NORM ESTIMATES AND UNIVALENCE CRITERIA FOR MEROMORPHIC FUNCTIONS

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Abstract. Norm estimates of the pre-Schwarzian derivatives are given for meromorphic functions in the outside of the unit circle. We deduce several univalence criteria for meromorphic functions from those estimates.

1. Introduction

Let \( \mathcal{A} \) denote the set of analytic functions \( f \) in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) normalized so that \( f(0) = 0 \) and \( f'(0) = 1 \). The set \( \mathcal{S} \) of univalent functions in \( \mathcal{A} \) has been intensively studied by many authors. It is well recognized that the set \( \Sigma \) of univalent meromorphic functions \( F \) in the domain \( \Delta = \{ \zeta : |\zeta| > 1 \} \) plays an indispensable role in the study of \( \mathcal{S} \).

In parallel with the analytic case, we consider the set \( \mathcal{M} \) of meromorphic functions in \( \Delta \) with the expansion (1.1) around \( \zeta = \infty \). For some technical reason, we also consider the set \( \mathcal{M}_n \) of functions \( F \) in \( \Sigma \) of the form

\[
F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}
\]

(1.1)

for each nonnegative integer \( n \). Note that \( \mathcal{M}_0 = \mathcal{M} \).

Practically, it is an important problem to determine univalence of a given function in \( \mathcal{A} \) or in \( \mathcal{M} \). The best known conditions for univalence are probably those involving pre-Schwarzian or Schwarzian derivatives, which are defined by

\[
T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2.
\]

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We define quantities for functions \( f \in \mathcal{A} \) and \( F \in \mathcal{M} \) by
\[
B(f) = \sup_{|z| < 1} (1 - |z|^2)|zT_f(z)|,
\]
\[
B^*(F) = \sup_{|\zeta| > 1} (|\zeta|^2 - 1)|\zeta T_F(\zeta)|,
\]
\[
N(f) = \sup_{|z| < 1} (1 - |z|^2)^2|S_f(z)|,
\]
\[
N^*(F) = \sup_{|\zeta| > 1} (|\zeta|^2 - 1)^2|S_F(\zeta)|.
\]

Note that these quantities may take \( \infty \) as their values. For example, if \( F \) has a pole at a finite point, then \( B^*(F) = \infty \). Those functions with finite norms constitute complex Banach spaces, which play a fundamental role in the universal Teichmüller space. See [19] for a survey on the universal Teichmüller space.

If \( f \in \mathcal{A} \) and \( F \in \mathcal{M} \) have the relation \( f(z) = 1/F(1/z) \), then we can easily see that the relation
\[
(1 - |z|^2)^2S_f(z) = (|\zeta|^2 - 1)^2S_F(\zeta)
\]
holds for \( z = 1/\zeta \). In particular, we have \( N(f) = N^*(F) \).

Nehari [18] proved the following univalence criteria except for the quasiconformal extension property, which is due to Ahlfors and Weill [1].

**Theorem A.** Every \( f \in \mathcal{I} \) satisfies \( N(f) \leq 6 \). Conversely, if \( f \in \mathcal{A} \) satisfies \( N(f) \leq 2 \), then \( f \) must be univalent. Moreover, if \( N(f) \leq 2k < 2 \), then \( f \) extends to a \( k \)-quasiconformal mapping of the extended plane. The constants 6 and 2 are best possible. The same is true for meromorphic \( F \).

Here and hereafter, a quasiconformal mapping \( g \) is called \( k \)-quasiconformal if its Beltrami coefficient \( \mu = g_\bar{z}/g_z \) satisfies \( \|\mu\|_\infty \leq k \). An extensive survey on those univalent functions in \( \mathcal{I} \) or \( \Sigma \) which extend to quasiconformal mappings of the Riemann sphere was recently supplied by Krushkal [16].

Though \( z f'(z)/f(z) = \zeta F'(\zeta)/F(\zeta) \), there is no such a simple relation between \( zT_f(z) \) and \( \zeta T_F(\zeta) \), and thus, between \( B(f) \) and \( B^*(F) \) for \( f(z) = 1/F(\zeta) \), \( \zeta = 1/z \). Indeed, we have the formula
\[
F'(\zeta) = \left( \frac{z}{f(z)} \right)^2 f'(z),
\]
which leads to
\[
\frac{\zeta F''(\zeta)}{F'(\zeta)} = 2 \left( 1 - \frac{zf'(z)}{f(z)} \right) + \frac{zf''(z)}{f'(z)}.
\]
Nevertheless, it is rather surprising that formally the same conclusion can be deduced for \( f \) and \( F \). Compare Theorem B with Theorem C.
Theorem B. Every $f \in \mathcal{S}$ satisfies $B(f) \leq 6$. Conversely, if $f \in \mathcal{A}$ satisfies $B(f) \leq 1$, then $f \in \mathcal{S}$. Moreover, if $B(f) \leq k < 1$, then $f$ extends to a $k$-quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.

The sufficiency of univalence and quasiconformal extendibility is due to Becker [7]. The sharpness of the constant 1 is due to Becker and Pommerenke [9]. The sharp inequality $B(f) \leq 6$ follows from a standard inequality appearing in coefficient estimation (see, e.g., [10, Theorem 2.4]).

Theorem C. Every $F \in \Sigma$ satisfies $B^*(F) \leq 6$. Conversely, if $F \in \mathcal{M}$ satisfies $B^*(F) \leq 1$, then $F \in \Sigma$. Moreover, if $B^*(F) \leq k < 1$, then $F$ extends to a $k$-quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.

The sufficiency of univalence and quasiconformal extendibility is due to Becker [8]. The sharpness of the constant 1 is again due to Becker and Pommerenke [9]. On the other hand, the estimate $B^*(F) \leq 6$ lies deeper. Avhadiev [4] first showed the sharp inequality $B^*(F) \leq 6$ by appealing to Goluzin’s inequality (see [11, p. 139]).

Note that many authors use a different norm for the pre-Schwarzian derivative of $f \in \mathcal{A}$, namely, $\|T_f\| = \sup_{|z| < 1} (1 - |z|^2)|T_f(z)|$, see [12], [13], [15] and [20]. By definition, we observe $B(f) \leq \|T_f\|$. The norm $\|T_f\|$ has some advantage such as invariance properties. For meromorphic functions, however, the corresponding norm is not suitable (see [19, §4.2]).

Recall that a plane domain $\Omega \subset \mathbb{C}$ is called hyperbolic if $\partial \Omega$ contains at least two points. The uniformization theorem ensures existence of the (complete) hyperbolic metric $\rho_{\Omega}(w)|dw|$ on $\Omega$ with constant Gaussian curvature $-4$. Let $\Omega$ be a hyperbolic plane domain such that $1 \in \Omega$ but $0 \notin \Omega$ and set

$$\Pi(\Omega) = \{ F \in \mathcal{M} : F'(\zeta) \in \Omega \text{ for all } \zeta \in \Delta \}.$$ 

Set also $\Pi_n(\Omega) = \Pi(\Omega) \cap \mathcal{M}_n$ for $n = 0, 1, 2, \ldots$.

In [14], the quantity

$$W(\Omega) = \sup_{w \in \Omega} \frac{1}{|w|\rho_{\Omega}(w)}$$

is studied and called the circular width of $\Omega$. Note that the circular width can also be expressed by $W(\Omega) = \sup_{z \in \mathbb{D}} (1 - |z|^2)|p'(z)/p(z)|$, where $p : \mathbb{D} \to \Omega$ is any analytic universal covering projection of $\mathbb{D}$ onto $\Omega$ (We do not demand the condition $p(0) = 1$). For concrete values of circular widths of specific domains, see [14].

One of our main results in the present paper is an estimate of $B^*(F)$ for $F \in \Pi_n(\Omega)$. The proof of the following theorem will be given in Section 2.
Theorem 1. Let $\Omega$ be a hyperbolic domain such that $1 \in \Omega$ but $0 \notin \Omega$. For every $F \in \Pi_n(\Omega)$, $n \geq 0$, the inequality
\[ B^*(F) \leq C_n W(\Omega) \]
holds, where $C_n$ are the constants given by
\[ C_0 = 2 \text{ and } C_n = \sup_{0 < r < 1} \frac{(n + 1)(1 - r^2)r^{n-1}}{1 - r^{2n+2}}, \quad n \geq 1. \] 

As we shall show later (see Proposition 5), we have $C_1 = 2$ and $1 < C_n < (n + 1)/n$ for $n \geq 2$. We note that an analytic counterpart of this theorem is known and it is much simpler to prove (see [13, Theorem 4.1]);
\[ B(f) \leq \|Tf\| \leq W(\Omega) \]
holds for $f \in A$ with $f'(\mathbb{D}) \subset \Omega$. 

The univalence criterion in the following is due to Aksent’ev [2] (see also [6, p. 11]). Later, Krzyż [17] gave quasiconformal extensions.

**Theorem D (Aksent’ev, Krzyż).** Let $0 \leq k \leq 1$. If $F \in M$ satisfies the inequality
\[ |F'(\zeta) - 1| \leq k, \quad |\zeta| > 1, \]
then $F$ is univalent. Furthermore, if $k < 1$, then $F$ extends to a $k$-quasiconformal mapping of the extended plane. The radii 1 and $k$ are best possible. 

The above criterion implies univalence of $F \in M$ when the range of $F'$ is contained in the disk $|w - 1| < 1$. We remind the reader of the fact that the Noshiro-Warschawski theorem asserts that the condition $\Re f' > 0$ is sufficient for $f \in A$ to be univalent (cf. [10, Theorem 2.16]). However, the meromorphic counterpart does not hold. Moreover, the range of $F'$ cannot be enlarged to any disk of the form $|w - r| < r$, $r > 1$, to ensure univalence of $F$ (Aksent’ev and Avhadiev [3], see also §4).

Applying Theorem 1 to specific domains $\Omega$, we have several results similar to Theorem D. The following are a couple of examples. Note that the univalence criteria in Theorems 2 and 3 for the case $n = 0$ were first given by Avhadiev and Aksent’ev [5].

Let $x_m$ be the unique solution to the equation
\[ 2F_1(1, -\frac{1}{m}; 1 - \frac{1}{m}; x) = \frac{1}{2} \]
in the interval $0 < x < 1$ for each integer $m \geq 2$ (see Section 4 for details). Put also $x_1 = x_2$.

**Theorem 2.** Let $n \geq 0$ and $0 \leq k \leq 1$. Suppose that a function $F \in M_n$ satisfies the condition
\[ |\arg F'(\zeta)| \leq \frac{kn}{4C_n}, \quad |\zeta| > 1, \]
then \( F \) must be univalent. Furthermore, if \( k < 1 \), then \( F \) extends to a \( k \)-quasiconformal mapping of the extended plane. As for univalence, the constant \( \pi/(4C_n) \) cannot be replaced by any larger number than \( (4/\pi) \arctan x_{n+1} \).

Note that \( x_1 = x_2 \approx 0.4198 \) and that \( (4/\pi) \arctan x_1 \approx 0.506057 \approx 1.28866(\pi/8) \).

In the following univalence criterion, \( F' \) is even allowed to take values with negative real part. Let \( \beta_m \) be the unique solution to the equation
\[
\frac{2\beta}{\pi/4} \int_0^{\pi/4} (\cot x)^{1/m} e^{2\beta(x-\pi/4)} dx = 1
\] in \( 0 < \beta < \infty \) for each integer \( m \geq 2 \) (see Example 11 in Section 4). Set \( \beta_1 = \beta_2 \).

**Theorem 3.** Let \( n \geq 0 \) and \( 0 \leq k \leq 1 \). Suppose that a function \( F \in \mathcal{M}_n \) satisfies the condition
\[
|\log |F'_{\infty}|| \leq \frac{k\pi}{4C_n}, \quad |\zeta| > 1,
\]
then \( F \) must be univalent. Furthermore, if \( k < 1 \), then \( F \) extends to a \( k \)-quasiconformal mapping of the extended plane. As for univalence, the constant \( \pi/(4C_n) \) cannot be replaced by any larger number than \( \pi\beta_{n+1}/2 \).

A numerical computation gives \( \pi\beta_{1}/2 \approx 0.719122 \approx 1.83123(\pi/8) \). These results can also be translated into those for the functions \( f \in \mathcal{A} \) by using the relation (1.2). The proofs of the above theorems and slightly more refined results will be presented in Section 5.

### 2. Proof of Theorem 1

Let \( \Omega \) be a plane domain with \( 1 \in \Omega \) and \( 0, \infty \in \hat{\mathbb{C}} \setminus \Omega \) and let \( p \) be an analytic universal covering map of \( \mathbb{D} \) onto \( \Omega \) with \( p(0) = 1 \). Let \( F \in \Pi_\Omega(\Omega) \) be given. When \( n = 0 \), the function \( F \) can be expressed in the form \( F = F_0 + b_0 \), where \( F_0 \in \mathcal{M}_1 \) and \( b_0 \) is a constant and hence \( F_0 \in \Pi_\Omega(\Omega) \). Recall that \( C_0 = C_1 = 2 \). Therefore, we may further assume that \( n \geq 1 \).

Let \( \omega : \mathbb{D} \to \mathbb{D} \), \( \omega(0) = 0 \), be the lift of the mapping \( z \mapsto F'(1/z) \) of \( \mathbb{D} \) into \( \Omega \) via the covering map \( p : \mathbb{D} \to \Omega \), namely,
\[
F'(\frac{1}{z}) = p(\omega(z)), \quad |z| < 1.
\]

Since \( F \in \mathcal{M}_n \), it can be expressed in the form
\[
F(\zeta) = \zeta + \sum_{k=n}^{\infty} b_k \zeta^{-k}, \quad |\zeta| > 1,
\]
we have

$$F'(1/z) = 1 - \sum_{k=n}^{\infty} k b_k z^{k+1} = 1 - \sum_{k=n+1}^{\infty} (k - 1) b_{k-1} z^{k}, \quad |z| < 1.$$ 

In particular, $\omega$ has a zero of at least order $n + 1$ at the origin. This implies that the function $\varphi(z) = \omega(z)/z^{n+1}$ is analytic and satisfies $|\varphi(z)| \leq 1$ by the maximum modulus principle. We now apply the Schwarz-Pick lemma to the function $\varphi$ to get

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}, \quad |z| < 1,$$

and equivalently,

$$|z \varphi'(z) - (n + 2) \omega(z)| \leq \frac{|z|^{2n+2} - |\omega(z)|^2}{|z|^n(1 - |z|^2)}, \quad |z| < 1.$$ 

In particular, we obtain

$$|z \omega'(z)| \leq (n + 2)|\omega(z)| + \frac{|z|^{2n+2} - |\omega(z)|^2}{|z|^n(1 - |z|^2)}, \quad |z| < 1.$$ 

The last inequality can be expressed by

$$|z|^{2n+2} |\omega'(z)| \leq (1 - |\omega(z)|^2) F(|z|, |\omega(z)|), \quad |z| < 1,$$

where the function $F(r, s)$ is defined by

$$F(r, s) = \frac{(n + 1)(1 - r^2)r^n s + r^{2n+2} - s^2}{r^{n+2}(1 - s^2)}.$$ 

Since $|\varphi(z)| \leq 1$, we see that $|\omega(z)| \leq |z|^{n+1}$ holds. We now show the following elementary result.

**Lemma 4.**

$$F(r, s) \leq F(r, r^{n+1}) = \frac{(n + 1)(1 - r^2)r^{n-1}}{1 - r^{2n+2}}, \quad 0 \leq s \leq r^{n+1}.$$ 

**Proof.** We first see the inequality

$$\frac{\partial F}{\partial s}(r, s) = \frac{1 + s^2}{r^{n+2}(1 - s^2)^2} \left[ (n + 1)r^n(1 - r^2) - 2(1 - r^{2n+2}) \right] \frac{s}{1 + s^2} \right] \geq \frac{1 + s^2}{r^{n+2}(1 - s^2)^2} \left[ (n + 1)r^n(1 - r^2) - 2(1 - r^{2n+2}) \right] \frac{r^{n+1}}{1 + r^{2n+2}} \right]$$

$$= \frac{(1 + s^2)}{r^2(1 - s^2)(1 + r^{2n+2})} G(r), \quad 0 \leq s \leq r^{n+1},$$
because the function \( s/(1 + s^2) \) is increasing in \( 0 < s < 1 \) and \( s \leq r^{n+1} \) is assumed. Here,

\[
G(r) = (n + 1)(1 - r^2)(1 + r^{2n+2}) - 2r(1 - r^{2n+2})
\]

\[
= (1 - r^2) \left[ (n + 1)(1 + r^{2n+2}) - 2r \sum_{j=0}^{n} r^{2j} \right]
\]

\[
= (1 - r^2) \left[ (n + 1)(1 + r^{2n+2}) - r \sum_{j=0}^{n} (r^{2j} + r^{2n-2j}) \right]
\]

\[
= (1 - r^2) \sum_{j=0}^{n} \left( 1 + r^{2n+2} - r(r^{2j} + r^{2n-2j}) \right)
\]

\[
= (1 - r^2) \sum_{j=0}^{n} (1 - r^{2j+1})(1 - r^{2n+1-2j}) > 0.
\]

Therefore, we conclude that \( (\partial F/\partial s)(r, s) > 0 \) in \( 0 < s < r^{n+1} \), which implies the monotonicity of the function \( F(r, s) \) in \( s \). Thus the inequality \( F(r, s) \leq F(r, r^{n+1}) \) holds in \( 0 \leq s \leq r^{n+1} \). □

We now complete the proof of Theorem 1. By taking the logarithmic derivative of the both sides of (2.1), we have the relation

\[
-F''(1/z) = \frac{p'(\omega(z))}{p(\omega(z))} \omega'(z), \quad |z| < 1.
\]

Letting \( \zeta = 1/z \), we thus obtain

\[
(\zeta^2 - 1) \left| \frac{\zeta F''(\zeta)}{F'(\zeta)} \right| = (1 - |\zeta|^2)|\zeta|^{-1} \left| \frac{p'(\omega(z))}{p(\omega(z))} \right| |\omega'(z)|.
\]

Recall here that \( C_n \) is nothing but the supremum of \( F(r, r^{n+1}) \) over \( 0 < r < 1 \).

We then make use of (2.4) and Lemma 4 to deduce the inequality

\[
(\zeta^2 - 1) \left| \frac{\zeta F''(\zeta)}{F'(\zeta)} \right| \leq C_n (1 - |\omega(z)|^2) \left| \frac{p'(\omega(z))}{p(\omega(z))} \right| F(|\zeta|, |z|^{n+1})
\]

\[
\leq C_n (1 - |\omega(z)|^2) \left| \frac{p'(\omega(z))}{p(\omega(z))} \right| W(\Omega).
\]

The assertion of the theorem now follows.

**Remark.** The theorem is sharp if the relation \( \rho_0 = r_0^{n+1} \) is satisfied by chance, where \( r = r_0 \) is the point where the maximum is attained in the definition of \( C_n \) and \( r = \rho_0 \) is the radius where the maximum is attained for \( (1 - |z|^2)|p'(z)/p(z)| \). Let \( w_0 \) be the maximum point of \( (1 - |z|^2)|p'(z)/p(z)| \) with \( |w_0| = \rho_0 \), and set \( z_0 = r_0 \). Then we choose \( \omega_0 \) so that \( \omega_0(z_0) = w_0 \) and equalities hold in (2.2).
and (2.3) at $z = z_0$ simultaneously (see the proof of Dieudonné’s lemma in [10, p. 198]). Then, we actually have $B^*(F) = C_n W(\Omega)$ in this case, where $F$ is determined by $F'(1/z) = p(\omega_0(z))$ in $|z| < 1$.

As we mentioned in Section 1, we give some information about the constants $C_n$.

**Proposition 5.** The constants $C_n$ given by (1.3) satisfy the following:

(2.5) \[ C_0 = C_1 = 2, \quad C_2 = \frac{3\sqrt{6(\sqrt{13} - 1)}}{5 + \sqrt{13}} \approx 1.37838, \]

(2.6) \[ 1 < C_n < \frac{n + 1}{n}, \quad n = 2, 3, \ldots. \]

**Proof.** The relations in (2.5) can be checked in a straightforward way. We omit the details. We show only (2.6). Let $n \geq 2$ and set

\[ g_n(x) = \frac{1 - x^{n+1}}{x^{(n-1)/2}(1 - x)}, \quad x \in (0, 1). \]

Then clearly, $C_n = (n + 1)/\inf_{0 < x < 1} g_n(x)$. First note that

\[ \lim_{x \to 1} g_n(x) = n + 1. \]

Therefore, we have $C_n \geq 1$. In order to show strictness, we set $x = 1 - \varepsilon$, $\varepsilon > 0$. Then

\[ g_n(1 - \varepsilon) = (n + 1) - \frac{n + 1}{2 \varepsilon} + O(\varepsilon^2), \quad \varepsilon \to 0, \]

which implies that $g_n(x)$ is smaller than $n + 1$ when $x < 1$ is close enough to 1. Therefore, $C_n > 1$.

We next show the reverse inequality. Since $g_n(x) \to +\infty$ as $x \to 0^+$, the function $g_n$ takes its minimum at a point in $(0, 1)$. We now estimate $g_n(x)$ from below:

\[
\begin{align*}
g_n(x) &= x^{(1-n)/2} \sum_{j=0}^{n} x^j \\
&> x^{(1-n)/2} \sum_{j=0}^{n-1} x^j \\
&= x^{(1-n)/2} \sum_{j=0}^{n-1} x^j + x^{n-1-j} \\
&= \sum_{j=0}^{n-1} x^{j-(n-1)/2} + x^{(n-1)/2-j} \\
&\geq \sum_{j=0}^{n-1} 1 = n.
\end{align*}
\]
Thus we get the inequality \( \min_{0 < x \leq 1} g_n(x) > n \), which in turn implies \( C_n < (n + 1)/n \).

3. A variant of Theorem 1

We give a variant of Theorem 1 in the present section. In the following theorem, the condition \( p(0) = 1 \) for the analytic universal covering map \( p \) of \( D \) onto \( \Omega \) is required and the constant involved might not be computed easily, but the estimate is independent of \( n \) and better than Theorem 1 at least when \( n = 0 \).

**Theorem 6.** Let \( \Omega \) be a plane domain with \( 1 \in \Omega \) but \( 0, \infty \not\in \Omega \) and let \( p \) be an analytic universal covering map of the unit disk \( D \) onto \( \Omega \) with \( p(0) = 1 \).

Then, for every \( F \in \Pi(\Omega) \) the inequality

\[
B^*(F) \leq 2 \sup_{|z| < 1} \frac{|p'(z)|}{|p(z)|}
\]

holds.

**Proof.** The proof proceeds basically in the same line as in the previous section. In order to show that the constant is really independent of \( n \) for which \( F \in \Pi_n(\Omega) \) holds, we prove the assertion under the additional assumption that \( F \in \Pi_n(\Omega) \) for a fixed \( n \geq 1 \). We replace the inequality (2.4) by

(3.1) \( (1 - |z|^2)|z|^{-1}\omega'(z)| \leq (1 - |\omega(z)|)H(|z|, |\omega(z)|), \quad |z| < 1, \)

where

\[
H(r, s) = \frac{(n + 1)(1 - r^2)r^n s + r^{2n+2} - s^2}{r^{n+2}(1 - s)^2}.
\]

Recall here that \( |\omega(z)| \leq |z|^{n+1} \) holds. Since the function \( s^2 - 2s \) is decreasing in \( 0 < s < r^{n+1} \), we have

\[
\frac{\partial H}{\partial s}(r, s) = \frac{s^2 - 2s + (n + 1)(1 - r^2)r^n + r^{2n+2}}{r^{n+2}(1 - s)^2}
\]

\[
\geq \frac{2n+2 - 2r^{n+1} + (n + 1)(1 - r^2)r^n + r^{2n+2}}{r^{n+2}(1 - s)^2}.
\]

The numerator of the last term can be written in the form

\[
r^n \left[(n + 1)(1 - r^2) - 2r(1 - r^{n+1})\right]
\]

\[
= r^n (1 - r) \left[(n + 1)(1 + r) - 2r(1 + r + r^2 + \cdots + r^n)\right]
\]

\[
= r^n (1 - r) \sum_{j=0}^{n} (1 + r - 2r^{j+1}).
\]

It is now clear that \( (\partial H/\partial s)(r, s) > 0 \) in \( 0 < s \leq r^{n+1} \). Thus \( H(r, s) \) is increasing in \( s \) and therefore

\[
H(r, s) \leq H(r, r^{n+1}) = \frac{(n + 1)(1 - r^2)r^{n-1}}{1 - r^{n+1}} = g(r).
\]
Since
\[
g'(r) = \frac{(n + 1)r^{n-2}((n - 1)(1 - r^2) - 2r^2(1 - r^{n-1}))}{(1 - r^{n+1})^2}
\]
\[
= \frac{(n + 1)r^{n-2}(1 - r)}{(1 - r^{n+1})^2} \sum_{j=0}^{n-2} [1 - r^{j+2} + r(1 - r^{j+1})] > 0,
\]
the function \( g(r) \) is increasing and thus \( g(r) < g(1-) = 2 \) for \( 0 \leq r < 1 \). Therefore, we obtain
\[
\sup_{0 < s \leq r^{n+1} < 1} H(r, s) = \sup_{0 < r < 1} g(r) = 2,
\]
which is, indeed, independent of \( n \).

The rest is same as in the previous section. We omit the details. \( \square \)

Since \( 1 - r \leq 1 - r^2 = (1 + r)(1 - r) \leq 2(1 - r) \), the inequalities
\[
\sup_{|z| < 1} (1 - |z|) \left| \frac{p'(z)}{p(z)} \right| \leq W(\Omega) = \sup_{|z| < 1} (1 - |z|^2) \left| \frac{p'(z)}{p(z)} \right| \leq 2 \sup_{|z| < 1} (1 - |z|) \left| \frac{p'(z)}{p(z)} \right|
\]
hold. Thus, when \( n = 0 \), the estimate in Theorem 6 is better than that in Theorem 1.

4. Examples of non-univalent functions

In this section, we present non-univalent meromorphic functions in the class \( \mathcal{M} \) to examine our univalence criteria given in introduction. First, we introduce the example given by Aksent’ev and Avhadiev [3].

Example 7. Let \( r > 1 \) be given and set \( \Omega = \{ w \in \mathbb{C} : |w - r| < r \} \). For a number \( c \in (0, 1/2] \), we set \( \Phi = G \circ F \), where \( F(\zeta) = \zeta + c/\zeta \) and \( G(\zeta) = \zeta + (1 + c)^2/\zeta \). Then
\[
\Phi'(\zeta) = 1 - \zeta^{-2} + c\psi(\zeta^{-2}),
\]
where
\[
\psi(z) = \psi_c(z) = -\frac{(c + 3) - (c^2 + 3)z + (c^2 - c)z^2}{(1 + cz)^2}.
\]
Note that the functions \( 1 - \zeta^{-2} \) and \( \psi(\zeta^{-2}) \) take the value \( 0 \) at \( \zeta = \pm 1 \). Since \( \psi_c \) is uniformly bounded in \( \mathbb{D} \) and \( \psi'(1) > 0 \), in order to see that \( F'(\mathbb{D}) \subset \Omega \) for sufficiently small \( c \), it is enough to check that the (signed) curvature of the curve \( \theta \mapsto \psi(e^{i\theta}) \) is positive at \( \theta = 0 \), in other words, \( \Re (1 + z\psi''(z)/\psi'(z))/|\psi'(z)| \) is positive at \( z = 1 \). A direct computation gives
\[
1 + z\psi''(z)/\psi'(z) = \frac{3 - 10c + 2(c^2 + c)z - c^2z^2}{(3 - cz)(1 + cz)},
\]
which shows \( \Re (1 + \psi''(1)/\psi'(1))/|\psi'(1)| > 0 \) for a small enough \( c > 0 \) as required.

We see now that \( \Phi \) is not univalent in \( \Delta \) by observing that the two points \( \pm i(1 + c + \sqrt{1 + 6c + c^2})/2 \) in \( \Delta \) are zeros of \( \Phi \).
The above example is qualitatively very nice but somewhat implicit because it is not simple to give a right value of \( c \) for a given \( r > 1 \). The next two examples are more concrete.

**Example 8.** We consider the function \( F_m \in \mathcal{M} \) given by

\[
F_m(\zeta) = \zeta - 2 \sum_{j=1}^{\infty} \frac{\zeta^{1-mj}}{mj - 1}
\]

for each integer \( m \geq 2 \), where \( _2F_1(a, b; c; x) \) stands for the hypergeometric function. Note that \( F_m \) has the \( m \)-fold symmetry

\[
F_m(e^{2\pi i/m} \zeta) = e^{2\pi i/m} F_m(\zeta)
\]

and belongs to the class \( \mathcal{M}_{m-1} \). Since the function \( h_m \) defined by

\[
h_m(x) = _2F_1(1, -\frac{1}{m}; 1 - \frac{1}{m}; x) - 1 \quad (x \in (0, 1))
\]

has the properties that \( h_m \) is monotone decreasing, \( h_m(0) = 1 \) and \( \lim_{x \to 1^{-}} h_m(x) = -\infty \), there is the unique point \( x_m \) such that \( h(x_m) = 0 \) in the interval \( 0 < x < 1 \). Hence, the function \( F_m \) has the \( m \) zeros \( e^{2\pi ij/m} x_m^{-1/m}, \ j = 0, 1, \ldots, m - 1 \), in \( \Delta \) and, in particular, is not univalent in \( \Delta \). On the other hand, we have

\[
F_m'(\zeta) = 1 + 2 \sum_{j=1}^{\infty} \zeta^{-mj} = p(\zeta^{-m}),
\]

where \( p(z) = (1 + z)/(1 - z) \). It is a standard fact that \( p \) maps the unit disk onto the right half-plane \( H = \{ w \in \mathbb{C} : \text{Re} w > 0 \} \). Therefore, \( F_m' \) maps \( \Delta \) onto \( H \) in an \( m \)-to-1 way and \( \text{Re} F_m' > 0 \) holds.

In particular, we have shown the following.

**Proposition 9.** For each integer \( n \geq 0 \), there is a non-univalent function \( F \) in the class \( \mathcal{M}_n \) such that \( \text{Re} F'(\zeta) > 0 \) in \( |\zeta| > 1 \).

Note that the function \( F_2 \) in the above example can be expressed also by

\[
F_2(\zeta) = \zeta - \log \frac{\zeta + 1}{\zeta - 1}, \quad |\zeta| > 1.
\]

A numerical computation yields, for instance,

\[
x_2 \approx 0.419798,
x_3 \approx 0.667508,
x_4 \approx 0.808289.
\]

The above functions can be used to examine univalence criteria. Note that, for a function \( F \in \mathcal{M} \), the new function

\[
F^t(\zeta) = t F\left( \frac{\zeta}{t} \right), \quad |\zeta| > 1,
\]
for \( t \in (0, 1) \) satisfies the relation \((F^t)'(\zeta) = F'(\zeta/t)\). For instance, for \( m \geq 2 \),
the function \( F^t_m(\zeta) = tF_m(\zeta/t) \) is not univalent as far as \( t^m > x_m \), because
\((\zeta/t)^{-m} = x_m \) has \( m \) roots in \( |\zeta| > 1 \) in this case. On the other hand, \((F^t_m)'\)
has the range \( \{ w \in \mathbb{C} : w = (1 + t^m z)/(1 - t^m z) \} \) for some \( z \in \mathbb{D} \) = \( \{ w \in \mathbb{C} : \)
\(|w - (1 + t^2m)/(1 - t^2m)| < 2t^m/(1 - t^2m)\} \). In this way, we have shown the
following.

**Proposition 10.** Let \( \Omega_s = \{ w \in \mathbb{C} : |w - (1 + s^2)/(1 - s^2)| < 2s/(1 - s^2) \} \) and \( n \geq 1 \). If \( s > x_{n+1} \), then the class \( \Pi_n(\Omega_s) \) contains non-univalent functions.

**Example 11.** The construction is similar to that of Example 8. First note that
the analytic function \((1 + z)/(1 - z))^{i\beta} \) gives a universal covering projection
of the unit disk onto the annulus \( \{ w = e^{-\pi \beta/2} < |w| < e^{\pi \beta/2} \} \) for a positive constant \( \beta \). Let \( G \in \mathcal{M}_{m-1} \) be the function determined by the relation
\( G'(\zeta) = ((\zeta^m + 1)/(|\zeta^m - 1|))^{i\beta} \) for an integer \( m \geq 2 \). Then \( G \) also has the \( m \)-fold symmetry. Let \( h_\beta(z) = 1/z - \int_0^z t^{m-2}q_\beta(t^m)dt \), where \((1 + z)/(1 - z))^{i\beta} = 1 + zq_\beta \), so that \( G(\zeta) = h_\beta(1/\zeta) \). Now take any root \( \omega \) of the polynomial
\( z^m + i \) and set \( \varphi(\beta) = \omega h_\beta(\omega) \). Since \( 1 + i \beta xq_\beta(ix) = ((1 + ix)/(1 - ix))^{i\beta} = \exp(2i\beta \arctan(ix)) = \exp(-2\beta \arctan x) \), we have for \( 0 < r \leq 1 \)
\begin{align*}
\omega h_\beta(\omega r) &= \frac{1}{r} + \int_0^r it^{m-2}q_\beta(-it^m)dt \\
&= \frac{1}{r} - \int_0^r (\exp(2\beta \arctan(t^m)) - 1) t^{-2}dt.
\end{align*}
Thus,
\begin{align*}
\varphi(\beta) &= 1 - \int_0^1 (\exp(2\beta \arctan(t^m)) - 1) t^{-2}dt.
\end{align*}
Since \( \varphi(0) = 1, \varphi(+\infty) = -\infty \) and
\begin{align*}
\varphi'(\beta) &= -\int_0^1 t^{-2} \arctan(t^m) \exp(2\beta \arctan(t^m))dt < 0,
\end{align*}
there exists a unique \( \beta_m \) such that \( \varphi(\beta_m) = 0 \). We now simplify the equation
\( \varphi(\beta) = 0 \). Performing integration by parts and then setting \( x = \arctan(t^m) \), we have
\begin{align*}
\varphi(\beta) &= e^{\pi \beta/2} - 2\beta \int_0^{\pi/4} e^{2\beta x}(\tan x)^{-1/m}dx \\
&= e^{\pi \beta/2} \left( 1 - 2\beta \int_0^{\pi/4} e^{2\beta(x-\pi/4)}(\cot x)^{1/m}dx \right).
\end{align*}
Thus we now have arrived at the form in (1.5).

We now fix any \( \beta > \beta_m \). Then \( \omega h_\beta(\omega r) > 0 \) for a small enough \( r > 0 \)
whereas \( \varphi(\beta) = \omega h_\beta(\omega) < 0 \). Therefore, there exists an \( \rho \in (0, 1) \) such that \( G(1/\omega \rho) = h_\beta(\omega \rho) = 0 \). In particular, \( G \) has at least \( m \) zeros in \( \Delta \) and thus
is not univalent. By the above observations, we have the following proposition.
Proposition 12. Let $n$ be an integer with $n \geq 1$ and let $\beta > \beta_{n+1}$. Then there exists a non-univalent function $G \in \mathcal{H}_n$ such that $e^{-\pi \beta/2} < |G'(\zeta)| < e^{\pi \beta/2}$ for $|\zeta| > 1$.

By a numerical computation, one has

\[
\beta_2 \approx 0.457807,
\beta_3 \approx 0.786518,
\beta_4 \approx 1.03144.
\]

5. Applications to univalence criteria

We combine Theorem 1 or Theorem 6 with Theorem C to deduce several univalence criteria for functions in $\mathcal{M}$. The same method can be applied also to $\mathcal{H}_n$ for $n \geq 1$, but we do not go into details here. In order to make statements concise, we introduce the notation $\Sigma(k)$ to designate the set of those functions in $\Sigma$ which can be extended to $k$-quasiconformal mappings of the extended plane. For $k = 1$, simply we define $\Sigma(1) = \Sigma$ for convenience.

To examine Theorems 1 and 6, we assume $\Omega$ to be a disk containing 1 but not containing 0. Then we can express $\Omega$ as $D(a, \rho) = \{ w : |w - a| < \rho \}$, where $0 < \rho \leq |a|$ and $|1 - a| < \rho$. If we put $p(z) = a + \rho z$, then we compute

\[
W(D(a, \rho)) = \sup_{z \in D} (1 - |z|^2) \frac{\rho}{|a + \rho z|}
= \sup_{0 < r < 1} (1 - r^2) \frac{\rho}{|a| - pr}
= \frac{\rho}{|a|} \sup_{0 < r < 1} \frac{1 - r^2}{1 - (\rho/|a|)^2}
= \frac{2\rho/|a|}{1 + \sqrt{1 - (\rho/|a|)^2}},
\]

where we have made a standard but tedious computation at the final step (see, for instance, [15, Lemma 4.2]). Therefore, by Theorem 1, we conclude that

\[
(5.1) \quad B^*(F) \leq \frac{2C_n \rho/|a|}{1 + \sqrt{1 - (\rho/|a|)^2}}
\]

for $F \in \Pi_n(D(a, \rho))$. It is easy to see that the right-hand side of the last inequality is less than or equal to $k$ if and only if $\rho/|a| \leq 4C_n k/(4C_n^2 + k^2)$. Thus we can show the following by appealing to Theorem C.

Theorem 13. Let $n$ be an integer with $n \geq 0$ and $a \in \mathbb{C}$, $\rho > 0$ satisfy $\rho \leq |a|$ and $|a - 1| < \rho$. Suppose that

\[
\frac{\rho}{|a|} \leq \frac{4C_n k}{4C_n^2 + k^2}
\]

for a constant $k$ with $0 \leq k \leq 1$. Then $\Pi_n(D(a, \rho)) \subset \Sigma(k)$. 
We recall that Theorem D gives the stronger assertion $\Pi(\mathbb{D}(1,k)) \subset \Sigma(k)$ when $a = 1$ and $\rho = k$.

We next consider to apply Theorem 6. It is not simple to treat the case when $a$ is not real. Therefore, we further assume that $a > 0$ for simplicity. Then the conformal map $p$ of $\mathbb{D}$ onto $\mathbb{D}(a,\rho)$ with $p(0) = 1$ can be taken in the form $p(z) = (1 + Az)/(1 + Bz)$, where $-1 < B < A \leq 1$. A simple computation gives us the relations $A = (\rho^2 - a^2 + a)/\rho$ and $B = (1 - a)/\rho$.

First observe the expression (see [15, Lemma 4.1])

$$W = \sup_{z \in \mathbb{D}} |p'(z)/p(z)| = \begin{cases} (A - B) \sup_{0 < r < 1} \frac{1 - r}{(1 - Ar)(1 - Br)} & \text{if } A + B \leq 0, \\ (A - B) \sup_{0 < r < 1} \frac{1 - r}{(1 + Ar)(1 + Br)} & \text{if } A + B \geq 0. \end{cases}$$

At any event, we can easily see that $W = A - B$. Therefore, by Theorem 6, we obtain the estimate

$$(5.2) \quad B^*(F) \leq 2(A - B) = 2(\rho^2 - (a - 1)^2)/\rho$$

for $F \in \Pi(\mathbb{D}(a,\rho))$. In the same way as above, we have the following.

**Theorem 14.** Let $a > 0$, $\rho > 0$ satisfy $\rho \leq a$ and $|a - 1| < \rho$. Suppose that

$$\rho^2 - (a - 1)^2 \leq \frac{kp}{2}$$

for a constant $k$ with $0 \leq k \leq 1$. Then $\Pi(\mathbb{D}(a,\rho)) \subset \Sigma(k)$.

As an example, let us consider the disk $\Omega_s = \{w \in \mathbb{C} : |w - (1+s^2)/(1-s^2)| < 2s/(1-s^2)\}$. In this case, $A = s, B = -s$, and therefore,

$$\frac{4\rho/|a|}{1 + \sqrt{1 - (\rho/|a|)^2}} = 4s = 2(A - B),$$

which means that the estimates (5.1) with $n = 0$ and (5.2) are identical in this case. In particular, Theorems 13 and 14 both imply that $\Pi(\Omega_s) \subset \Sigma$ if $s \leq 1/4$. This is, however, weaker than Theorem D because $\Omega_s \subset \mathbb{D}(1,1)$ for $s \leq 1/3$. On the other hand, Proposition 10 implies that $\Pi(\Omega_s)$ is not contained in $\Sigma$ for $s > x_2 \approx 0.4198$.

However, Theorems 13 and 14 may imply the inclusion $\Pi(\mathbb{D}(a,\rho)) \subset \Sigma$ for a disk $\mathbb{D}(a,\rho)$ which is not contained in $\mathbb{D}(1,1)$. For instance, by Theorem 14, we see that $\Pi(\mathbb{D}(3/2,4/5)) \subset \Sigma$ but $\mathbb{D}(3/2,4/5)$ is not a subset of $\mathbb{D}(1,1)$. By the way, this is not implied by Theorem 13.

We next recall basic results for the values of $W(\Omega)$ for special domains $\Omega$. We set

$$S(\alpha, \gamma) = \{w \in \mathbb{C} : |\arg w - \gamma| < \pi\alpha/2\}$$

$$A(r_1, r_2) = \{w \in \mathbb{C} : r_1 < |w| < r_2\},$$

where $0 < \alpha \leq 2$, $\gamma \in \mathbb{R}$ and $0 < r_1 < r_2 < \infty$. The domain $S(\alpha, \gamma)$ is called a sector with opening $\pi\alpha$ and vertex at 0. The domain $A = A(r_1, r_2)$ is
called a round annulus centered at 0 with modulus \( m = \log(r_2/r_1) \). We write \( m = \text{mod } A \). Then we have the following.

**Lemma 15** ([14]).

\[
W(S(\alpha, \gamma)) = 2\alpha, \quad 0 < \alpha \leq 2,
\]

\[
W(A(r_1, r_2)) = \frac{2}{\pi} \log \frac{r_2}{r_1} = \frac{2}{\pi} \text{mod } A(r_1, r_2), \quad 0 < r_1 < r_2 < \infty.
\]

Combining this lemma with Theorems 1 and C, we can prove the following two results. Theorems 2 and 3 are just special cases of them up to non-univalent examples, which were supplied in the previous section.

**Theorem 16.** Let \( 0 \leq k \leq 1 \). If \( \Omega \) is a sector with opening \( k\pi/4 \) and vertex at 0 such that \( 1 \in \Omega \), then \( \Pi(\Omega) \subset \Sigma(k) \).

**Theorem 17.** Let \( 0 \leq k \leq 1 \). If \( \Omega \) is a round annulus centered at 0 with modulus \( k\pi/4 \) such that \( 1 \in \Omega \), then \( \Pi(\Omega) \subset \Sigma(k) \).

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