected components form complex manifolds.

Next, an analytic set $A$ in $\mathbb{C}^d$ is irreducible if it cannot be written as a union of two proper analytic subsets. An irreducible analytic subset $A'$ of an analytic set $A$ is called an irreducible component of $A$ if every analytic subset $A'' \subseteq A$ with $A' \subset A''$ is reducible. It follows that every irreducible analytic subset of $A$ is contained in an irreducible component. In fact, if $\bigcup_i R_i$ is the decomposition of $\text{reg} A$ into its connected components $R_i$, then the irreducible components of $A$ are given by the sets $\bar{R}_i$ (= closure in $A$, which, since any analytic set is closed, equals the closure in $C^d$). By the density of $\text{reg} A$ in $A$, we have that $A = \bigcup_i \bar{R}_i$, and this is the decomposition of $A$ into irreducible components. In particular, if $p = p_1^{r_1} \cdots p_m^{r_m}$ is the factorization of a nonconstant polynomial $p$ into irreducible factors $p_i$ with corresponding multiplicities $r_i$, the irreducible components of $Z(p)$ are formed by $Z(p_i)$, $i = 1, \ldots, m$. 


If $f \in H$, we denote by $\text{ord}_f(a)$ the order of the zero $a \in Z(f)$. Thus $\text{ord}_f(a)$ is the largest natural number $n$ for which $D^n f(a) = 0$ whenever $|a| < n$.

**Lemma 1.** Let $p = p_1^{r_1} \cdots p_m^{r_m}$ be the factorization of a polynomial $p$ into irreducible factors $p_i$ with corresponding multiplicities $r_i \geq 1$. Let $U \subseteq \mathbb{C}^d$ be any open set such that $U$ meets every irreducible component $Z(p_i)$ of $Z(p)$. Then

\[(1) \quad E_U = \{ qe_a : a \in U \cap Z(p), \ q \text{ is a polynomial with } \deg p < \text{ord}_p(a) \}\]

forms a total subset of $\ker p(D)$.

**Proof.** First of all, $E_U$ is indeed a subset of $\ker p(D)$. For let $qe_a$ be any element of $E_U$. Then $p(D)qe_a = e_ap(a + D)q$. Now $p(a + D)$ is a sum of derivatives $D^\alpha$ of order $|\alpha| \geq \text{ord}_p(a)$, and so $p(a + D)q = 0$.

Next, we intend to prove the statement by induction over the sum $\sum r_i$. For the starting value $\sum r_i = m$, we shall apply the following (cf. [10, Chapter 0.2]):

**Sublemma.** Let $p = p_1 \cdots p_m$ be a polynomial with distinct irreducible factors $p_i$ and assume $U \cap Z(p_i) \neq \emptyset$ for all $i$, where $U \subseteq \mathbb{C}^d$ is open. Then $p \cdot \text{Exp} = \{ \varphi : U \cap Z(p) \subseteq Z(\varphi) \}$.

**Proof of Sublemma.** Assume first that $U = \mathbb{C}^d$. We must then prove that $p \cdot \text{Exp} = \{ \varphi : Z(p) \subseteq Z(\varphi) \}$. To this end we assume first that $p$ is irreducible. So suppose $Z(p) \subseteq Z(\varphi)$. We must prove that $\varphi = p\psi$ for some $\psi \in \text{Exp}$. But the proof of [18, Lemma 29.2] shows that $\varphi = p\psi$ for some (unique) entire $\psi \in H$. Thus we only have to prove that $\psi$ is of exponential type, which follows by [18, Lemma 28.1]. Next, let $p = p_1 \cdots p_m$, where $p_i$ are distinct irreducible polynomials, and assume $Z(p) \subseteq Z(\varphi)$. Then $Z(p_i) \subseteq Z(\varphi)$ for all $i$ and so, from what we just have proved, $\varphi = p_1\psi_1 = p_2\psi_2 = \cdots = p_m\psi_m$ for some unique $\psi_i \in \text{Exp}$. By virtue of [13, Lemma 4] we conclude that $p|\varphi$ in $\text{Exp}$. Hence the Sublemma holds when $U = \mathbb{C}^d$.

Let now $U \subset \mathbb{C}^d$. From what we just have proved, it suffices to prove that if $\varphi \in \text{Exp}$ vanishes on $U \cap Z(p)$, then $\varphi$ vanishes on $Z(p)$. But let $Z_0$ be any connected component of $\text{reg}Z(p)$. Then $Z_0$ is densely contained in some $Z(p_i)$. Now, $U$ meets $Z(p_i)$, and hence $Z_0$, so $U_0 \equiv U \cap Z_0$ forms a nonempty open set in the complex manifold $Z_0$. But $\varphi$ vanishes on $U_0$, and hence on all of $Z_0$. Since $Z_0$ was arbitrary we conclude that $\varphi|_{\text{reg}Z(p)} = 0$ and finally, by density, $\varphi|_{Z(p)} = 0$ and the Sublemma follows.

The Sublemma shows now that $\{ e_a : a \in U \cap Z(p) \}$ is dense in $\ker p(D)$ if $\sum r_i = m$. Indeed, assume $\varphi \in \text{Exp} \approx H'$ is orthogonal to $\{ e_a : a \in U \cap Z(p) \}$. Then $0 = \{ e_a, \varphi \} = \varphi(a)$ for all $a \in U \cap Z(p)$. Hence, $\varphi \in p \cdot \text{Exp}$ which equals $\ker p(D)^{\perp}$, since $p : \varphi \mapsto p\varphi$ is the transpose of (the surjective operator [18, Theorem 28.2]) $p(D)$. Thus the lemma holds when $\sum r_i = m$. Assume that the lemma is proved for $\sum r_i = m, \ldots, n$. Let now $\sum r_i = n$, we must then prove the statement for $p' \equiv p_1^{r_1'} \cdots p_m^{r_m'}$ where $r'_1 \equiv r_1 + 1$. Assume $\varphi$ is
orthogonal to \( \{q_e : a \in U \cap Z(p') \}, \ deg q < \text{ord}_{p'}(a) \}. \) But then \( \varphi \) is orthogonal

to the smaller set \( \{q_e : a \in U \cap Z(p), \ deg q < \text{ord}_p(a) \} \) where \( p \equiv p_1^{r_1} \cdots p_m^{r_m} \),

and so, by the inductive hypothesis, \( \varphi = p\psi \) for some \( \psi \in \text{Exp}. \) We must thus prove

that \( p_1 | \psi \) which, by the Sublemma, is equivalent to \( U \cap Z(p_1) \subseteq Z(\psi) \).

So let \( a \in U \cap Z(p_1) \) be arbitrary and put \( \nu \equiv \text{ord}_{p_1}(a) \). Then \( 1 \leq \nu < \text{ord}_{p'}(a) \)

and there exists a polynomial \( q \) with \( deg q = \nu \) such that \( q(D)p_1(a) \neq 0 \), while \( q_0(D)p_1(a) = 0 \) when \( deg q_0 < \nu \). Hence, by Leibniz’ Formula \( q(D)(fg) = \sum_{\alpha} (q(\alpha)(D)f)D^\alpha g/\alpha! \), we obtain

\[
0 = \langle q_e, \varphi \rangle = q(D)\varphi(a) = q(D)(p_1\psi)(a) = q(D)p_1(a) \cdot \psi(a).
\]

Thus \( \psi(a) = 0 \) and we are done. \( \square \)

**Theorem 1.** Let \( p = p_1^{r_1} \cdots p_m^{r_m} \) be the factorization of a nonconstant polynomial \( p \) into irreducible factors \( p_i \) with corresponding multiplicities \( r_i \geq 1 \). A necessary and sufficient condition for \( \varphi(D) \) to be \( p \)-hypercyclic is that the restriction \( \varphi|_{Z(p_i)} \) is nonconstant for all \( i \).

**Proof.** First we prove the sufficient part, and we shall apply the HC. The proof of the following Sublemma is due to Edgar Lee Stout [17].

**Sublemma.** Let \( \varphi, p \in \text{Exp} \), where \( p \) is an irreducible polynomial. Then \( \varphi|_{Z(p)} \) is nonconstant if and only if the sets \( \Phi_0 \equiv \{z : |\varphi(z)| < 1\} \) and \( \Phi_\infty \equiv \{z : |\varphi(z)| \geq 1\} \) meet \( Z(p) \).

**Proof of Sublemma.** Assume \( \Phi_\infty \) does not meet \( Z(p) \). This means that \( \varphi \) is bounded on \( Z(p) \), and we must prove the analogue of Liouville’s Theorem, that \( \varphi \) must be constant on \( Z(p) \). But \( Z(p) \) is an irreducible algebraic variety \( V \) and hence, there exists a “projection” \( \pi : V \to \mathbb{C}^k \) that is an analytic cover [3, Prop. 7.3.2]. Now the algebra \( H(V) \) of analytic functions on \( V \) (i.e., analytic in a neighborhood of \( V \)), is integral over \( \pi^*H(\mathbb{C}^k) \equiv \{f \circ \pi : f \in H(\mathbb{C}^k)\} \subseteq H(V) \). (This is essentially Noether’s Normalization Theorem.) In particular \( \varphi \in H(V) \) and so there is a monic polynomial \( m(x) = x^\nu + g_{\nu-1}x^{\nu-1} + \cdots + g_0 \), with coefficients \( g_i \in \pi^*H(\mathbb{C}^k) \), such that \( m(\varphi) = 0 \). The coefficients \( g_i \) are symmetric functions in the values of \( \varphi \). For example, if \( p \in V \), \( g_0(p) = \) the product of the values of \( \varphi \) on the fiber \( \pi^{-1}(\pi(p)) \), taken with appropriate multiplicities. Since \( \varphi \) is bounded on \( V \), every \( g_i \) must be bounded and hence, by Liouville’s Theorem, they must be constant. But then, since \( V \) is irreducible and \( \varphi \) is continuous, \( \varphi \) must be constant on \( V = Z(p) \).

That \( \Phi_0 \cap Z(p) = \emptyset \) implies that \( \varphi \) is constant on \( Z(p) \), follows from the arguments above by considering the function \( 1/\varphi \), which is analytic in a neighborhood of \( Z(p) \) if \( \Phi_0 \cap Z(p) = \emptyset \).

Since it is evident that \( \varphi|_{Z(p)} \) is nonconstant whenever \( \Phi_0 \) and \( \Phi_\infty \) meet \( Z(p) \), the Sublemma follows.

Hence, our hypothesis is equivalent to that \( \Phi_0 \) and \( \Phi_\infty \) meet every irreducible component \( Z(p_i) \) of \( Z(p) \). But \( \Phi_0 \) and \( \Phi_\infty \) are open so, by Lemma 1, the sets \( Z \equiv \text{span } E_{\Phi_0} \) and \( \mathcal{Y} \equiv \text{span } E_{\Phi_\infty} \) are dense in \( \mathcal{X} \equiv \ker p(D) \). We prove that
\( \varphi(D)^n \to 0 \) pointwise on \( Z \). It is enough to prove that \( \varphi(D)^n z^\alpha e_a \to 0 \) for any \( \alpha \) and \( a \in \Phi_0 \). But \( \varphi(D)^n z^\alpha e_a = e_a \varphi^n(a + D) z^\alpha \) and, with usual multi-index notation,

\[
\varphi^n(a + D) z^\alpha = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} z^{\alpha - \beta} D^\beta \varphi^n(a).
\]

Hence it suffices to prove that \( D^\alpha \varphi^n(a) \to 0 \) \((n \to \infty)\) for any \( \alpha \) and \( a \in \Phi_0 \), in fact, it is easier to prove the following more general:

**Claim.** Assume \( \varphi \in \text{Exp} \) and \( |\varphi(a)| < 1 \). Then \( n^\nu D^\alpha(\varphi^n \psi)(a) \to 0 \) \((n \to \infty)\) for all \( \nu > 0, \psi \in \text{Exp} \) and multi-indices \( \alpha \).

**Proof of Claim.** When \( |\alpha| = 0 \) the statement is evidently true. Assume now the claim holds for \( |\alpha| = 0, \ldots, m \). Then for any \( i \leq d \) and \( a \) with \( |\alpha| = m \):

\[
n^\nu D_i D^\alpha[\varphi^n \psi](a) = n^\nu D^\alpha[n \varphi^{n-1} \psi D_i \varphi + \varphi^n D_i \psi](a) \to 0
\]

by the inductive hypothesis. Hence the claim.

Next we construct a right inverse \( S : \mathcal{Y} \to \mathcal{Y} \) to \( \varphi(D) \) in the following recursive way. (We apply the fact that the exponential polynomials \( z^\alpha e_a, \alpha \in \mathbb{N}^d, a \in \mathbb{C}^d \), form a linearly independent set in \( H \).) Let \( a \in \Phi_\infty \cap Z(p) \). Then we define \( S e_a \equiv e_a / \varphi(a) \), and extend \( S \) linearly to \( \text{span}\{e_a\} \). Assume now that \( S \) is defined on \( \mathcal{Y}_{a,n} \equiv \text{span}\{z^\alpha e_a : |\alpha| \leq n\} \) in such a way that \( S \) forms a right inverse to \( \varphi(D) \) on this set and \( S \) maps \( \mathcal{Y}_{a,n} \) invariantly. Then if \( |\alpha| = n + 1 < \text{ord}_p(a) \) we define

\[
S(z^\alpha e_a) \equiv z^\alpha e_a / \varphi(a) - S([\varphi(D) - \varphi(a)]z^\alpha e_a / \varphi(a)) = z^\alpha e_a / \varphi(a) - S(e_a [\varphi(a + D) - \varphi(a)] z^\alpha / \varphi(a)),
\]

and extend \( S \) linearly to all of \( \mathcal{Y}_{a,n+1} \). In this way we obtain a right inverse to \( \varphi(D) \) on \( \mathcal{Y}_n \equiv \text{span}\{z^\alpha e_a : |\alpha| < \text{ord}_p(a)\} \). By doing this for all \( a \in \Phi_\infty \cap Z(p) \), we obtain a right inverse \( S \) to \( \varphi(D) \) on \( \cup_{a \in \mathcal{Y}_{a,n}} \mathcal{Y}_n \), which we finally extend linearly to a right inverse \( S \) on \( \mathcal{Y} \). By our construction \( S \) maps \( \mathcal{Y} \) invariantly, and trivially we have now \( \varphi(D)^n S^n \to \text{Id}_\mathcal{Y} \) pointwise. It remains thus only to prove that \( S^n \to 0 \) on \( \mathcal{Y} \), i.e., \( S^n(z^\alpha e_a) \to 0 \) for any \( a \in Z(p) \cap \Phi_\infty \) and \( \alpha \). To prove this, we consider what happens with the lowest degree elements, i.e., when \( |\alpha| \) is small. When \( |\alpha| = 0 \) we have \( S^n e_a = e_a / \varphi(a)^n \to 0 \). Next we derive that \( S(z_1 e_a) = z_1 e_a / \varphi(a) - D_1 \varphi(a) e_a / \varphi(a)^2 \). The general formula is \( S^n(z_1 e_a) = z_1 e_a / \varphi(a)^n - n D_1 \varphi(a) e_a / \varphi(a)^{n+1} \), which thus goes to zero. Computation gives

\[
S^n(z_1^2 e_a) = \frac{1}{\varphi(a)^n} z_1^2 e_a - 2n \frac{D_1 \varphi(a)}{\varphi(a)^{n+1}} z_1 e_a - n \frac{D_1^2 \varphi(a)}{\varphi(a)^{n+2}} e_a + (n^2 + n) \frac{(D_1 \varphi(a))^2}{\varphi(a)^{n+2}} e_a,
\]
and
\[ S^n(z_1 z_2 e_a) = \frac{1}{\varphi(a)^n} z_1 z_2 e_a - n \frac{D_1 \varphi(a)}{\varphi(a)^{n+1}} z_2 e_a - n \frac{D_2 \varphi(a)}{\varphi(a)^{n+1}} z_1 e_a - n \frac{D_1 D_2 \varphi(a)}{\varphi(a)^{n+2}} e_a + (4n - 2) \frac{D_1 \varphi(a) D_2 \varphi(a)}{\varphi(a)^{n+2}} e_a. \]

The general feature is as follows. For any \( n, S^n(z^\alpha e_a) \) is a linear combination of elements \( z^\beta e_a, \beta \leq \alpha \), and the coefficient for \( z^\beta e_a \) is of type \( q(n)O(|\varphi(a)|^{-n}) \) where \( q \) is a polynomial. From this we conclude that \( S^n \to 0 \) pointwise on \( Y \). Hence \( \varphi(D) \) is \( p \)-hypercentric by the HC.

It remains to prove that it is necessary that \( \varphi|_{Z(p_i)} \) is nonconstant for all \( i \).
To this end we recall that a necessary condition for an operator \( T \in L(X) \) to be hypercyclic is that \( T - \lambda \) has dense range for all scalars \( \lambda \), i.e., the adjoint \( T^* \) lacks eigenvalues. Indeed, suppose \( y \in X^* \) is an eigenvector for \( T^* \), with corresponding eigenvalue \( \lambda \), and that \( T \) has a hypercyclic vector \( x \). Then \( y \neq 0 \) and has thus dense range. The image of \( \{T^n x\}_{n \geq 0} \) must therefore be dense, but \( \{T^n x, y\}_{n \geq 0} = \{\lambda^n \langle x, y \rangle\}_{n \geq 0} \) is not dense, hence a contradiction. Now, by the way of contradiction, assume \( \varphi(D) \) is \( p \)-hypercentric but \( \varphi(z) = \lambda \) for all \( z \in Z(p_i) \) for some \( i \) and \( \lambda \), we may assume \( i = 1 \). Then \( \varphi - \lambda \) vanishes on \( Z(p_1) \) and is thus divisible by \( p_1 \) (Sublemma of Lemma 1). We know that the set (1) with \( U = \mathbb{C}^d \), which we denote by \( A \), is total in ker \( p(D) \). Since \( \varphi(D) - \lambda \) has dense range, \( [\varphi(D) - \lambda] A \) must be total in ker \( p(D) \). But
\[ [\varphi(D) - \lambda] A \subseteq B + C \equiv \text{span}\{q e_a : a \in Z(p_1), \ deg q < \text{ord}_p(a) - \text{ord}_{p_1}(a)\} + \text{span}\{q e_a : a \in Z(p) \setminus Z(p_1), \ deg q < \text{ord}_p(a)\}, \]
where, in the definition of \( B \), we tacitly assume \( q = 0 \) in the case \( \text{ord}_p(a) = \text{ord}_{p_1}(a) \). To obtain a contradiction, it suffices to prove that there exists \( \varphi_0 \in \text{Exp} \) that is orthogonal to \( B + C \), but not to \( A \) (i.e., not to ker \( p(D) \)). We claim that \( \varphi_0 \equiv p_1^{m_1 - 1} p_2^{m_2} \cdots p_m^{m_m} \) will due. Indeed, if \( a \in Z(p) \setminus Z(p_1) \), we have that \( \text{ord}_{\varphi_0}(a) = \text{ord}_p(a) \) and so if \( q e_a \in C \),
\[ \langle q e_a, \varphi_0 \rangle = q(D) \varphi_0(a) = 0. \]
Next, if \( a \in Z(p_1) \), then \( \text{ord}_{\varphi_0}(a) = \text{ord}_p(a) - \text{ord}_{p_1}(a) \) which implies \( \langle q e_a, \varphi_0 \rangle = q(D) \varphi_0(a) = 0 \) if \( \text{deg} q < \text{ord}_p(a) - \text{ord}_{p_1}(a) \). Thus \( \varphi_0 \) is orthogonal to \( B + C \).
Now, pick \( a \in Z(p_1) \) arbitrarily. Then \( \text{ord}_p(a) > \text{ord}_{\varphi_0}(a) \), so there exists a polynomial \( q \) with \( \text{deg} q < \text{ord}_p(a) \) but \( q(D) \varphi_0(a) \neq 0 \). But then \( q e_a \in A \) and \( \langle q e_a, \varphi_0 \rangle = q(D) \varphi_0(a) \neq 0 \).

Let \( HC(p) \) denote the set of all \( \varphi \in \text{Exp} \) for which \( \varphi(D) \) is \( p \)-hypercentric.
From Theorem 1 it is immediate that \( HC(p q) = HC(p) \cap HC(q) \) for any pair \( p, q \) of nonconstant polynomials, or more generally:

**Corollary 1.** Let \( p_1, \ldots, p_m \) be nonconstant polynomials. Then
\[ HC(p_1 \cdots p_m) = \cap_i HC(p_i) = HC(\text{lcm}\{p_1, \ldots, p_m\}), \]
where \( \text{lcm}\{p_1, \ldots, p_m\} \) denotes the least common multiple of \( p_1, \ldots, p_m \) (which, of course, is uniquely determined up to a constant factor).

**Corollary 2.** Let \( p, q \) be nonconstant polynomials in \( d \geq 2 \) variables. Then \( HC(p) \subseteq HC(q) \) if and only if \( Z(q) \subseteq Z(p) \). Thus, \( HC(p) = HC(q) \) if and only if \( Z(q) = Z(p) \).

**Proof.** Since neither \( Z(\cdot) \) nor (by Theorem 1) \( HC(\cdot) \) change if we only change the multiplicities of the polynomial under consideration, we can assume \( p \) and \( q \) have no multiple irreducible factors.

Suppose first that \( Z(q) \subseteq Z(p) \). Then every irreducible factor \( q' \) of \( q \) is an irreducible factor of \( p \), and so \( HC(p) \subseteq HC(q) \) by Theorem 1.

Next, assume \( HC(p) \subseteq HC(q) \), i.e., by Theorem 1 (Corollary 1), \( HC(p) \subseteq HC(q') \) for every irreducible factor \( q' \) of \( q \). This means that if \( \varphi \in \text{Exp} \) is constant on \( Z(q') \), then \( \varphi \) is constant on some component \( Z(p') \) of \( Z(p) \). We must prove that \( q' \) is a factor of \( p \). Assume not. To obtain a contradiction we must construct a \( \varphi \) that is constant on \( Z(q') \) but not on any \( Z(p') \), where \( p' \) is an irreducible factor of \( p \). If \( q' \) is nonconstant on every \( Z(p') \), we may put \( \varphi \equiv q' \). If this is not the case, let \( p'_1, \ldots, p'_n \) be the irreducible factors of \( p \) for which \( q'|_{Z(p'_i)} \) is constant. Thus \( q'(z) = \alpha_i \) for all \( z \in Z(p'_i) \) for some constants \( \alpha_i \). Since \( q' \) is distinct from every \( p'_i, \alpha_i \neq 0 \) for all \( i \).

**Claim.** Let \( d \geq 2 \) and \( q \) be an irreducible polynomial that is not a factor of a polynomial \( p \). Then there exists a \( \psi \in \text{Exp} \) such that \( p\psi \) is not constant on \( Z(q) \).

**Proof of Claim.** Assume the claim does not hold, i.e., for every \( \psi \in \text{Exp} \) there exist a constant \( \alpha \) and \( \varphi \in \text{Exp} \) such that \( p\psi = \alpha + q\varphi \). This means that image of the operator \( T : \mathbb{C} \times \text{Exp} \rightarrow \text{Exp} \), defined by \( (\alpha, \varphi) \mapsto \alpha + q\varphi \), contains the ideal \( p \cdot \text{Exp} = \ker p(D) \). Hence

\[
\ker T = \text{Im} T^\perp \subseteq (p \cdot \text{Exp})^\perp = \ker p(D),
\]

where \( T \) is the transpose of \( T \) with respect to the duality \( (H, \text{Exp}) \) and the natural duality between \( \mathbb{C} \times \text{Exp} \) and \( \mathbb{C} \times H \). We derive that \( T^t f = (f(0), q(D)f) \).

Hence, to obtain a contradiction, it suffices to prove that there exist a \( f \in H \) such that \( q(D)f = 0 \) and \( f(0) = 0 \) but \( p(D)f \neq 0 \). But since \( q \) is not a factor of \( p \), there exists \( a \in Z(q) \setminus Z(p) \). Since \( d \geq 2 \), the zeros are not isolated so we may find two different points \( a, b \in Z(q) \setminus Z(p) \). Consider now \( f \equiv e_a - e_b \). It is evident that \( q(D)f = 0 \) and \( f(0) = 0 \) hold, and it follows that \( p(D)f = p(a)e_a - p(b)e_b \neq 0 \). Indeed, if \( p(a)e_a - p(b)e_b = 0 \), the linearly independence of \( e_a, e_b \) yields that \( p(a) = p(b) = 0 \), which is a contradiction. Hence the claim.

By the claim there exist for every \( p'_i \), a \( \psi_i \in \text{Exp} \) such that \( \psi p/p'_i \) is not constant on \( Z(p'_i) \). It follows now that

\[
\varphi \equiv q'\left(\frac{\psi_1 p}{p'_1} + \cdots + \frac{\psi_n p}{p'_n} + 1\right)
\]
is a required function.

Corollary 3. Let $p$ be a nonconstant polynomial and assume $\varphi(D)$ is $p$-hypercyclic. Then $\varphi_\nu(D)$, where $\varphi_\nu \equiv \nu \circ \varphi$, is $p$-hypercyclic for any nonconstant $\nu \in \text{Exp}(\mathbb{C})$ such that $\varphi_\nu \in \text{Exp}$. (In particular, $\varphi_\nu \in \text{Exp}$ if $\varphi, \nu$ are polynomials or if $\nu = az + b$ and $\varphi$ is arbitrary.)

Proof. Let $p'$ be any irreducible factor of $p$. Since $\varphi$ is nonconstant and analytic on $\text{reg} Z(p')$, the image $\varphi(Z(p'))$ contains an open set. Thus, since $\nu$ is nonconstant, $\nu$ cannot be constant on this set, which proves that $\varphi_\nu$ is nonconstant on $Z(p')$.

When $d = 1$, $\ker p(D)$ is finite dimensional for any polynomial $p$ and thus $HC(p)$ is empty. (We also see that no $\varphi \in \text{Exp}$ satisfies the necessary and sufficient condition of Theorem 1.) We shall now prove that $HC(\psi) = \emptyset$ in general, except for the trivial cases.

Proposition 4. If $d = 1$, $\psi \in \text{Exp}$ admits a $\psi$-hypercyclic operator $\varphi(D)$ if and only if $\psi = Ae^a$, for some $A \in \mathbb{C}$, $a \in \mathbb{C}$, i.e., if and only if $\ker \psi(D) = \{0\}, H$.

Proof. If $\psi = Ae^a$, then $\ker \psi(D) = H$ (if $A = 0$) or $\ker \psi(D) = \{0\}$ (if $A \neq 0$), and so $HC(\psi) \neq \emptyset$ (see the discussion following Definition 1). So we must prove the converse.

Assume $\psi$ is not of the form $Ae^a$. In particular this means that $Z(\psi)$ is countable and not empty. Indeed, any zero free entire function is of the form $e^g$, where $g \in H$, and $e^g \in \text{Exp}$ if and only if $g(z) = az + b$. Now, the key is to note that Lemma 1 (we assume $U = \mathbb{C}^d$) extends to the non polynomial case. That is, we claim: $\{qe_a : a \in Z(\psi), \deg q < \text{ord}_\psi(a)\}$ is total in $\ker \psi(D)$. To see this we show first that the Sublemma of Lemma 1 extends in the sense that if $\phi \in \text{Exp}$ has only simple zeros and $Z(\phi) \subset Z(\psi)$ ($\psi \in \text{Exp}$), then $\phi$ divides $\psi$ in $\text{Exp}$. Indeed, any element $\phi \in \text{Exp}$ can be written in the form $\phi(z) = Az^m e^{az} \Pi(1 - z/a_n) e^{az/a_n}$, see [8, Chapter 2.1]. Here $(a_n)$ are the zeros $\neq 0$ for $\phi$, multiple zeros are repeated in the sequence, ordered so that $|a_n| \leq |a_{n+1}|$. From this it evident that $Z(\phi) \subset Z(\psi)$ implies $\psi \in \phi \cdot \text{Exp} = \ker \phi(D)^{1/\psi}$. From this point the proof of our claim goes parallel to that of Lemma 1.

Now, let $\varphi \in \text{Exp}$ and let $a_0$ be any zero point of $\psi$, and put $\lambda \equiv \varphi(a_0)$. From the proof of Theorem 1, we recall that in order that $\varphi(D)$ is $\psi$-hypercyclic, $[\varphi(D) - \lambda]A$, where $A \equiv \{qe_a : a \in Z(\psi), \deg q < \text{ord}_\psi(a)\}$, must be total in $\ker \psi(D)$. That this is not the case follows, as in the proof of Theorem 1, by the absence of terms $z^{a} e_{a_0}$, $\alpha = |\alpha| = \text{ord}_\psi(a_0) - 1$, in $\text{span}[\varphi(D) - \lambda]A$. 

3. On Problem 2

In the previous section we studied hypercyclicity of restrictions, and in this section we consider the converse problem; the problem of finding hypercyclic
extensions of hypercyclic operators on smaller spaces. We continue to work with the space $H(\mathbb{C}^d)$.

By $\| \cdot \|_r$ ($r > 0$) we denote the generating family of seminorms on $H(\mathbb{C}^d)$ defined by $\| f \|_r \equiv \sum_{n \geq 0} \| H_n f \| r^n$, where $H_n f$ denotes the $n$:th homogeneous part of the Taylor expansion of $f \in H(\mathbb{C}^d)$ about the origin and $\| \varphi \| \equiv \sup_{\| z \| \leq 1} | \varphi(z) |$. Next, $E$ denotes the subspace of $H(\mathbb{C}^d \times \mathbb{C}^d) \simeq H(\mathbb{C}^{2d})$ formed by all $f = f(z, \xi)$ of the form $\sum_{n=0}^{\infty} f_n(\xi) \langle z, \xi \rangle^n / n!$ ($f_n \in H(\mathbb{C}^d)$) with absolute convergence in $H(\mathbb{C}^{2d})$, or equivalently, for which

$$\lim_{n \to \infty} \sup_{n \geq 0} \left( \frac{\| f_n \|_r}{n!} \right)^{1/n} = 0$$

for all $r > 0$. Recall that $\langle z, \xi \rangle \equiv \sum z_i \xi_i$. Note that the functions $f_i$ are unique for $f \in E$, and we shall see that $E$ forms an infinite dimensional closed (and when $d = 1$, complemented) subspace of $H(\mathbb{C}^{2d})$.

The main result in this section reads:

**Theorem 2.** Any operator $T$ on $H(\mathbb{C}^d)$ that satisfies the HC (e.g. any non-trivial $\varphi(D)$), defines a hypercyclic operator $T_E : E \to E$ by

$$\sum_{n \geq 0} f_n(\xi) \langle z, \xi \rangle^n / n! \to \sum_{n \geq 0} (T f_n)(\xi) \langle z, \xi \rangle^n / n!.$$

Further, if $d = 1$, $T_E$ extends to a hypercyclic operator $\hat{T}_E : H(\mathbb{C}^2) \to H(\mathbb{C}^2)$.

We shall also establish an explicit hypercyclic extension $\hat{T}_E$ of $T_E$ when $d = 1$ (see Corollary 5). However, before we prove Theorem 2, we shall give another characterization of the space $E$, which is of independent interest. Further, our proof of Theorem 2 goes via general results applicable for Problem 2.

We equip $\text{Exp} = \text{Exp}(\mathbb{C}^d)$ with its standard inductive limit topology. That is, by $\text{Exp}_r = \text{Exp}_r(\mathbb{C}^d)$ we denote the Banach space of all functions $\varphi \in H = H(\mathbb{C}^d)$ such that $\| \varphi \|_r \equiv \sup_{r \geq 0} \| \varphi(z) e^{-r\|z\|} \| < \infty$, provided with the norm $\| \cdot \|_r$. Then $\text{Exp} = \cup_{r>0} \text{Exp}_r$ and we endow $\text{Exp}$ with the corresponding inductive locally convex topology. It follows then that the identification $\text{Exp} \simeq H'$ (see the previous section) is a topological isomorphism, where $H'$ carries the strong topology, and so $\text{Exp}$ is a nuclear (because $H$ is) DF-space. By $L(\text{Exp}, H)$ we denote the space of all continuous linear mappings $\text{Exp} \to H$ provided with the topology of uniform convergence on bounded (or equivalently, since $\text{Exp}$ is nuclear, compact) sets. Since $\text{Exp}$ is a DF-space and $H$ a Fréchet space, $L(\text{Exp}, H)$ is a Fréchet space. In fact:

**Proposition 5.** Let, as before, $e_\xi \equiv e^{i \langle \cdot, \xi \rangle} \in \text{Exp} = \text{Exp}(\mathbb{C}^d)$, $\xi \in \mathbb{C}^d$. Then the map $T \mapsto f(z, \xi) \equiv T e_\xi(z)$ is a topological isomorphism between $L(\text{Exp}, H)$ and $H(\mathbb{C}^{2d})$.

**Proof.** Let $T \in L(\text{Exp}, H)$ and consider $f(z, \xi) \equiv T e_\xi(z)$. We prove that $f \in H(\mathbb{C}^{2d})$. It is clear that $f(\cdot, \xi) \in H(\mathbb{C}^d)$ for any fixed $\xi \in \mathbb{C}^d$. But we note
that $T e_\xi(z) = T e_\xi(\xi)$, and so $f(z, \cdot) \in H(\mathbb{C}^d)$ for fixed $z$, and by Hartog’s Theorem we conclude that $f \in H(\mathbb{C}^{2d})$. That $i : T \to f$ is one-to-one follows from the fact that the elements $e_\xi, \xi \in \mathbb{C}^d$, form a total set in $\text{Exp}$. Indeed, since $H$ is reflexive, $\text{Exp}' \simeq H$ in the sense of the duality between $H$ and $\text{Exp}$. So if $f \in H$ is orthogonal to $\{ e_\xi : \xi \in \mathbb{C}^d \}$, then $0 = \langle f, e_\xi \rangle = f(\xi)$ for all $\xi$ and hence $f = 0$. This shows that $\{ e_\xi : \xi \in \mathbb{C}^d \}$ is total and thus $i$ is injective. Next, let $f \in H(\mathbb{C}^{2d})$ and define $T$ by $T \varphi(z) = \langle f(z, \cdot), \varphi \rangle$. It is easily checked that $T \in L(\text{Exp}, H)$, and we note that $i T(z, \xi) = \langle f(z, \cdot), e_\xi \rangle = f(z, \xi)$. Hence $i$ is a bijection and therefore an isomorphism by the Open-Mapping Theorem.

It follows that the set of convolution operators on $\text{Exp}(\mathbb{C}^d)$ is given by all $f(D) = \sum_{n \in \mathbb{N}} f_n D^n$ (pointwise convergence in $\text{Exp}$) where $f = \sum_{n} f_n z^n \in H(\mathbb{C}^d)$. (Given a convolution operator $T$, we have that $T = f(D)$ where $f(z) \equiv T e_\xi(0) = T 1(z)$.) In particular we have that $f(D)$ is “multiplication by $f$” and $f(D) g(D) = (fg)(D)$, so convolution operators on $\text{Exp}$ commute.

**Definition 2.** An operator $T \in L(\text{Exp}, H)$ is said to be PDE-preserving for a set $A \subseteq \text{Exp} \times H$ if $T \ker f(D) \subseteq \ker \varphi(D)$ for all $(\varphi, f) \in A$. The set of PDE-preserving operators for $A$ is denoted by $O(A)$.

Note that $O(A)$ forms a closed subspace of $L(\text{Exp}, H) (\simeq H(\mathbb{C}^{2d}))$ for any set $A$. PDE-preserving operators, in other settings, have been studied in e.g. [11, 12, 13].

**Proposition 6.** Let $\mathbb{H}$ denote the set of homogeneous polynomials ($d \geq 1$ variables) and $E \equiv \{(p, p) : p \in \mathbb{H}\} \subset \text{Exp} \times H$. Then $O(\mathbb{E})$ is formed by all operators $T$ of the form $T = \sum_{n \geq 0} H_n \circ f_n(D)$, where the sequence $(f_n)$ satisfies (2) and is unique for $T$. In particular, $E = O(\mathbb{E})$ in the sense of the isomorphism in Proposition 5, and so $E$ is closed.

**Proof.** By the fact that any $f_n(D)$ commutes with any $p(D), p \in \mathbb{H}$, and, if $p \in \mathbb{H}$ is $m$-homogeneous, $p(D) H_n$ equals $H_{n-m} p(D)$ if $n \geq m$ and 0 otherwise, it is easily checked that any operator $T$ of the form $\sum_{n \geq 0} H_n \circ f_n(D)$ (where $(f_n)$ satisfies (2)), belongs to $O(\mathbb{E})$. In fact, the growth condition (2) implies that $\sum_{n \geq 0} H_n \circ f_n(D)$ converges pointwise and, by Banach-Steinhaus Theorem, defines thus a continuous operator. Further, $f_n(D) e_\xi = f_n(\xi) e_\xi$, and hence $T e_\xi(z) = \sum f_n(\xi)(z, \xi)^n / n!$. It remains thus only to prove that every $T \in O(\mathbb{E})$ is of this form.

For any $n \in \mathbb{N}$ and $z \in \mathbb{C}^d$, $\varphi \mapsto H_n \varphi(z)$ is a continuous linear functional on $\text{Exp} = \text{Exp}(\mathbb{C}^d)$. Hence, there exists a unique $g_{n,z} \in H \simeq \text{Exp}'$ such that $\langle g_{n,z}, \varphi \rangle = H_n \varphi(z)$ for all $\varphi \in \text{Exp}$. We prove that $g_{n,z} \in \langle z, \cdot \rangle^n \cdot H$ if $n \geq 1$ and $z \neq 0$. Multiplication by $\langle z, \cdot \rangle^n$, $f \mapsto \langle z, \cdot \rangle^n f$, is the transpose of $p_{n,z}(D) : \text{Exp} \to \text{Exp}$ where $p_{n,z}$ is the homogeneous polynomial $\langle z, \cdot \rangle^n$ (i.e., $p_{n,z}(D) \varphi$ is the $n$:th directional derivative of $\varphi$ in $\text{Exp}$ along $z$). Now, since $T \in O(\mathbb{E})$, it is evident that $g_{n,z} \in \text{ker} p_{n,z}(D)^\perp$, which equals $\text{Im} \langle z, \cdot \rangle^n$, because $p_{n,z}(D)$ is surjective [18, Theorem 28.2]. So, for every $n \geq 1$ and $z \neq 0$,
there exists a unique \( f_{n,z} \in H \) with \( g_{n,z} = \langle z, \cdot \rangle^n f_{n,z}/n! \). Hence
\[
H_n f_{n,z}(D) \varphi(z) = \langle \langle z, \cdot \rangle^n /n!, f_{n,z}(D) \varphi \rangle = \langle f_{n,z}(z, \cdot)^n/n!, \varphi \rangle = H_n T \varphi(z)
\]
if \( z \neq 0 \), and we note that the identity \( H_n f_{n,z}(D) \varphi(z) = H_n T \varphi(z) \) holds even if \( z = 0 \) (both sides vanish). We must prove that, for fixed \( n \geq 1 \), the \( f_{n,z} \) are independent of \( z \neq 0 \), i.e., equal to some \( f_n \in H \). To this end we only have to refer to the proof of Theorem 2 in [11], where an analogous step is solved. When \( n = 0 \), \( g_{n,z} \) is independent of \( z \) and we put \( f_0 \equiv g_{0,z} \) and notice that \( H_0 f_0(D) \varphi = H_0 T \varphi \). Thus, for any \( \varphi \in \text{Exp} \) we have
\[
T \varphi = \sum H_n T \varphi = \sum H_n f_n(D) \varphi
\]
pointwise. It remain thus only to prove that \( (f_n) \) satisfies the growth condition (2). From the identity \( H_n f_n(D) \xi_\varepsilon = H_n T \xi_\varepsilon \) (consider the \((n+m)\)-homogeneous part in \( \xi \) of both sides) we derive that
\[
\langle z, \xi \rangle^n f_{n,m}(\xi) = \langle T^{(n+m)}/(n + m)! \delta, \cdot \rangle^n \langle z, \cdot \rangle^n,
\]
where \( f_{n,m} = H_m f_n \). Now, for any \( \varepsilon > 0 \) we have that
\[
B \equiv \{ \varepsilon^{-n-m} \langle \cdot, \xi \rangle^{n+m}/(n + m)! : n, m \in \mathbb{N}, ||\xi|| = 1 \}
\]
forms a bounded set in \( \text{Exp} \) (it is contained and bounded in some \( \text{Exp}_r \)). Hence, \( TB \) is bounded in \( H \) and so, for every \( r > 0 \) there exists \( M_r = M_r(\varepsilon) > 0 \) such that
\[
||T^{(n+m)}/(n + m)! \delta, \cdot \rangle^n \langle z, \cdot \rangle^n ||_r \leq \varepsilon^{n+m} M_r
\]
for all \( \xi \) on the unit sphere and all \( n, m \geq 0 \). Next we need the following simple consequence of Cauchy’s estimates, whose proof we omit:

**Sublemma.** If \( \varphi \in \text{Exp}_r(\mathbb{C}^d) \), then \( ||\langle f, \varphi \rangle|| \leq ||f||_{d \varepsilon \varepsilon} ||\varphi||_r \) for all \( f \in H(\mathbb{C}^d) \).

We see now that \( \langle z, \cdot \rangle^n \in \text{Exp}_1 \) with \( \langle \langle z, \cdot \rangle^n, 1 \rangle \leq n! ||z||^n \). Hence, from (3), (4) and the Sublemma, we conclude that
\[
||\langle z, \xi \rangle^n f_{n,m}(\xi) || \leq \varepsilon^{n+m} M_d \varepsilon n! ||z||^n, \quad ||\xi|| = 1.
\]
If \( p \) and \( q \) are an \( n \) and \( m \)-homogeneous polynomial respectively, we have that \( ||pq|| \leq (2\varepsilon)^{n+m} ||p|| ||q|| \) (see [4, p. 72]). This yields with \( p \equiv \langle z, \cdot \rangle^n \) and \( q \equiv f_{n,m} \):
\[
||z||^n ||f_{n,m}|| \leq (2\varepsilon)^{n+m} ||\langle z, \cdot \rangle^n f_{n,m}|| \leq (2\varepsilon)^{n+m} M_d \varepsilon n! ||z||^n.
\]
Accordingly, for any given \( r > 0 \) (take \( z \) with \( ||z|| = r \) in (5) and \( \varepsilon = \varepsilon, \) small enough), we have
\[
\tau^n ||f_n||_r = \sum_{m \geq 0} r^n ||f_{n,m}|| \leq M_d \varepsilon n! (2\varepsilon r)^n \sum_{m \geq 0} (2\varepsilon r)^m \leq Nn!
\]
for some constant \( N = N(r, \varepsilon) \). Hence \( (f_n) \) satisfies (2). \( \Box \)
Note that when $d = 1$, $\mathcal{O}(\mathbb{E})$ consists precisely of the continuous operators $T : \text{Exp}(\mathbb{C}) \to H(\mathbb{C})$ that, for every $m$, maps polynomials of degree at most $m$ onto polynomials of degree at most $m$.

Next we prove some general results related to our problem, Problem 2.

We recall that a sequence $(T_n)_{n \geq 0} \subseteq L(X)$ is called hypercyclic if, for some (hypercyclic) $x \in X$, $(T_n x)_{n \geq 0}$ is dense, and the HC (Proposition 1) extends to sequences of operators on a separable Fréchet space in the sense that we may replace $T^n$ by $T_n$ [1]. (Thus an operator $T$ is hypercyclic if and only if the sequence $(T^n)_n$ is.)

**Proposition 7.** Let $X$ be a separable Fréchet space and assume we have a topological decomposition $X = \mathcal{Z} \oplus \mathcal{Y}$. If $S \in L(\mathcal{Z})$ and $T \in L(\mathcal{Y})$ satisfies the HC with respect to some common sequence $(n_k) \subseteq \mathbb{N}$, then $S \oplus T : z + y \mapsto Sz + Ty$ forms a hypercyclic operator on $X$. In fact, for any hypercyclic vector $y \in \mathcal{Y}$ for $(T^{n_k})$, there exists a hypercyclic vector $z \in \mathcal{Z}$ for $S$ such that $z + y$ is hypercyclic for $S \oplus T$.

**Proof.** It suffices to prove the last part, i.e., given a hypercyclic vector $y \in \mathcal{Y}$ for $(T^{n_k})$, there exists a hypercyclic vector $z \in \mathcal{Z}$ for $S$ such that $z + y$ is hypercyclic for $S \oplus T$. Let $(y_i)_{i \geq 0}$ be a countable dense set in $\mathcal{Y}$. There exists a subsequence $(n_i)_{j}$ of $(n_k)$ such that $T^{n_i} y \to y_i$ ($j \to \infty$). Consider now the denumerable family of sequences $(S^{n_i})_{j} \subseteq L(\mathcal{Z})$, $i = 0, 1, \ldots$. We know that $(S^{n_k})$, and hence every $(S^{n_i})$, satisfies the HC. Accordingly, by [6, Prop. 3], $(S^{n_i})$, $i = 0, 1, \ldots$ have a common hypercyclic vector $z \in \mathcal{Z}$. We prove that $z + y$ is a required hypercyclic vector for $S \oplus T$. So let $x = x_{\mathcal{Z}} + x_{\mathcal{Y}} \in \mathcal{Z} \oplus \mathcal{Y} = X$ be arbitrary, and choose a continuous seminorm $p$ and $\varepsilon > 0$ arbitrarily. Pick $i_0$ so that $p(y_{i_0} - x_{\mathcal{Y}}) \leq \varepsilon/3$. Next we may find a $j_0$ such that $p(T^{n_i} y - y_{i_0}) \leq \varepsilon/3$ for all $j \geq j_0$. Now $z$ is hypercyclic for $(S^{n_{i_0}})$, so there exists a $j_0 \geq j_0$ with $p(S^{n_{i_0}} z - x_{\mathcal{Z}}) \leq \varepsilon/3$. The triangle inequality gives

$$p((S \oplus T)^{n_{i_0}} z + y - x) \leq p(S^{n_{i_0}} z - x_{\mathcal{Z}}) + p(T^{n_{i_0}} y - y_{i_0}) + p(y_{i_0} - x_{\mathcal{Y}}) \leq \varepsilon,$$

thus $z + y$ is indeed a hypercyclic vector. \hfill \Box

Thus, in order to extend an operator $T \in L(\mathcal{Y})$ that satisfies the HC to a hypercyclic operator on $X$, where $\mathcal{Y} \subseteq X$ is complemented, we only have to construct an operator $S \in L(\mathcal{Z})$, where $X = \mathcal{Z} \oplus \mathcal{Y}$, that satisfies the HC with respect to a sequence for which $T$ satisfies the HC.

**Corollary 4.** Let $X$ be a separable Fréchet space and assume $\mathcal{Y} \subseteq X$ is complemented in $X$ and codim $\mathcal{Y} = \infty$. Then every operator $T \in L(\mathcal{Y})$ that satisfies the HC with respect to the full sequence $(n_k) = k$, has a hypercyclic extension $\hat{T} \in L(X)$.

**Proof.** Let $X = \mathcal{Z} \oplus \mathcal{Y}$ be a topological decomposition. Since codim $\mathcal{Y} = \infty$, $\mathcal{Z}$ forms an infinite dimensional separable Fréchet space and hence supports a hypercyclic operator $S \in L(\mathcal{Z})$ [2]. In fact, from the proof in [2], we may choose $S$ so that $S$ satisfies the HC with respect to some sequence $(n_k)$. Now
$T$ satisfies the HC with respect to the full sequence and hence with respect to $(n_k)$. Thus $S \oplus T$ is a required extension by Proposition 7.

Remark 1. In some settings, Corollary 4 can strengthened. Indeed, let $X$ be a separable Hilbert space and assume $\mathcal{Y} \subseteq X$ is closed and has infinite codimension. Then $\mathcal{Y}$ is complemented and if $X = Z \oplus \mathcal{Y}$ is a topological decomposition, $\dim Z = \infty$. Thus $Z \cong \ell_2$ and so $Z$ supports (because $\ell_2$ does) an operator $S \in L(Z)$ that satisfies the HC with respect to the full sequence. From this we conclude: Every hypercyclic operator $T \in L(\mathcal{Y})$, where $\mathcal{Y}$ is an infinite codimensional closed subspace of a separable Hilbert space $X$, has a hypercyclic extension $\hat{T} \in L(X)$.

Example 1. We recall that any kernel $\ker \psi(D)$, $\psi \in \Exp$, is complemented in $H = H(\mathbb{C}^d)$. We claim that $\ker \psi(D)$ has infinite codimension if $\psi \neq 0$. Indeed, the complement of $Z(\psi)$ is infinite for any such $\psi$, and we have that $\{ e_a \equiv e_a + \ker \psi(D) : a \notin Z(\psi) \}$ forms a linearly independent set in $H/\ker \psi(D)$. For if $\sum_{i=1}^n A_i e_{a_i} = 0$, $a_i \notin Z(\psi)$, then $\sum A_i \psi(a_i) e_{a_i} = 0$. But $\{ e_a : a \in \mathbb{C}^d \}$ is a linearly independent set in $H$ and so, since $\psi(a_i) \neq 0$, $A_i = 0$ for all $i$. Hence any operator $T : \ker \psi(D) \to \ker \psi(D)$ that satisfies the HC with respect to the full sequence admits a hypercyclic extension $\hat{T} \in L(H)$. In particular, let $p$ be a nonconstant homogeneous polynomial. Then we have that $H = \tilde{p} \cdot H \oplus \ker p(D)$ is a topological decomposition of $H$ [16]. Here $\tilde{p}$ denotes the homogeneous polynomial obtained from $p$ by conjugating the coefficients. Now, assume $T : \ker p(D) \to \ker p(D)$ satisfies the HC with respect to the full sequence, and let $S \in L(H)$ be any operator that satisfies the HC. (In particular we may have $T = \varphi(D)|_{\ker p(D)}$ where $\varphi \in \Exp$ satisfies the hypothesis of Theorem 1 with respect to $p$.) Then $S$ induces an operator $\hat{S}$ on $pH$, that satisfies the HC, by $\hat{p}f \mapsto \hat{p}Sf$. Thus, by Proposition 7, $\hat{S} \oplus T \in L(H)$ forms a hypercyclic extension of $T$. (Note that all this latter is a generalization of Proposition 2, which corresponds to the case $p = \tilde{p} = z_2$, $T = D_1|_{\ker D_2}$ and $S = D_1$, because in this case $\hat{S} = S|_{pH}$.)

Proposition 8. If $d = 1$, $E$ is complemented in $H(\mathbb{C}^2)$. In fact, $H = E \oplus F$ forms a topological decomposition of $H = H(\mathbb{C}^2)$, where $F$ is formed by all $p = \sum_{n \geq 1} p_n(\xi)z^n/n!$ where each $p_n$ is a polynomial with $\deg p_n < n$ and the sequence $(p_n)$ satisfies (2).

Proof. Since $H = H(\mathbb{C}^2)$ is a Fréchet space and $E$ is closed, it suffices, by the Closed-Graph Theorem, to prove that $F$ is closed and $H = E \oplus F$. Let $f \in H$. Then $f$ can uniquely be written in the form $\sum_{n \geq 0} f_n(\xi)z^n/n!$ where the sequence $(f_n) \subset H(\mathbb{C})$ satisfies (2). But each $f_n$ has a unique decomposition $f_n(\xi) = g_n + \gamma_n \xi^{n}$ where $g_n \in H(\mathbb{C})$ and $\gamma_n = \sum_{m < n} \frac{f_n(\xi^m)(0)}{m!} \xi^{m}$ is a polynomial of degree $\geq n$. If $n \geq 1$ and $p_0 = 0$. We prove that $(\gamma_n)$ and $(p_n)$ satisfy (2). Put $h_n \equiv g_n \xi^n = \sum_{m \geq 0} f_n^{(m)}(0) \xi^{m}/m!$. We note that $\|h_n\|_r, \|p_n\|_r \leq \|p_n\|_r + \|h_n\|_r = \|f_n\|_r$ for any $r > 0$. Hence $(p_n)$ and $(h_n)$,
and therefore \((g_n)\), satisfy (2). This implies that \(g \equiv \sum g_n(\xi)z^n/\xi^n/n!\) and 
\(p \equiv \sum p_n(\xi)z^n/\xi^n/n!\) converge in \(H\) and belong to \(E\) and \(F\) respectively. Thus \(f = g + p\) and so \(H = E + F\). It is evident that \(E \cap F = \{0\}\) and hence \(H = E \oplus F\) holds. Finally, the fact that the space of polynomials of degree \(\leq m\) is closed in \(H\) for all \(m\), implies \(F\) is closed in \(H\), and we are done. \(\square\)

**Proof of Theorem 2.** It is clear that \(T_E\) indeed defines a continuous operator on \(E\), and we prove that \(T_E\) is hypercyclic. We shall apply the HC. Let \(Z, Y\) be dense subspaces of \(H = H(\mathbb{C}^d)\) and \(S_k : Y \to H\) linear maps for which for some sequence \((n_k)\); \(T^{n_k} \to 0\) on \(Z\) and \(S_k \to 0\), \(T^{n_k} S_k \to \text{Id}_Y\) on \(Y\). By \(Z\) (\(Y\) resp.) we denote the set in \(E\) of all finite sums \(\sum f_n(\xi)(z, \xi)^n/n!\) where \(f_i \in Z\) (\(Y\) resp.). It is easily checked that \(Z\) and \(Y\) are dense in \(E\), and \(T^{n_k}\) goes to zero on \(Z\). Next we define mappings \(S^k_E : Y \to E\) by \(\sum f_n(\xi)(z, \xi)^n/n!\) \(\mapsto\) \(\sum f_n(\xi)(z, \xi)^n/n!\). Again it is easily seen that \(T^{n_k} S^k_E \to \text{Id}_Y\) and \(S^k_E \to 0\) pointwise on \(Y\), hence \(T_E\) satisfies the HC with respect to \((n_k)\).

It remains to prove that, when \(d = 1\), \(T_E\) can be extended to a hypercyclic operator on \(H(\mathbb{C})\), and we shall apply Proposition 7. In view of Proposition 8, we only have to construct an operator \(S \in L(F)\) that satisfies the HC with respect to the full sequence. Indeed, then \(T_E \oplus S\) forms a required extension of \(T_E\) as \(T_E\) satisfies the HC with respect to some sequence. We shall prove that \(S\) defined by

\[
\sum_{n \geq 1} p_n(\xi)z^n/n! \mapsto \sum_{n \geq 1} p'_{n+1}(\xi)z^n/n!
\]

(6)

is a required hypercyclic operator. Note that \(\deg p'_{n+1} \leq n - 1\), and it is an easy exercise to show that \(S \in L(F)\). Let \(Z = Y\) be the dense subspace of \(F\) formed by all \(\sum_{n \geq 1} p_n(\xi)z^n/n!\) \(\in F\) where \(p_n = 0\) for all but a finite number of \(n\). It is then obvious that \(S^n \to 0\) pointwise on \(Z\). (In fact, \(S^n f = 0\) for all \(n\) sufficiently large if \(f \in Z\).) Next we define a map \(\rho : P \to P\) on the space \(P\) of one-variable polynomials in the following way. Let \(\rho(z^n) \equiv z^{n+1}/(n+1), n \geq 0\), and then extend \(\rho\) linearly. Note that \(\rho\) forms a right inverse to the differentiation operator \(D\) on \(P\), and \(\rho\) maps polynomials of degree \(\leq n\) to polynomials of degree at most \(n+1\). From this we conclude that \(\sum_{n \geq 1} p_n(\xi)z^n/n! \mapsto \sum_{n \geq 2} \rho(p_{n-1})(\xi)z^n/n!\) defines a right inverse \(R : Y \to Y\) to \(S\). Hence, we only have to prove that \(R^n \to 0\) pointwise on \(Y\). It suffices to prove that \(R^ny_{i,j} \to 0\) for all \(0 \leq i < j\) where \(y_{i,j} \equiv \xi^i z^j \in Y\). We derive that \(R^n y_{i,j} = i! \xi^{i+n} z^{j+n}/[(i+n)!/(j+n)!] \to 0\) \((n \to \infty)\). \(\square\)

**Corollary 5.** The operator \(S\) defined by (6) is hypercyclic on \(F \subseteq H(\mathbb{C})\), and \(\hat{T}_E \equiv T_E \oplus S\) is a hypercyclic extension of \(T_E\). If \(T\) satisfies the HC with respect to \((n_k)\), then so does \(T_E\) (thus \((T_E^n)k\) is hypercyclic). For any hypercyclic vector \(f\) for \(S\) \((T_E^n)_k\) resp.), there exists a hypercyclic vector \(g\) for \(T_E\) \((S\) resp.) such that \(f + g\) is hypercyclic for \(\hat{T}_E\).
Remark 2. From the proof of Theorem 2 we see that the theorem extends in the following way: If \((T_n)_{n \geq 0}\) is an equicontinuous sequence of operators \(T_n \in L(H(\mathbb{C}^d))\) that satisfy the HC with respect to some common sequence \((n_k)\), then \(\sum f_n(\xi)(z, \xi)^n/n! \mapsto \sum (T_n f_n)(\xi)(z, \xi)^n/n!\) defines a hypercyclic operator on \(E\). (Note that equicontinuous means that there for every \(r > 0\) exist \(M, R > 0\) such that \(\|T_n f\|_r \leq M \|f\|_R\) for all \(n\) and \(f \in H\).) In particular, any nontrivial \(\varphi(D)\) satisfies the HC with respect to the full sequence, and we conclude that any sequence \((\varphi_n)_{n \geq 0} \subset \text{Exp} \setminus \mathbb{C}\) such that \(\sup_{n,z} |\varphi_n(z)|e^{-r\|z\|} < \infty\) for some \(r > 0\), defines a hypercyclic operator on \(E\) by \(\sum f_n(\xi)(z, \xi)^n/n! \mapsto \sum (\varphi_n(D) f_n)(\xi)(z, \xi)^n/n!\) on \(E\).

4. Some extensions

The results of Section 2 extend to other spaces, i.e., we may replace \(H\) by some other suitable function or power series space. Indeed, consider for example the space \(C^\infty = C^\infty(\mathbb{R}^d)\) of all complex-valued smooth functions on \(\mathbb{R}^d\) equipped with its usual Fréchet space topology [7, p. 44]. From Paley-Wiener-Schwartz’ Theorem [7, p. 181], we know that the dual of \(C^\infty\) can be identified with the space \(\text{Exp} = \text{Exp}(\mathbb{C}^d)\) of all \(\varphi \in H\) such that (for some \(C, r, n > 0\) \(|\varphi(z)| \leq C(1 + \|z\|)^n e^{r\|z\|}\), via the Fourier-Laplace transform \(\lambda \mapsto \varphi(z) = \lambda(e_z) \in \text{Exp}\). Here \(imz \equiv (imz_i) \in \mathbb{R}^d\) and \(e_z(x) = e_z(z) = e^{-i(x,z)}\), \((x, z) \in \mathbb{R}^d \times \mathbb{C}^d\). Thus \(C^\infty\) and \(\text{Exp}\) form a dual pair and, given \(\varphi \in \text{Exp}\), we define the convolution operator \(\varphi(\partial)f(x) = \langle f, \varphi e_z \rangle\) on \(C^\infty\). It follows that we obtain all the convolution operators on \(C^\infty\) in this way [14, Prop. 2]. In particular, if \(p\) is a polynomial, \(p(\partial)\) is the differential operator obtained by replacing each variable in \(z_j\) in \(p\) by \(i\partial_j = i\partial/\partial x_j\). Moreover, Godefroy and Shapiro’s result extends in the sense that any nontrivial convolution operator \(\varphi(\partial)\) is hypercyclic on \(C^\infty\) [14, Theorem 3]. Now, the key-lemma, Lemma 1, remains true for any kernel \(\text{ker} p(\partial)\), and we have the following analogue of Theorem 1:

Theorem 3. Let \(p = p_1^{r_1} \cdots p_m^{r_m}\) be the factorization of a nonconstant polynomial \(p\) into irreducible factors \(p_i\) with corresponding multiplicities \(r_i \geq 1\). A necessary and sufficient condition for the restriction \(\varphi(\partial) : \text{ker} p(\partial) \rightarrow \text{ker} p(\partial)\) to be hypercyclic is that \(\varphi|_{Z(p_i)}\) is nonconstant for all \(i\).

Theorem 1 (and Theorem 3) extends also naturally to spaces such as the ring, and Fréchet space, \(F\) of formal power series (\(d\) variables, complex coefficients) equipped with the topology of convergence at each coefficient. (Replace \(\varphi\) in Theorem 1 by a nonconstant polynomial, so \(\varphi(D)\) is a well-defined continuous operator on \(F\).)

We conclude by considering another type of extension of our study. In [5] the authors showed that an operator \(T\) acting on a separable Fréchet space is chaotic, in the sense of Devaney, if and only if \(T\) is hypercyclic and has a dense set of periodic points. (Recall that a periodic point is a point for which \(T^n x = x\).)
for some $n$.) In the same paper they extended Proposition 3 by proving that any nontrivial convolution operator on $H(\mathbb{C}^d)$ is in fact chaotic. Thus, we are led to the question: is the restriction $\varphi(D)|_{\ker p(D)}$ of any convolution operator $\varphi(D)$, satisfying the hypothesis of Theorem 1 with respect to $p$, in fact chaotic? We shall not here give a full treatment of this problem, this chaotic problem seems to be more delicate in the sense that multiple factors in the polynomial $p$ affects. However, we prove the following:

**Theorem 4.** Let $p = p_0 \cdots p_m$ be a nonconstant polynomial whose irreducible factors $p_i$ are distinct. Then $\varphi(D)|_{\ker p(D)}$ is chaotic if and only if $\varphi|_{Z(p_i)}$ is nonconstant for all $i$.

**Proof.** We must prove that $\varphi(D)|_{\ker p(D)}$ has a dense set of periodic points provided $\varphi|_{Z(p_i)}$ is nonconstant for all $i$. In particular we have that $d \geq 2$. We know, from the proof of Lemma 1, that $\ker p(D)^{\perp} = \{ \psi : Z(p) \subseteq Z(\psi) \}$. Thus we must show that there exists a set of periodic points for $\varphi(D)$ in $\ker p(D)$ such that if $\psi \in \text{Exp}$ is orthogonal to this set, then $\psi$ vanishes on $Z(p)$. From this point the proof goes very close to that of [5, Theorem 6.2], however, we give some details. Assume first that $d = 2$. Our hypothesis on $\varphi$ implies that for every $i$ there is a point $a_i \in \text{reg} Z(p_i)$ such that $|\varphi(a_i)| = 1$. Since every $\text{reg} Z(p_i)$ forms a one dimensional complex manifold, there exist for every $i$ a neighborhood $U_i$ of $a_i$ in $\text{reg} Z(p_i)$, a domain $\Omega_i \subseteq \mathbb{C}$ and a biholomorphic map $u_i : \Omega_i \rightarrow U_i$ such that $\varphi_i \equiv \varphi \circ u_i \in H(\Omega_i)$. Now, $\varphi_i(\Omega_i)$ meets the unit circle, and we may find an open relatively compact set $G_i \subseteq \Omega_i$ such that $\varphi_i(G_i)$ meets the unit circle. Since a holomorphic map is open, $\varphi_i(G_i)$ contains a nontrivial arc of the unit circle, and hence contains infinitely many roots of unity. Thus, there exists an infinite set $E_i \subseteq G_i$ for which $\varphi_i(E_i) = \varphi(u_i(E))$ is formed by roots of unity. We now claim that $P \equiv \bigcup_i \{ e_a : a \in u_i(E_i) \}$ is a total set of periodic points in $\ker p(D)$ and hence, since linear combinations of periodic points are again periodic, its span is the required dense set of periodic points. Indeed, if $\psi$ is orthogonal to $P$ then $\psi$ vanishes on every $u_i(E_i)$. But, since $G_i$ is relatively compact, $E_i$ has a limit point in $\Omega_i$. Hence $\psi \circ u_i = 0$ and so $\psi$ vanishes on every $\text{reg} Z(p_i)$. Thus $\psi|_{Z(p)} = 0$ and our claim follows.

When $d > 2$ we only have to refine the proof in the same way as in the proof in [5].

**Acknowledgements.** I would like to thank Edgar Lee Stout for his proof of the Sublemma in the proof of Theorem 1.

**References**


School of Mathematical Sciences
Chalmers, SE-412 96, Gteborg

E-mail address: hpe@usi.vxu.se