A CONSTRUCTION OF ONE-FACTORIZATION

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Abstract. In this paper, we construct one-factorizations of given complete graphs of even order. These constructions partition the edges of the complete graph into one-factors and triples. Our new constructions of one-factors and triples can be applied to a recursive construction of Steiner triple systems for all possible orders $\geq 15$.

1. Introduction

A graph $G = (V, E)$ consists of a finite set $V$ of objects called vertices together with a set $E$ of unordered pairs of vertices called edges. A subgraph of a graph $G$ that contains every vertex of $G$ is called a factor (or a spanning subgraph in [10]). A factorization of $G$ is a set of factors of $G$ which are pairwise edge-disjoint (that is, no two have a common edge) and whose union is all of $G$. Since $G$ is a factor of itself, $\{G\}$ is a factorization of $G$ so that every graph has a factorization. However, it is more interesting to consider factorizations in which the factors satisfy certain conditions. For a given graph $G$, a one-factor is a factor which is a regular graph of degree one. In other words, a one-factor is a set of pairwise disjoint edges of $G$ which between them contain every vertex. A one-factorization of $G$ is a partition of all the edges into one-factors each of which is, in its turn, a partition of the set of vertices.

One-factors and one-factorizations of the complete graph of even order $2n$, written as $K_{2n}$, have been studied by several authors in [4, 7, 8]. A cyclic graph of order $2n$ (see [10]) is a graph whose vertices are the integers modulo $2n$ with the property that if $\{x, y\}$ is an edge then so is $\{x + i, y + i\}$ for $1 \leq i \leq 2n - 1$ when $|x - y| \neq n$; for $1 \leq i \leq n - 1$ when $|x - y| = n$. As general backgrounds on one-factorizations of the complete graphs, we refer to [1, 5, 9, 10]. The existence of a one-factor is known for all $K_{2n}$ and it has been known that a one-factorization exists for $K_{2n}$ as well (see [6]). As an application, it is

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well-known that a round robin tournament can be expressed as a proper one-factorization of a complete graph of even order (see \[8, 10\]). In the theory of designs, one-factorization of \(K_{2n}\) has been used for constructing a Steiner triple system of order \(v\) (written as \(STS(v)\)), which is a 2-design with \(v\) points and blocks of size 3, called triples. We sometimes denote a Steiner triple system by \(STS(v) = (V, T)\) for the set of vertices \(V\) and the set of triples. In a given Steiner triple system \(S\), a Steiner triple system \(T\) is called a subsystem of \(S\) if every block of \(T\) is a block of \(S\). In [3], J. Doyen and R. Wilson show that a \(STS(u)\) is a subsystem of \(STS(v)\) for every \(u\) and \(v\) such that \(u, v \equiv 1, 3 \pmod{6}\) and \(v > 2u\). Similar recursive construction method of \(STS(2v + s)\) for \(s = 1, 7\) based on a given \(STS(v)\) can be found in [2, 8, 10], which shows the existence of Steiner triple system for all feasible orders.

In this paper, we give new constructions of one-factorizations of a cyclic complete graph \(K_{2n}\) depending on whether \(2n / \gcd(\alpha, 2n)\) is even or odd, where \(\alpha\) is a positive integer in the set of all edge differences in \(K_{2n}\). For the odd case of \(\alpha\), we next give a factorization by modifying our one-factorization in order to obtain an infinite family of \(STS(v)\) by applying Doyen-Wilson theorem [3] on a recursive construction scheme. Then, we obtain a family of \(STS(v)\)s for all possible orders \(v \equiv 1, 3 \pmod{6}\) derived from our modified one-factorizations for odd \(\alpha\)'s and the one-factorizations given in [10] for even \(\alpha\)'s. These Steiner triple systems are different from the ones given by Bose, seen in [2].

2. Constructions of one-factorization

Let \(K_{2n}\) be the cyclic complete graph whose vertex-set is the additive group of residue classes of integers modulo \(2n\), that is,

\[ \mathbb{Z}_{2n} = \{0, 1, \ldots, 2n - 1\}. \]

For \(\alpha = 1, 2, \ldots, n\), let

\[ E_\alpha = \{i, j\} \mid i - j \equiv \pm \alpha \pmod{2n}. \]

Then \(\{E_\alpha \mid \alpha = 1, 2, \ldots, n\}\) is a partition of the edge-set of \(K_{2n}\) and \(E_\alpha\) is always a one-factor of \(K_{2n}\). In this case, \(\alpha\) is called the difference of an edge \(\{i, j\}\) and \(E_\alpha\) is said to have \(\alpha\)-difference.

Let \(D\) be the set of all differences of \(K_{2n}\). Then \(D = D_e \cup D_o\), where

\[ D_e = \{\alpha \mid 2n / \gcd(\alpha, 2n)\ \text{is even, } 1 \leq \alpha \leq n\} \]

and

\[ D_o = \{\alpha \mid 2n / \gcd(\alpha, 2n)\ \text{is odd, } 1 \leq \alpha \leq n\}. \]

Note that if \(\alpha\) is odd, then \(\alpha\) must be in \(D_e\).

We now give a construction of one-factors of \(K_{2n}\) as follows.
Construction 1. (1) Let $\alpha \in D_o$, $\alpha \not= n$ and let $g_\alpha = \gcd(\alpha, 2n)$. Then, as a graph, $E_\alpha$ consists of $g_\alpha$ component cycles of length $\frac{2n}{g_\alpha}$ (see [10]) which means

$$E_\alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{j + \alpha i, j + \alpha + \alpha i\} \mid i = 0, 1, \ldots, \frac{2n}{g_\alpha} - 1 \right\}.$$  

By taking alternate members of the cycles, we have two one-factors $F_\alpha$ and $F_\alpha + \alpha$ of the complete graph $K_{2n}$ as follows:

$$F_\alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{j + \alpha i, j + \alpha + \alpha i\} \mid i = 0, 2, \ldots, \frac{2n}{g_\alpha} - 2 \right\},$$  

$$F_\alpha + \alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{j + \alpha i, j + \alpha + \alpha i\} \mid i = 1, 3, \ldots, \frac{2n}{g_\alpha} - 1 \right\}.$$  

That is,

$$F_\alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{j + 2\alpha i, j + 2\alpha + \alpha\} \mid i = 0, 1, \ldots, \frac{n}{g_\alpha} - 1 \right\},$$  

$$F_\alpha + \alpha = \bigcup_{j=0}^{g_\alpha-1} \left\{ \{j + 2\alpha i + \alpha, j + 2\alpha + 2\alpha\} \mid i = 0, 1, \ldots, \frac{n}{g_\alpha} - 1 \right\}.$$  

(2) Let $\alpha \in D_o$ and $g_\alpha = \gcd(\alpha, 2n)$. For each $j = \frac{2ak}{g_\alpha}$, $l = 0, 1, \ldots, \frac{2n}{g_\alpha} - 1$ and each $k = 1, 2, \ldots, \frac{2n}{g_\alpha} - 1$, define

$$A_\alpha(j, k) = \bigcup_{i=0}^{\frac{2ak}{g_\alpha}-1} \left\{ \left\{ \frac{g_\alpha k}{2} + i + j, 2n - \frac{g_\alpha k}{2} + i + j \right\}, \{i + j, n + i + j\} \right\}.$$  

Then for each $j = \frac{2ak}{g_\alpha}$, $l = 0, 1, \ldots, \frac{2n}{g_\alpha} - 1$,

$$F_\alpha + j = \bigcup_{k=1}^{\frac{2n}{g_\alpha}-1} A_\alpha(j, k)$$  

is a one-factor of $K_{2n}$.

The one-factors shown in (2) of Construction 1 satisfies the following proposition.

Proposition 2. Let $K_v$ be the cyclic complete graph. If $v = 2^r p$ for some odd number $p > 1$ and some $r \in \mathbb{N}$, then

$$\bigcup \{F_{2^r} + j \mid j = 2^r - l, 0 \leq l \leq p - 1\} = \bigcup \{E_\alpha \mid \alpha \in D_o \cup \left\{ \frac{n}{2} \right\} \}.$$  

Proof. Let $v = 2^r p$ for some odd number $p > 1$ and some $r \in \mathbb{N}$. Then $D_o = \{2^r, 2 \times 2^r, 3 \times 2^r, \ldots, \frac{p-1}{2} \times 2^r\}$. For a difference $\alpha = 2^r$, let $g_\alpha = \gcd(2^r, 2^r p) = 2^r$. For each $j = 2^r - l$, $l = 0, 1, \ldots, p - 1$ we define $F_{2^r} + j$ as (2) in Construction 1. Then for each $j = 2^r - l$ and $0 \leq l \leq p - 1$, $F_{2^r} + j$
is one-factor and the difference of each edge of $F_{2^r} + j$ is one of $2^r$, $2 \times 2^r$, $3 \times 2^r, \ldots, \left(\frac{p - 1}{2}\right) \times 2^r$, and $2^{r-1}p$. Hence there are $p$ one-factors which are pairwise edge-disjoint and the set of differences of all edges of $F_{2^r} + j$ is equal to $D_0 \cup \left\{ \frac{v}{2} \right\}$, so that

$$\bigcup \left\{ F_{2^r} + j \mid j = 2^{r-1}l, \ 0 \leq l \leq p - 1 \right\} \subseteq \bigcup \left\{ E_\alpha \mid \alpha \in D_0 \cup \left\{ \frac{v}{2} \right\} \right\}.$$ 

Note that the cardinal number of the right side is $(p - 1)v/2 + v/2$ from the definition of $E_\alpha$. Since the cardinal number of left side is $pv/2$, we have the equality of the statement. This completes the proof.

Note that if $v = 2n$ is a power of 2, then $D_0 = \emptyset$ so that $D = D_e$. The following theorem, which is a construction of one-factorization for $v = 2^r$, is known in [4, 7, 8, 10].

**Theorem 3.** If $v = 2^r$ for some $r \in \mathbb{N}$, then the cyclic graph $K_v$ has a one-factorization

$$\mathfrak{F}_{2^r} = \left\{ F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2} \right\} \cup \left\{ E_\frac{v}{2} \right\},$$

where $F_\alpha, F_\alpha + \alpha$ and $E_\frac{v}{2}$ are defined in (1) of Construction 1.

We now have a new construction of one-factorizations of the cyclic complete graph $K_{2^n}$ from Proposition 2 and Theorem 3.

**Theorem 4.** If $v = 2^r p$ for some $r \in \mathbb{N}$ and odd number $p$, then the cyclic graph $K_v$ has a one-factorization

$$\mathfrak{F}_{2^r p} = \left\{ F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2} \right\} \cup \left\{ F_{2^r} + 2^{r-1}l \mid l = 0, 1, \ldots, p - 1 \right\},$$

where $F_\alpha, F_\alpha + \alpha$ and $F_{2^r} + 2^{r-1}l$ ($l = 0, 1, \ldots, p - 1$) are one-factors defined in Construction 1.

**Proof.** If $p = 1$, it is obvious from Theorem 3. Now we suppose $v = 2^r p$ for $p > 1$ and $r \in \mathbb{N}$ so that $D = \{1, 2, 3, \ldots, 2^{r-1}p\}$.

For $\alpha \in D$, let $g_\alpha = \gcd(\alpha, v)$. For $\alpha = 2^r$, by Proposition 2, note that

$$\left\{ F_{2^r} + 2^{r-1}l \mid l = 0, 1, \ldots, p - 1 \right\}$$

consists of $p$ edge-disjoint one-factors of $K_v$ and

$$\bigcup \left\{ F_{2^r} + 2^{r-1}l \mid l = 0, 1, \ldots, p - 1 \right\} = \bigcup \left\{ E_\alpha \mid \alpha \in D_0 \cup \left\{ \frac{v}{2} \right\} \right\}.$$

For each $\alpha \in D_e - \left\{ \frac{v}{2} \right\}$, the one-factors $F_\alpha$ and $F_\alpha + \alpha$ have $\alpha$-difference by (1) of Construction 1. Since

$$D_0 \cup \left\{ \frac{v}{2} \right\} = \left\{ 2^r, 2 \times 2^r, 3 \times 2^r, \ldots, \left(\frac{p - 1}{2}\right) \times 2^r, 2^{r-1}p \right\},$$

the cardinal number of $D_e - \left\{ \frac{v}{2} \right\}$ is

$$|D - \left\{ 2^r, 2 \times 2^r, 3 \times 2^r, \ldots, \left(\frac{p - 1}{2}\right) \times 2^r, 2^{r-1}p \right\}| = 2^{r-1}p - \frac{p - 1}{2} - 1.$$
Thus there are

\[ \left( 2^{r-1}p - \frac{p-1}{2} - 1 \right) \times 2 = 2^r p - p - 1 \]

one-factors of \( K_v \) and the set of all edges of \( \{ F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2} \} \) is equal to the set \( \bigcup \{ E_\alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2} \} \). Hence the total number of one-factors of \( K_v \) is \( p + 2^r p - p - 1 = 2^r p - 1 = v - 1 \) and the set of all edges of \( \mathcal{F}_{2^r} \) is equal to the set \( \bigcup \{ E_\alpha \mid \alpha \in D \} \), where

\[ \mathcal{F}_{2^r} = \{ F_{2^r} + 2^{r-1}l \mid l = 0, 1, \ldots, p - 1 \} \cup \{ F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e, \alpha \neq \frac{v}{2} \}. \]

Therefore, the cyclic complete graph \( K_v \) has a one-factorization \( \mathcal{F}_{2^r} \) which consists of \( v - 1 \) one-factors.

We now apply the previous Theorem 4 to a complete graph \( K_{10} \).

**Example 5.** Consider the cyclic complete graph \( K_{10} \). Then \( D_e = \{1, 3, 5\} \), and \( D_o = \{2, 4\} \).

If \( \alpha = 1 \in D_e \), then \( g_\alpha = \gcd(1, 10) = 1 \) and \( j = 0 \); so, we have

\[
F_1 = \{\{2i, 2i + 1\} \mid i = 0, 1, 2, 3, 4\} = \{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}\},
F_1 + 1 = \{\{2i + 1, 2i + 2\} \mid i = 0, 1, 2, 3, 4\} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 0\}\}.
\]

If \( \alpha = 3 \in D_e \), then \( g_\alpha = \gcd(3, 10) = 1 \) and \( j = 0 \); so, we have

\[
F_3 = \{\{6i, 6i + 3\} \mid i = 0, 1, 2, 3, 4\} = \{\{0, 3\}, \{6, 9\}, \{2, 5\}, \{8, 1\}, \{4, 7\}\},
F_3 + 3 = \{\{6i + 3, 6i + 6\} \mid i = 0, 1, 2, 3, 4\} = \{\{3, 6\}, \{9, 2\}, \{5, 8\}, \{1, 4\}, \{7, 0\}\}.
\]

Now, since \( 10 = 2^1 \times 5 \) for the difference \( \alpha = 2 \) we have \( g_2 = \gcd(2, 10) = 2 \) and \( j = 0, 1, 2, 3, 4 \). Thus

\[
F_2 + 0 = \{\{k, (10 - k)\}, \{0, 5\} \mid k = 1, 2, 3, 4\} = \{\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{0, 5\}\},
F_2 + 1 = \{\{k + 1, (10 - k) + 1\}, \{0 + 1, 5 + 1\} \mid k = 1, 2, 3, 4\} = \{\{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}, \{1, 6\}\},
F_2 + 2 = \{\{k + 2, (10 - k) + 2\}, \{0 + 2, 5 + 2\} \mid k = 1, 2, 3, 4\} = \{\{3, 1\}, \{4, 10\}, \{5, 9\}, \{6, 8\}, \{2, 7\}\},
F_2 + 3 = \{\{k + 3, (10 - k) + 3\}, \{0 + 3, 5 + 3\} \mid k = 1, 2, 3, 4\} = \{\{4, 2\}, \{5, 1\}, \{6, 10\}, \{7, 9\}, \{3, 8\}\},
F_2 + 4 = \{\{k + 4, (10 - k) + 4\}, \{0 + 4, 5 + 4\} \mid k = 1, 2, 3, 4\} = \{\{5, 3\}, \{6, 2\}, \{7, 1\}, \{8, 10\}, \{4, 9\}\}.
\]

Hence \( \{ F_\alpha, F_\alpha + \alpha \mid \alpha = 1, 3 \} \cup \{ F_2 + j \mid j = 0, 1, 2, 3, 4 \} \) forms a one-factorization of \( K_{10} \) which consists of 9 one-factors of \( K_{10} \).

By modifying one-factors in Construction 1, we obtain a new factorization consisting of one-factors and triples for \( v \equiv 0 \pmod{6} \) and \( v > 3 \).
Construction 6. Take $\mathcal{F}_{2^p}$ the one-factorization of $K_v$ stated in Theorem 4.

If $v \equiv 0 \pmod{6}$, then for each $j = 2^{r-1}l$ ($l = 0, 1, \ldots, p - 1$) and each $k = 1, 2, \ldots, p - 1$, define

$$A_{2^r}(j, k) = A_{2^r}(j, k) - \bigcup_{i=0}^{2^a-1} \left\{ \left\{ \frac{v}{6} + j + i, \frac{5v}{6} + j + i \right\}, \left\{ \frac{v}{3} + j + i, \frac{2v}{3} + j + i \right\} \right\}$$

$$\bigcup_{i=0}^{2^a-1} \left\{ \left\{ \frac{v}{6} + j + i, \frac{v}{3} + j + i \right\}, \left\{ \frac{5v}{6} + j + i, \frac{2v}{3} + j + i \right\} \right\}.$$  

For each $j = 2^{r-1}l$ ($l = 0, 1, \ldots, p - 1$), let

$$F_{2^r}^* + j = \bigcup_{k=1}^{p-1} A_{2^r}(j, k)$$

be a one-factor of the cyclic complete graph $K_v$. Define a $\frac{v}{3}$-set of triples as

$$T_{\frac{v}{3}} = \left\{ \left\{ i, \frac{v}{3} + i, \frac{2v}{3} + i \right\} \mid i = 0, 1, \ldots, \frac{v}{3} - 1 \right\}.$$  

Let

$$\mathcal{F}_{2^*}^* = \left\{ F_{2^*}^* + 2^{r-1}l \mid l = 0, 1, \ldots, p - 1 \right\}$$

and

$$\mathcal{F}_{2}^* = \left\{ F_{\alpha}, F_{\alpha} + \alpha \mid \alpha \in D_e - \left\{ \frac{v}{6}, \frac{v}{2} \right\} \right\},$$

where $F_{\alpha}$ and $F_{\alpha} + \alpha$ are one-factors defined in Construction 1. Finally, we define $\mathcal{F}_v^*$ to be

$$\mathcal{F}_v^* = \mathcal{F}_{2^*}^* \cup \mathcal{F}_{2}^* \cup T_{\frac{v}{3}}$$

which consists of one-factors and triples.

The set $\mathcal{F}_v^*$ defined in Construction 6 satisfies the following theorem.

Theorem 7. If $v \equiv 0 \pmod{6}$, then the edges of the cyclic $K_v$ are partitioned into $\frac{v}{3}$ triples and $v - 3$ one-factors.

Proof. By the definition in Construction 6, the set $\mathcal{F}_{2^*}^*$ consists of $p$ one-factors in which each edge of the set $\bigcup \{ E_{\alpha} \mid \alpha \in D_e - \left\{ \frac{v}{6}, \frac{v}{2} \right\} \}$ occurs exactly once. Note that for each $\alpha \in D_e - \left\{ \frac{v}{6}, \frac{v}{2} \right\}$, there are two edge-disjoint one-factors. Thus $\mathcal{F}_{2}^*$ defined in Construction 6 consists of

$$2 \times (2^{r-1}p - \frac{p-1}{2} - 1) - 1 = 2^r p - p - 3$$

pairwise edge-disjoint one-factors, and the set of all edges of $\mathcal{F}_v^*$ is equal to the set $\bigcup \{ E_{\alpha} \mid \alpha \in D_e - \left\{ \frac{v}{6}, \frac{v}{2} \right\} \}$. Hence the total number of pairwise edge-disjoint one-factors is

$$p + (2^r p - p - 3) = v - 3.$$
From the definition of $T_{\frac{2}{3}}$ in Construction 6, $T_{\frac{2}{3}}$ consists of $\frac{2}{3}$ triples in which each edge of $E_{\frac{2}{3}}$ occurs exactly once.

From Construction 6, we have the following example for $K_{12}$.

**Example 8.** Consider the cyclic complete graph $K_{12}$. Firstly we have an one-factorization which consists of 11 one-factors from Construction 1. Then $D_e = \{1,2,3,5,6\}$, and $D_o = \{4\}$.

If $\alpha = 1 \in D_e$, then we have

$$F_1 = \{\{0,1\}, \{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}, \{10,11\}\},$$

$$F_1 + 1 = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}, \{11,0\}\}.$$  

If $\alpha = 2 \in D_e$, then we have

$$F_2 = \{\{0,2\}, \{4,6\}, \{8,10\}, \{1,3\}, \{5,7\}, \{9,11\}\},$$

$$F_2 + 2 = \{\{2,4\}, \{6,8\}, \{10,0\}, \{3,5\}, \{7,9\}, \{11,1\}\}.$$  

If $\alpha = 3 \in D_e$, then we have

$$F_3 = \{\{0,3\}, \{6,9\}, \{1,4\}, \{7,10\}, \{2,5\}, \{8,11\}\},$$

$$F_3 + 3 = \{\{3,6\}, \{9,0\}, \{4,7\}, \{10,1\}, \{5,8\}, \{11,2\}\}.$$  

If $\alpha = 5 \in D_e$, then we have

$$F_5 = \{\{0,5\}, \{10,3\}, \{8,1\}, \{6,11\}, \{4,9\}, \{2,7\}\},$$

$$F_5 + 5 = \{\{5,10\}, \{3,8\}, \{1,6\}, \{11,4\}, \{9,2\}, \{7,0\}\}.$$  

Now, for the differences $\alpha = 4 \in D_o$ we have 3 one-factors

$$F_4 + 0 = \{\{2,10\}, \{0,6\}, \{3,11\}, \{1,7\}, \{4,8\}, \{5,9\}\},$$

$$F_4 + 2 = \{\{4,0\}, \{2,8\}, \{5,1\}, \{3,9\}, \{6,10\}, \{7,11\}\},$$

$$F_4 + 4 = \{\{6,2\}, \{4,10\}, \{7,3\}, \{5,11\}, \{8,0\}, \{9,1\}\}$$

which consist of edges with differences $\alpha = 4,6$. Hence

$$\{F_\alpha, F_\alpha + \alpha \mid \alpha = 1,2,3,5\} \cup \{F_4 + j \mid j = 0,2,4\}$$

forms a one-factorization of $K_{12}$ which consists of 11 one-factors of $K_{12}$.

Applying Construction 6 to this one-factorization, we have 9 one-factors and 4 triples as follows.

$$F_4^* + 0 = \{\{2,4\}, \{10,8\}, \{3,5\}, \{11,9\}, \{0,6\}, \{1,7\}\},$$

$$F_4^* + 2 = \{\{4,6\}, \{0,10\}, \{5,7\}, \{1,11\}, \{2,8\}, \{3,9\}\},$$

$$F_4^* + 4 = \{\{6,8\}, \{2,0\}, \{7,9\}, \{3,1\}, \{4,10\}, \{5,11\}\},$$

$$F_4^* + 1, F_1^*, F_3^* + 3, F_5^* + 5 \text{ and }$$

$$T_4 = \{\{0,4,8\}, \{1,5,9\}, \{2,6,9\}, \{3,7,11\}\}.$$

We define

$$\mathcal{F}_1^* = \{F_4^* + 0, F_4^* + 2, F_4^* + 4\},$$

$$\mathcal{F}_2^* = \{F_1^*, F_1^* + 1, F_3^*, F_3^* + 3, F_5^*, F_5^* + 5\}.$$  

and

$$\mathcal{F}_{12}^* = \mathcal{F}_1^* \cup \mathcal{F}_2^* \cup T_4.$$
Then $\mathcal{F}^*_1$ is a factorization which consists of 9 one-factors and 4 triples from Construction 6.

Next, by modifying one-factorization in Construction 1, we obtain a new factorization consisting of one-factors and triples for $v \equiv 0 \pmod{2}$, $v > 7$ and $v \neq 10$ as follows.

**Construction 9.** Take $\mathcal{F}^{2p}$ the one-factorization of $K_v$ stated in Theorem 4. If $v \equiv 0 \pmod{2}$, $v > 7$ and $v \neq 10$, then for each $j = 2^{r-1}l \ (l = 0, 1, \ldots, p-1)$ we have the following one-factor,

$$F^*_2 + j = (F_2 + j - \{1 + j, (v - 1) + j\}, \{(\frac{v}{2} - 1) + j, (\frac{v}{2} + 1) + j\}) \cup \{1 + j, (\frac{v}{2} - 1) + j\}, \{(v - 1) + j, (\frac{v}{2} + 1) + j\}.$$

Define a $v$-set of triples as

$$T_{1,2,3} = \{i, i + 1, i + 3 \mid i = 0, 1, 2, \ldots, v - 1\}.$$

Let

$$\mathcal{F}_1 = \{F_2 + 2^{r-1}l \mid r > 1, \ l = 0, 1, \ldots, p - 1\},$$

$$\mathcal{F}^*_1 = \{F_2 + j \mid j = 0, 1, \ldots, \frac{v}{2} - 1\},$$

$$\mathcal{F}^*_2 = \{F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e - \{1, 3, \frac{v}{2} - 2, \frac{v}{2}\}\}, \text{ and}$$

$$\mathcal{F}^*_3 = \{F_\alpha, F_\alpha + \alpha \mid \alpha \in D_e - \{1, 2, 3, \frac{v}{2}\}\}.$$  

We define $\mathcal{F}^*_v$ to be

$$\mathcal{F}^*_v = \begin{cases} \mathcal{F}^*_1 \cup \mathcal{F}^*_2 \cup T_{1,2,3} & \text{if } v = 2p, v \neq 10, \\ \mathcal{F}^*_1 \cup \mathcal{F}^*_3 \cup T_{1,2,3} & \text{if } v = 2^3p, r > 1 \end{cases}$$

which consists of one-factors and triples.

The set $\mathcal{F}^*_v$ defined in Construction 9 satisfies the following theorem.

**Theorem 10.** If $v \equiv 0 \pmod{2}$ and $v > 7, v \neq 10$, then the edges of the cyclic $K_v$ are partitioned into $v$ triples and $v - 7$ one-factors.

**Proof.** For $v = 2^rp$, we have two cases: $r = 1$ and $r > 1$. If $r = 1$ and $v \neq 10$, then $2 \in D_e$. From the definition of $\mathcal{F}^*_1$ in Construction 9, $\mathcal{F}^*_1$ consists of $p$ pairwise edge-disjoint one-factors and the set of all edges of $\mathcal{F}^*_1$ is equal to the set

$$\bigcup \{E_\alpha \mid \alpha \in D_e \cup \{\frac{v}{2}, 2, \frac{v}{2}\} - \{2\}\}.$$  

Note that for each $\alpha \in D_e - \{1, 3, \frac{v}{2} - 2, \frac{v}{2}\}$, there are edge-disjoint two one-factors $F_\alpha, F_\alpha + \alpha$. Thus $\mathcal{F}^*_2$ defined in Construction 9 contains $2 \times (2^{r-1}p - \frac{v-1}{2} - 1 - 3)$ pairwise edge-disjoint one-factors and the set of all edges of $\mathcal{F}^*_2$ is equal to

$$\bigcup \{E_\alpha \mid \alpha \in D_e - \{1, 3, \frac{v}{2} - 2, \frac{v}{2}\}\}.$$  

Hence for the case $r = 1$, $\mathcal{F}^*_1 \cup \mathcal{F}^*_2$ consists of

$$p + 2 \times (2^{r-1}p - \frac{v-1}{2} - 1 - 3) = v - 7.$$
pairwise edge-disjoint one-factors in which each edge of the set

\[ \bigcup \{ E_\alpha \mid \alpha \in D - \{1, 2, 3\} \} \]

appears exactly once.

Now, we suppose \( r > 1 \); then, \( 2 \in D_e \). From the definition of \( \mathfrak{F}_3^* \) in Construction 9, there are \( 2 \times (2^{r-1}p - \frac{p-1}{2} - 1 - 3) = 2^rp - p - 7 \) one-factors in which every edge of

\[ \bigcup \{ E_\alpha \mid \alpha \in D_e - \{1, 2, 3, \frac{v}{2}\} \} \]

appears exactly once. From Proposition 2, \( \mathfrak{F}_1 \) defined in Construction 9 consists of \( p \) one-factors in which each edge of \( \bigcup \{ E_\alpha \mid \alpha \in D_o \cup \{\frac{v}{2}\} \} \) occurs exactly once. Hence, for \( r > 1 \), \( \mathfrak{F}_1 \cup \mathfrak{F}_3^* \) consists of

\[ p + (2^kp - p - 7) = v - 7 \]

one-factors in which all edges of \( \bigcup \{ E_\alpha \mid \alpha \in D - \{1, 2, 3\} \} \) appears exactly once. Then, from the definition of \( T_{1,2,3} \), we have \( v \) edge-disjoint triples and the set of all edges of them is equal to \( \{ E_\alpha \mid \alpha = 1, 2, 3 \} \). In all, for \( v \equiv 0 \pmod{2} \), \( v > 7 \) and \( v \neq 10 \),

\[ \mathfrak{F}_v^* = \begin{cases} \mathfrak{F}_1^* \cup \mathfrak{F}_2^* \cup T_{1,2,3} & \text{if } v = 2p, v \neq 10, \\ \mathfrak{F}_1 \cup \mathfrak{F}_3^* \cup T_{1,2,3} & \text{if } v = 2^rp \end{cases} \]

consists of \( v - 7 \) one-factors and \( v \) triples in which each edge of \( K_v \) appears exactly once.

Note that Construction 6 and 9 directly imply the Theorem 7 and 10, respectively, which can be seen in [2, 7, 9].

Now we remark the excluded case that \( v = 10 \) from Construction 9. In \( K_{10} \), we have three one-factors and ten triples as follows;

\[
\begin{align*}
\{ (0,5), (1,7), (2,6), (3,9), (4,8) \}, \\
\{ (1,6), (2,8), (3,7), (4,0), (5,9) \}, \\
\{ (2,7), (3,8), (4,9), (0,6), (1,5) \}, \text{ and} \\
\{ \{i, i+1, i+3\} \mid i = 0, 1, 2, \ldots, 10 - 1 \}.
\end{align*}
\]

Including this construction for \( K_{10} \), we finally have the following theorem.

**Theorem 11.** If \( v \equiv 0 \pmod{2} \) and \( v > 7 \), then the edges of the cyclic \( K_v \) are partitioned into \( v \) triples and \( v - 7 \) one-factors.

We remark that the result in Theorem 11 can be also seen in [2, 7, 9] with different approaches.

As an example of Construction 9, we have the following factorization of \( K_{14} \).

**Example 12.** For the cyclic complete graph \( K_{14} \), we have a factorization which consists of 7 one-factors and 14 triples.
From (2) in Construction 1, we first have 7 one-factors

\[ F_2^* + 0 = \{\{1, 13\}, \{2, 12\}, \{3, 11\}, \{4, 10\}, \{5, 9\}, \{6, 8\}, \{0, 7\}\}, \]
\[ F_2^* + 1 = \{\{2, 0\}, \{3, 13\}, \{4, 12\}, \{5, 11\}, \{6, 10\}, \{7, 9\}, \{1, 8\}\}, \]
\[ F_2^* + 2 = \{\{3, 1\}, \{4, 0\}, \{5, 13\}, \{6, 12\}, \{7, 11\}, \{8, 10\}, \{2, 9\}\}, \]
\[ F_2^* + 3 = \{\{4, 2\}, \{5, 1\}, \{6, 0\}, \{7, 13\}, \{8, 12\}, \{9, 11\}, \{3, 10\}\}, \]
\[ F_2^* + 4 = \{\{5, 3\}, \{6, 2\}, \{7, 1\}, \{8, 0\}, \{9, 13\}, \{10, 12\}, \{4, 11\}\}, \]
\[ F_2^* + 5 = \{\{6, 4\}, \{7, 3\}, \{8, 2\}, \{9, 1\}, \{10, 0\}, \{11, 13\}, \{5, 12\}\}, \]
\[ F_2^* + 6 = \{\{7, 5\}, \{8, 4\}, \{9, 3\}, \{10, 2\}, \{11, 1\}, \{12, 0\}, \{6, 13\}\} \]

for the edge difference \(\alpha \in D_o \cup \{7\} = \{2, 4, 6\} \cup \{7\}\) and 6 one-factors

\[ F_1^*, F_1^* + 1, F_3^*, F_3^* + 3, F_5^*, F_5^* + 5 \]

for the edge difference \(\alpha \in D_o - \{7\} = \{1, 3, 5, 7\} - \{7\}\).

By modifying these 13 one-factors, we construct new factorization consisting of 7 one-factors and 14 triples as follows.

\[ F_2^* + 0 = \{\{1, 6\}, \{2, 12\}, \{3, 11\}, \{4, 10\}, \{5, 9\}, \{13, 8\}, \{0, 7\}\}, \]
\[ F_2^* + 1 = \{\{2, 7\}, \{3, 13\}, \{4, 12\}, \{5, 11\}, \{6, 10\}, \{0, 9\}, \{1, 8\}\}, \]
\[ F_2^* + 2 = \{\{3, 8\}, \{4, 0\}, \{5, 13\}, \{6, 12\}, \{7, 11\}, \{1, 10\}, \{2, 9\}\}, \]
\[ F_2^* + 3 = \{\{4, 9\}, \{5, 1\}, \{6, 0\}, \{7, 13\}, \{8, 12\}, \{2, 11\}, \{3, 10\}\}, \]
\[ F_2^* + 4 = \{\{5, 10\}, \{6, 2\}, \{7, 1\}, \{8, 0\}, \{9, 13\}, \{3, 12\}, \{4, 11\}\}, \]
\[ F_2^* + 5 = \{\{6, 11\}, \{7, 3\}, \{8, 2\}, \{9, 1\}, \{10, 0\}, \{4, 13\}, \{5, 12\}\}, \]
\[ F_2^* + 6 = \{\{7, 12\}, \{8, 4\}, \{9, 3\}, \{10, 2\}, \{11, 1\}, \{5, 0\}, \{6, 13\}\}, \]
\[ T_{1,2,3}^* = \{\{i, i + 1, i + 3\} \mid i = 0, 1, \ldots, 13\}. \]

We define

\[ \mathfrak{F}^*_1 = \{F_2^* + j \mid j = 0, 1, \ldots, 6\}, \]
\[ \mathfrak{F}^*_2 = \emptyset. \]

Then we have a factorization

\[ \mathfrak{F}^*_s = \mathfrak{F}^*_1 \cup \mathfrak{F}^*_2 \cup T_{1,2,3} \]

which consists of 7 one-factors and 14 triples.

We now apply our one-factors and triples given from Constructions 1, 6, and 9 to the well-known recursive construction of STS suggested by J. Doyen and R. Wilson [3] described as follows.

**Theorem 13.** If there is a STS(v), then there is a STS(2v+s) with the original STS(v) as a subsystem for s = 1, or 7. If v ≡ 3 (mod 6) and s = 3, then there is a STS(2v+s) with the original STS(v) as a subsystem.

Theorem 13 (also shown in [2, 8, 10]) guarantees that STS(2v+s) is obtained from combining the given subsystem STS(v) with our one-factorization of K_v+s for each s = 1, 7, and STS(2v+s) is also guaranteed for the case when s = 3 and v ≡ 3 (mod 6).


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