ORTHOGONAL MULTI-WAVELETS FROM MATRIX FACTORIZATION

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Abstract. Accuracy of the scaling function is very crucial in wavelet theory, or correspondingly, in the study of wavelet filter banks. We are mainly interested in vector-valued filter banks having matrix factorization and indicate how to choose block central symmetric matrices to construct multi-wavelets with suitable accuracy.

1. Introduction

We consider the case of compactly supported multi-wavelets. That is, suppose \( \Phi = (\phi_1, \ldots, \phi_r)^T \) are scaling functions, and \( \Psi = (\psi_1, \ldots, \psi_r)^T \) are the corresponding wavelet functions, so that the following two-scale equations hold for all \( x \in \mathbb{R} \):

\[
\Phi(x) = \sum_{n=0}^{M} S_n \Phi(2x - n),
\]

\[
\Psi(x) = \sum_{n=0}^{M} T_n \Phi(2x - n).
\]

Define the corresponding symbol functions as

\[
m_0(x) := \frac{1}{2} \sum_{n \in \mathbb{Z}} S_n x^n, \quad m_1(x) := \frac{1}{2} \sum_{n \in \mathbb{Z}} T_n x^n.
\]

As is well known, the orthonormality of the multi-wavelets implies the following PR condition

\[
H(x) H^T(1/x) = I_2,
\]

with

\[
H(x) := \begin{pmatrix}
m_0(x) & m_0(-x) \\
m_1(x) & m_1(-x)
\end{pmatrix}.
\]
Up to now, it seems very difficult to find all solutions of the matrix equation (1.3) so people hope to construct some special solutions. A class of PR wavelet filter banks was given in [2] for the general case. In particular, one solution for the multi-wavelets case is as follows:

\[
\begin{align*}
  m_k(x) &= \frac{1}{2} (I_r, xI_r) \left[ \prod_{j=1}^{N} U_j \text{diag} (I_r, x^2I_r) U_j^T \right] V_k, \quad k = 0, 1.
\end{align*}
\]

Here \( N \) is a fixed positive integer, \( U_j \) is any arbitrary \( 2r \times 2r \) real orthogonal matrix, and

\[
\begin{align*}
  V := (V_0, V_1) &= \begin{pmatrix}
    I_r & I_r \\
    I_r & -I_r
  \end{pmatrix}.
\end{align*}
\]

The following theorem could be found in [2].

**Theorem 1.1.** Suppose filter banks are constructed as in (1.5)-(1.6). Then the PR condition (1.3) is satisfied.

It is well known that the linear phase of filter banks corresponds to symmetry of the related functions. It was also pointed out in [2] that to ensure the uniform linear phase, i.e., to ensure that there exists a natural number \( s \) such that \( m_k(x) = x^s m_k(1/x), \) \( k = 0, 1, \) we should choose \( U_j \) to be \( r \)-block central symmetric matrices:

\[
\begin{align*}
  U_j &= S \begin{pmatrix}
    P_j & 0 \\
    0 & Q_j
  \end{pmatrix} S^T, \\
  S &= \begin{pmatrix}
    I_r & -J_r \\
    J_r & I_r
  \end{pmatrix},
\end{align*}
\]

where \( P_j, Q_j \) are \( r \)-th real orthogonal matrices and \( J_r \) is the \( r \)-th reversal matrix.

For later convenience, let \( G(x) = \frac{1}{2} (I_r, xI_r) \prod_{j=1}^{N} U_j \text{diag} (I_r, x^2I_r) U_j^T. \)

2. Accuracy conditions of multiple scaling functions

In the following sections, we concentrate on sufficient and necessary conditions so that the scaling functions have accuracy of order \( p \). That is, all polynomials with total degree at most \( p - 1 \) can be reproduced from linear combinations of the multi-integer translates of function \( \Phi \).

Gilbert Strang stated in [5] that to ensure accuracy, one must check the value of function \( m_0 \) and its derivatives at all aliasing frequencies which seems difficult to compute. By only imposing some conditions on the functions \( m_0, m_1 \) at \( x = 1 \), the author of this paper produced the following theorem, [9].

**Theorem 2.1.** If \( \Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T \) are scaling functions, and the integer translates of \( \phi_1, \ldots, \phi_r \) are linearly independent, moreover, if the corresponding filter bank satisfies the PR condition (1.3), then \( \Phi \) have accuracy \( p \) if and only
if there are \( p \) vectors \( \nu_0, \ldots, \nu_{p-1} \), each \( \nu_l \) being \( r \times 1 \) vector and \( \nu_0 \neq 0 \), such that for all \( j \in \mathbb{Z}_p = \{0, 1, \ldots, p-1\} \):

\[
\sum_{l=0}^{j} \binom{j}{l} (2i)^{l-j} m_0^{(j-l)}(1) \nu_l = 2^j \nu_j,
\]

(2.1)

\[
\sum_{l=0}^{j} \binom{j}{l} (2i)^{l-j} m_1^{(j-l)}(1) \nu_l = 0.
\]

(2.2)

By using this theorem, the next proposition is obtained for the filter banks constructed as above.

**Proposition 2.2.** If \( \Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T \) are scaling functions with the integer translates of \( \phi_1, \ldots, \phi_r \) being linearly independent, and, the corresponding filter banks are constructed as in (1.5)-(1.6), then \( \Phi \) have accuracy \( p \) if and only if there are \( p \) vectors \( \nu_0, \ldots, \nu_{p-1} \), each \( \nu_l \) being \( r \times 1 \) vector and \( \nu_0 \neq 0 \), such that for all \( j \in \mathbb{Z}_p \):

\[
\sum_{l=0}^{j} \binom{j}{l} (2i)^{l-j} G^{(j-l)}(1) V_0 \nu_l = 2^j \nu_j,
\]

(2.3)

\[
\sum_{l=0}^{j} \binom{j}{l} (2i)^{l-j} G^{(j-l)}(1) V_1 \nu_l = 0.
\]

(2.4)

By this proposition, we must find \( p \) vectors \( \nu_0, \ldots, \nu_{p-1} \) to meet the requirements of equations (2.3) and (2.4). The procedure is simplified as follows so that one must only find a nonzero vector \( \nu_0 \) which is the common eigen-vector corresponding to eigenvalue \( \lambda = 0 \) of several matrices.

**Theorem 2.3.** Under the assumptions of Proposition 2.2, the scaling functions have accuracy \( p \) if and only if there exist a \( r \times 1 \) vector \( \nu_0 \neq 0 \) such that \( \nu_0 \) are common eigenvector corresponding to eigenvalue \( 0 \) of matrices \( B_1, \ldots, B_{p-1} \), that is,

\[
B_n \nu_0 = 0, \quad n = 1, 2, \ldots, p-1.
\]

(2.5)

The matrices \( B_j \) are constructed iteratively as

\[
B_1 = N_1,
\]

(2.6)

\[
B_n = N_n + \sum_{j=1}^{n-1} \frac{\binom{n}{j}}{2^{j-1}} N_{n-j} A_j, \quad 2 \leq n \leq p-1
\]

and

\[
A_1 = M_1,
\]

(2.7)

\[
A_n = M_n + \sum_{j=1}^{n-1} \frac{\binom{n}{j}}{2^{j-1}} M_{n-j} A_j, \quad 2 \leq n \leq p-1
\]
with
\[(2.8)\]
\[M_j := G^{(j)}(1) V_0, \quad N_j := G^{(j)}(1) V_1.\]
Furthermore, the solutions of equations (2.3), (2.4) are given as
\[(2.9)\]
\[\nu_n = \frac{(2i)^{-n}}{2^n - 1} A_n \nu_0, \quad n = 1, 2, \ldots, p - 1.\]

Proof. If the assumptions of Proposition 2.2 are satisfied, one easily checks that
\[(2.10)\]
\[G(1) V_0 = I_r, \quad G(1) V_1 = 0_r,\]
so that for any \(r \times 1\) vector \(\nu_0 \neq 0\), equations (2.3) and (2.4) for \(p = 1\). Thus, the corresponding scaling functions have at least accuracy of \(p = 1\).

From Proposition 2.2, \(\Phi\) have accuracy \(p = 2\) if and only if there exists \(r \times 1\) vector \(\nu_0, \nu_1\) with \(\nu_0 \neq 0\) such that
\[(2.11)\]
\[G(1) V_0 \nu_0 = \nu_0, \quad G(1) V_1 \nu_0 = 0,\]
\[(2i)^{-1} G^{(1)}(1) V_0 \nu_0 + G(1) V_0 \nu_1 = 2 \nu_1, \quad (2i)^{-1} G^{(1)}(1) V_1 \nu_0 + G(1) V_1 \nu_1 = 0.\]
By using the relations (2.10) and the notations in (2.6)-(2.8), the last equations are equivalent to
\[B_1 \nu_0 = 0, \quad \nu_1 = (2i)^{-1} A_1 \nu_0,\]
this is just what equations (2.5) and (2.9) states for \(p = 2\).

Suppose this theorem holds for some \(p \geq 2\), next we will prove by induction that it also holds for \(p + 1\). Proposition 2.2 states that \(\Phi\) have accuracy \(p + 1\) if and only if there exists \(r \times 1\) vector \(\nu_0, \ldots, \nu_{p+1}\) with \(\nu_0 \neq 0\) such that (2.3) and (2.4) holds for all \(j = 0, 1, \ldots, p\). By induction, this is equivalent to
\[(2.12)\]
\[B_n \nu_0 = 0, \quad \nu_n = \frac{(2i)^{-n}}{2^n - 1} A_n \nu_0, \quad n = 1, 2, \ldots, p - 1;\]
\[(2.13)\]
\[\sum_{l=0}^{p} \binom{p}{l} (2i)^{l-p} G^{(p-l)}(1) V_0 \nu_l = 2^p \nu_p;\]
\[(2.14)\]
\[\sum_{l=0}^{p} \binom{p}{l} (2i)^{l-p} G^{(p-l)}(1) V_1 \nu_l = 0.\]
By using (2.10) and (2.12), the left side of equation (2.14) equals to
\[\sum_{l=1}^{p} \binom{p}{l} (2i)^{l-p} N_{p-l} \nu_l + (2i)^{-p} N_p \nu_0\]
\[= \sum_{l=1}^{p} \binom{p}{l} (2i)^{l-p} N_{p-l} \frac{(2i)^{-l}}{2^l - 1} A_l \nu_0 + (2i)^{-p} N_p \nu_0\]
\[= (2i)^{-p} \left\{ \sum_{l=1}^{p} \frac{\binom{p}{l}}{2^l - 1} N_{p-l} A_l + N_p \right\} \nu_0 = (2i)^{-p} B_p \nu_0.\]
Similarly, by using (2.10) and (2.12), the left side of equations (2.13) equals to
\[
\sum_{l=1}^{p} \binom{p}{l} (2i)^{-p} M_{p-l} v_l + (2i)^{-p} M_p v_0 + v_p
\]
\[
= \sum_{l=1}^{p} \binom{p}{l} (2i)^{-p} M_{p-l} \frac{(2i)^{-l}}{2^l - 1} A_l v_0 + (2i)^{-p} M_p v_0 + v_p
\]
\[
= (2i)^{-p} \left\{ \sum_{l=1}^{p} \frac{\binom{p}{l}}{2^l - 1} M_{p-l} A_l + M_p \right\} v_0 + v_p = (2i)^{-p} A_p v_0 + v_p.
\]
Thus, equations (2.12)-(2.14) are equivalent to
\[
B_n v_0 = 0, \quad v_n = \frac{(2i)^{-n}}{2^n - 1} A_n v_0, \quad n = 1, 2, \ldots, p.
\]
So we have proved this theorem. \(\square\)

3. Computation of the derivatives \(G^{(j)}(1)\)

Theorem 2.3 propose a sufficient and necessary condition for the corresponding scaling functions to have accuracy of degree \(p\). Note that this implies the necessity of computing the derivatives of \(G(x)\), that is, the derivatives of the product of several functions. Next, we give the following results concerning the computation of derivatives.

3.1. Derivation of products of functions

The first lemma is classical in mathematical analysis which is called Leibniz’s formula.

**Lemma 3.1.** Let \(f(x) = f_1(x) f_2(x)\). Then, for any natural number \(n\), the \(n\)-th derivative of function \(f\) is
\[
(f^{(n)}(x) = \sum_{m=0}^{n} \binom{n}{m} f_1^{(m)}(x) f_2^{(n-m)}(x).
\]

**Lemma 3.2.** Let \(f(x) = f_1(x) \cdots f_M(x)\). Then, for any natural number \(n\), the \(n\)-th derivative of \(f\) is
\[
(f^{(n)}_1 \cdots f^{(n)}_M)(x) = \sum_{j_1+\cdots+j_M=n} \frac{n!}{j_1! j_2! \cdots j_M!} f_1^{(j_1)}(x) \cdots f_M^{(j_M)}(x).
\]

**Proof.** We will prove this theorem by induction of \(M\). For \(M = 1\), this theorem holds naturally. And, Lemma 3.1 states that (3.2) holds for \(M = 2\).
Suppose this theorem holds for some natural number $M \geq 2$, then, by Lemma 3.1,
\[
(f_1 \cdots f_{M+1})^{(n)}(x)
= \sum_{m=0}^{n} \binom{n}{m} (f_1 \cdots f_{M})^{(m)}(x) f_{M+1}^{(n-m)}(x)
= \sum_{m=0}^{n} \binom{n}{m} \sum_{j_1+\cdots+j_M=m, \ j_i \geq 0} \frac{m!}{j_1! j_2! \cdots j_M!} f_1^{(j_1)}(x) \cdots f_M^{(j_M)}(x) f_{M+1}^{(n-m)}(x)
= \sum_{j_1+\cdots+j_M+1=n, \ j_i \geq 0} \frac{(n+1)!}{j_1! j_2! \cdots j_{M+1}!} f_1^{(j_1)}(x) \cdots f_{M+1}^{(j_{M+1})}(x).
\]

Thus, this theorem also holds for $M + 1$. □

3.2. Computation of the derivatives $G^{(j)}(1)$

In this section we will concentrate on the filter banks which are constructed in (1.5)-(1.7). Let $f_0 = \frac{1}{2} (I_r, x I_r)$, and for $j = 1, \ldots, N$,

\[
f_j(x) = U_j \text{diag} (I_r, x^2 I_r) U_j^T.
\]

Then their derivatives are

\[
f_0^{(k)}(1) = \begin{cases}
\frac{1}{2} (I_r, I_r), & k = 0, \\
\frac{1}{2} (0_r, I_r), & k = 1, \\
0_{r \times 2r}, & k \geq 2,
\end{cases}
\]

\[
f_j^{(k)}(1) = \begin{cases}
I_{2r}, & k = 0, \\
2U_j \text{diag} (0_r, I_r) U_j^T, & k = 1, 2, \\
0_{2r}, & k \geq 3.
\end{cases}
\]

Or equivalently,

\[
f_0^{(k)}(1) = \frac{1}{2} (\delta_k I_r, (\delta_k + \delta_{k-1})I_r),
\]

\[
f_j^{(k)}(1) = U_j \text{diag} (\delta_k I_r, (\delta_k + 2\delta_{k-1} + 2\delta_{k-2})I_r) U_j^T.
\]

Thus, by Lemma 3.2, the derivative of $G$ at $x = 1$ is given as in the next theorem.

**Theorem 3.3.** The derivatives of $G(x)$ is given as

\[
G^{(k)}(1) = \frac{1}{2} (I_r, I_r) \sum_{k_1+\cdots+k_M = k, \ 0 \leq k_i \leq 2} \frac{k!}{k_1! \cdots k_M!} \prod_{j=1}^{N} U_j \text{diag} (\delta_{k_j} 0_r, (\delta_{k_j} + 2\delta_{k_j-1} + 2\delta_{k_j-2})I_r) U_j^T
\]

\[+ \frac{1}{2} (0_r, I_r) \sum_{k_1+\cdots+k_M = k-1, \ 0 \leq k_i \leq 2} \frac{(k-1)!}{k_1! \cdots k_M!} \prod_{j=1}^{N} U_j \text{diag} (\delta_{k_j} 0_r, (\delta_{k_j} + 2\delta_{k_j-1} + 2\delta_{k_j-2})I_r) U_j^T.
\]
Proof. By using Lemma 3.2 and relation (3.3), we have
\[
G^{(k)}(1) = \frac{1}{2}(I_r, I_r) \sum_{k_1+\ldots+k_N=k \atop 0 \leq k_i \leq 2} \frac{k!}{k_1! \cdots k_M!} f_1^{(k_1)}(1) \cdots f_N^{(k_N)}(1)
\]
(3.6)
\[
+ \frac{1}{2}(0_r, I_r) \sum_{k_1+\ldots+k_N=k-1 \atop 0 \leq k_i \leq 2} \frac{(k-1)!}{k_1! \cdots k_M!} f_1^{(k_1)}(1) \cdots f_N^{(k_N)}(1).
\]
This combined with relations (3.4)-(3.5) concludes the proof of this theorem. □

Although this theorem gives a close form of the derivative \(G^{(j)}(1)\), it is not very convenient for computational purpose. It is implied from relations (3.3) and (3.6) that to compute \(G^{(k)}(1)\), we should consider matrix multiplication of the following form \(\prod_{j=1}^{M} U_j \text{ diag} (0_r, I_r) U_j^T\) where each \(U_j\) is characterized as in (1.7).

Proposition 3.4. Let \(U_j\) be characterized as in (1.7). Then for any \(2 \leq M \leq N\), we have
\[
\prod_{j=1}^{M} U_j \begin{pmatrix} 0_r & I_r \end{pmatrix} U_j^T = U_1 \begin{pmatrix} 0_r & 0_r \\ 0_r & \prod_{j=1}^{M-1} (Q_j^T Q_{j+1} + J_r P_j^T P_{j+1} J_r) \end{pmatrix} U_M^T.
\]
(3.7)
Proof. We will prove this proposition by induction of \(M\).

(1) For \(M = 2\), by using the fact that
\[
U_j \begin{pmatrix} 0_r & I_r \end{pmatrix} U_j^T = S \begin{pmatrix} P_j & 0_r \\ 0_r & Q_j \end{pmatrix} S^T \begin{pmatrix} 0_r & I_r \\ 0_r & L_r \end{pmatrix} S \begin{pmatrix} P_j^T & 0_r \\ 0_r & Q_j^T \end{pmatrix} S^T,
\]
we have
\[
U_1 \begin{pmatrix} 0_r & I_r \end{pmatrix} U_1^T U_2 \begin{pmatrix} 0_r & I_r \end{pmatrix} U_2^T = S \begin{pmatrix} P_1 & 0_r \\ 0_r & Q_1 \end{pmatrix} S^T \begin{pmatrix} 0_r & I_r \\ 0_r & L_r \end{pmatrix} S \begin{pmatrix} P_1^T & 0_r \\ 0_r & Q_1^T \end{pmatrix} S^T \begin{pmatrix} 0_r & I_r \\ 0_r & L_r \end{pmatrix} S \begin{pmatrix} P_2 & 0_r \\ 0_r & Q_2 \end{pmatrix} S^T.
\]
Note that for any \(r \times r\) matrices \(A, B, C, D\), the following identities hold:
\[
S \begin{pmatrix} P & 0_r \\ 0_r & Q \end{pmatrix} S^T \begin{pmatrix} 0_r & I_r \\ 0_r & D \end{pmatrix} = \begin{pmatrix} 0_r & 0_r \\ 0_r & D \end{pmatrix}, \quad (3.9)
\]
\[
S \begin{pmatrix} P & 0_r \\ 0_r & Q \end{pmatrix} S^T = \begin{pmatrix} * & * \\ * & Q_1^T Q_2 + J_r P_1^T P_2 J_r \end{pmatrix}.
\]
(3.10)
Thus, we have
\[
U_1 \begin{pmatrix} 0_r & I_r \end{pmatrix} U_1^T U_2 \begin{pmatrix} 0_r & I_r \end{pmatrix} U_2^T = U_1 \begin{pmatrix} 0_r & Q_1^T Q_2 + J_r P_1^T P_2 J_r \\ 0_r & D \end{pmatrix} U_2^T.
\]
That is, we have proved the relation (3.7) for \(M = 2\).
(2) Suppose relation (3.7) holds for some $M \geq 2$, then, by induction,

$\prod_{j=1}^{M+1} U_j \left( \begin{smallmatrix} 0_r & I_r \end{smallmatrix} \right) U_j^T$

(3.11)

$= U_1 \left( \begin{smallmatrix} 0_r & \prod_{j=1}^{M-1} (Q_j^T Q_{j+1} + J_r P_{j+1} J_r) \end{smallmatrix} \right) U_1^T U_{M+1} \left( \begin{smallmatrix} 0_r & I_r \end{smallmatrix} \right) U_{M+1}^T.$

Note that for any $r \times r$ matrices $A, B, C, D, F$
the following identities hold:

$U_M^T U_{M+1} = S \left( \begin{smallmatrix} P_M^T & Q_{M+1}^T \end{smallmatrix} \right) \left( \begin{smallmatrix} P_{M+1}^T & Q_M^T \end{smallmatrix} \right)^T$

$= \left( \begin{smallmatrix} * & * \\ * & Q_{M+1}^T Q_M + J_r P_M^T P_{M+1} J_r \end{smallmatrix} \right).$

Consequently,

$\prod_{j=1}^{M+1} U_j \left( \begin{smallmatrix} 0_r & I_r \end{smallmatrix} \right) U_j^T = U_1 \left( \begin{smallmatrix} 0_r & 0_r \\ 0_r & \prod_{j=1}^{M} (Q_j^T Q_{j+1} + J_r P_{j+1} J_r) \end{smallmatrix} \right) U_1^T U_{M+1}.$

That is, we have proved that this proposition holds for $M + 1$. □

Another question concerning the formula (3.6) is as follows: how to characterize the set $\{(j_1, \ldots, j_N) : j_1 + \cdots + j_N = k, j_i \in \{0, 1, 2\}\}$. In fact we only have to consider the following simpler sets

(3.12)

$S_{N,k} := \{(j_1, \ldots, j_N) : j_1 \geq j_2 \geq \cdots \geq j_N, j_1 + \cdots + j_N = k, j_i \in \{0, 1, 2\}\}$.

**Proposition 3.5.** Given a natural number $N$, suppose $S_{N,k}$ are defined in (3.12) for all nonnegative integers $k$. Then, we have

(3.13)

$\#(S_{N,k}) = \begin{cases} [\frac{k}{2}] + 1 & 0 \leq k \leq N, \\
N - [\frac{k}{2}] & N + 1 \leq k \leq 2N, \\
0 & k \geq 2N + 1. \end{cases}$
Proof. To verify the equalities (3.13), we only have to prove that the third equality holds for all \( 2 \leq k \leq N \), that is, \( S_{N,k} = \{(1,\ldots,1,0,\ldots,0)\} \cup \tilde{S}_{N,k-2} \). On the one hand, for any \((j_1,\ldots,j_N) \in S_{N,k}\), we have either \( j_1 = 2 \) or \( j_1 = 1 \). If \( j_1 = 2 \), then it is easy to verify that \((j_2,\ldots,j_N,0) \in S_{N,k-2}\), thus, \((j_1,\ldots,j_N) \in \tilde{S}_{N,k-2}\); in the case that \( j_1 = 1 \), we have \((j_1,\ldots,j_N) = (1,\ldots,1,0,\ldots,0)\). So we have prove that

\[
S_{N,k} \subseteq \left\{(1,\ldots,1,0,\ldots,0)\right\} \cup \tilde{S}_{N,k-2}.
\]

On the other hand, firstly we know that \((1,\ldots,1,0,\ldots,0) \in S_{N,k}\). For any \((2,j_1,\ldots,j_{N-1}) \in \tilde{S}_{N,k-2}\) where \((j_1,\ldots,j_N) \in S_{N,k-2}\), one can check easily that \( j_N = 0 \). Otherwise, if \( j_N \geq 1 \), then it is implied that \( j_1 + \cdots + j_N \geq N > k - 2 \). Thus, we have \((2,j_1,\ldots,j_{N-1}) \in S_{N,k}\). Consequently, we have proved that \( S_{N,k} \supseteq \{(1,\ldots,1,0,\ldots,0)\} \cup \tilde{S}_{N,k-2}\). Combining the two result, the proof of (3.13) is finished. The equality (3.14) can be easily checked by using the results of (3.13). \(\square\)

Proposition 3.5 produces a recursive method to construct the sets \( S_{N,k} \). In what follows we will show exactly what \( S_{N,k} \) are.

**Proposition 3.6.** Let the sets \( S_{N,k} \) be defined as in (3.12). Then for any \( 0 \leq k \leq N \), we have

\[
S_{N,k} = \bigcup_{j=0}^{[N/k]} \left\{(2^{j-1}2^j1^{N-k-j},0,\ldots,0)\right\}.
\]

And, for those \( N + 1 \leq k \leq 2N \), we have

\[
S_{N,k} = \bigcup_{j=0}^{[N/k]} \left\{2^{j-N}k\cdot\cdot\cdot2^j1\cdot\cdot\cdot1,0,\ldots,0\right\}.
\]

**Proof.** One can easily checks that equality (3.16) is implied by (3.13) and (3.15). We will prove by induction of \( k \) that equality (3.15) holds for all \( 0 \leq k \leq N \). Firstly, it is easy to verify that this holds for \( k = 0,1 \). Furthermore, the fact that \( S_{N,2} = \{(2,0,0,\ldots,0),(1,1,0,\ldots,0)\} \) implies that this equality also holds for \( k = 2 \).

Suppose there are some \( 2 \leq k \leq N - 1 \), such that equality (3.15) holds for all \( 0 \leq n \leq k \). By (3.13), we have

\[
S_{N,k+1} = \{(1,\ldots,1,0,\ldots,0)\} \cup \tilde{S}_{N,k-1},
\]

with \( \tilde{S}_{N,k-1} = \{(2,j_1,\ldots,j_{N-1}) : (j_1,\ldots,j_N) \in S_{N,k-1}\} \).
As was pointed out in the proof of the last proposition, for any \((j_1, \ldots, j_N) \in S_{N,k-1}\), we have \(j_N = 0\). Thus, by induction,

\[
S_{N,k-1} = \bigcup_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left\{ \begin{array}{l}
\frac{j+1}{2} \\
(2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)
\end{array} \right\},
\]

so that

\[
S_{N,k+1} = \left\{ \begin{array}{l}
k+1
\end{array} \right\} \cup \bigcup_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left\{ \begin{array}{l}
k+1
\end{array} \right\} \bigcup_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left\{ \begin{array}{l}
k+1
\end{array} \right\} \bigcup_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left\{ \begin{array}{l}
k+1
\end{array} \right\} \bigcup_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left\{ \begin{array}{l}
k+1
\end{array} \right\}.
\]

(3.18)

Note that to prove the equality (3.18), we have used the equality \([k+1] - 1 = [k\cdot\frac{k}{2}]\).

\[\square\]

Combining the results in relation (3.6), and Propositions 3.4, 3.6, we propose the following theorem which seems more convenient to compute \(G^{(1)}(1)\).

**Theorem 3.7.** For \(0 \leq k \leq N\), we have

\[
\sum_{k_1 + \cdots + k_N = k, \quad 0 \leq h \leq 2} \frac{k!}{k_1! \cdots k_M!} f^{(k_1)}(1) \cdots f^{(k_N)}(1)
\]

\[= \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-j}{j} \sum_{(1 \leq p < \cdots < p_{k-j} \leq N)} 2^{k-2j} k! U_{p^\prime} \left( \begin{array}{l}
0, 0, 0, 0, \prod_{i=1}^{k-j-1} (Q_{p_{i+1}}^T Q_{p_{i+1}} + J P_{p_{i+1}}^T P_{p_{i+1}} J)
\end{array} \right) U_{p^\prime}^T.
\]

And, for \(N + 1 \leq k \leq 2N\), we have

\[
\sum_{k_1 + \cdots + k_N = k, \quad 0 \leq h \leq 2} \frac{k!}{k_1! \cdots k_M!} f^{(k_1)}(1) \cdots f^{(k_N)}(1)
\]

\[= \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-j}{j} \sum_{(1 \leq p < \cdots < p_{k-j} \leq N)} \frac{k!}{2^{k-j-N}} U_{p^\prime} \left( \begin{array}{l}
0, 0, \prod_{i=1}^{k-j-1} (Q_{p_{i+1}}^T Q_{p_{i+1}} + J P_{p_{i+1}}^T P_{p_{i+1}} J)
\end{array} \right) U_{p^\prime}^T.
\]

**Proof.** For any \(n\)-tuples \((a_1, \ldots, a_n)\), denotes by \(P(a_1, \ldots, a_n)\) the set of permutations of \(a_1, \ldots, a_n\). First consider the case \(0 \leq k \leq N\), we have

\[
\sum_{k_1 + \cdots + k_N = k, \quad 0 \leq h \leq 2} \frac{k!}{k_1! \cdots k_M!} f^{(k_1)}(1) \cdots f^{(k_N)}(1)
\]

\[= \sum_{(j_1, \ldots, j_N) \in S_{N,k}} \sum_{(k_1, \ldots, k_N) \in P(j_1, \ldots, j_N)} \frac{k!}{k_1! \cdots k_M!} f^{(k_1)}(1) \cdots f^{(k_N)}(1)
\]
\[
\sum_{(j_1, \ldots, j_N) \in \mathbb{S}_N, (k_1, \ldots, k_N) \in P(j_1, \ldots, j_N)} \frac{k!}{j_1! \cdots j_M!} f^{(k_1)}(1) \cdots f^{(k_N)}(1)
\]

\[=
\sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \sum_{(k_1, \ldots, k_N) \in P(\epsilon_{j,k})} \frac{k!}{2^j j^{k-2j}} f^{(k_1)}(1) \cdots f^{(k_N)}(1)
\]

where the \(N\)-tuples \(e_{j,k}\) are defined as \(e_{j,k} := (2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)\), note that the last equality holds due to Proposition 3.6. Next, we will show clearly what the summation in the bracket is. For any given \(j = 0, 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\), let

\[
C_1 := \sum_{(k_1, \ldots, k_N) \in P(\epsilon_{j,k})} \frac{k!}{2^j} f^{(k_1)}(1) \cdots f^{(k_N)}(1),
\]

\[
C_2 := \sum_{(q_1, \ldots, q_{k-j}) \in P(\tilde{e}_{j,k})} \sum_{1 \leq p_1 < \cdots < p_{k-j} \leq N} \frac{k!}{2^j} f^{(q_1)}(1) \cdots f^{(q_{k-j})}(1),
\]

where \(\tilde{e}_{j,k} := (2, \ldots, 2, 1, \ldots, 1, 1)\), we claim that \(C_1 = C_2\). On the one hand, it is straightforward to prove that the terms of summations are equal since \(\binom{N}{j} \binom{N-j}{k-2j} = \binom{k-j}{j} \binom{N}{k-j}\). On the other hand, we will verify that any summation term of \(C_1\) emerges also in \(C_2\). For fixed permutation of \(e_{j,k}\), it is implied from relation (3.3) that only \(k-j\) terms in the product \(\frac{k!}{2^j} f^{(k_1)}(1) \cdots f^{(k_N)}(1)\) counts. Thus, there exists \(1 \leq p_1 < \cdots < p_{k-j} \leq N\), and \(q_1, \ldots, q_{k-j}\) being permutation of \(e_{j,k}\) so that \(\frac{k!}{2^j} f^{(k_1)}(1) \cdots f^{(k_N)}(1) = \frac{k!}{2^j} f^{(q_1)}(1) \cdots f^{(q_{k-j})}(1)\).

This concludes the proof of identity \(C_1 = C_2\). So we have

\[
\sum_{k_1 + \cdots + k_N = k, 0 \leq k_i \leq 2} \frac{k!}{k_1! \cdots k_M!} f^{(k_1)}(1) \cdots f^{(k_N)}(1)
\]

\[=
\sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \sum_{(q_1, \ldots, q_{k-j}) \in P(\tilde{e}_{j,k})} \sum_{1 \leq p_1 < \cdots < p_{k-j} \leq N} \frac{k!}{2^j} f^{(q_1)}(1) \cdots f^{(q_{k-j})}(1)
\]

\[=
\sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-j}{j} \sum_{1 \leq p_1 < p_2 < \cdots < p_{k-j} \leq N} \frac{k!}{2^j} f^{(q_1)}(1) \cdots f^{(q_{k-j})}(1)
\]

\[=
\sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-j}{j} \sum_{1 \leq p_1 < p_2 < \cdots < p_{k-j} \leq N} \mathbf{w}_{n}^{(j)}(0, 0, 0, \ldots, 0, \pi_{k-j}^{-1}) (Q^n_{0} Q_{p} + J P_{p}^{T} P_{n+j}) U_{p_{k-j}}^{T}.
\]
It should be noted that we have used Proposition 3.4 and the following facts:

1. for any $j = 1, \ldots, N$, $f_j^{(1)}(1) = f_j^{(2)}(1) = 2U_j \left( \begin{smallmatrix} 0_r & \cdots & 0_r \end{smallmatrix} \right) U_j^T$;

2. the number of permutations of $(2, \ldots, 2, 1, \ldots, 1)$ is $\binom{m+n}{m}$.

By the same trick we can prove the theorem for the case $N+1 \leq k \leq 2N$. □

4. Numerical examples

Consider the case $r = 2$. Assume that filter banks are constructed as in (1.5)-(1.7) where the $2 \times 2$ real orthogonal matrices $P_j, Q_j$ are

$$
P_j = \begin{pmatrix} \cos \alpha_j & \sin \alpha_j \\
-\sin \alpha_j & \cos \alpha_j \end{pmatrix}, \quad Q_j = \begin{pmatrix} \cos \beta_j & \sin \beta_j \\
-\sin \beta_j & \cos \beta_j \end{pmatrix}.
$$

Let $\gamma_j = \alpha_j + \beta_j$, then the matrices $B_1, B_2$ defined in Theorem 2.3 are

$$
B_1 = -\lambda_3 I_2, \quad B_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix},
$$

with parameters

$$
\lambda_1 = \frac{1}{2} + \sum_{j=1}^{N} \cos \gamma_j,
$$

$$
\lambda_2 = -N + \sum_{j=1}^{N} \sin \gamma_j - (N+1) \sum_{j=1}^{N} \cos \gamma_j + \sum_{1 \leq k < j \leq N} \sin (\gamma_j - \gamma_k),
$$

$$
\lambda_3 = -N - \sum_{j=1}^{N} \sin \gamma_j - (N+1) \sum_{j=1}^{N} \cos \gamma_j - \sum_{1 \leq k < j \leq N} \sin (\gamma_j - \gamma_k).
$$

Thus, by Theorem 2.3, we have the following sufficient and necessary conditions for the corresponding scaling functions to have accuracy $p = 2, 3$.

**Theorem 4.1.** When $r = 2$, and wavelet filter banks are constructed as in (1.5)-(1.7) and (4.19), then the corresponding scaling functions have at least accuracy of order $p = 2$ if and only if

$$
\sum_{j=1}^{N} \cos \gamma_j = -\frac{1}{2}.
$$

Moreover, the corresponding scaling functions have at least accuracy of order $p = 3$ if and only if in addition to (4.20), the following equality holds:

$$
\sum_{j=1}^{N} \sin \gamma_j + \sum_{1 \leq k < j \leq N} \sin (\gamma_j - \gamma_k) = \pm \frac{1}{2}.
$$
Proof. By Theorem 2.3, the scaling functions have at least second accuracy if and only if there exists nonzero $2 \times 1$ vector $\nu_0$ such that $B_1 \nu_0 = -\lambda_1 \nu_0 = 0$, this reduces to $\lambda_1 = 0$.

Similarly, the scaling functions have at least third accuracy if and only if there exists nonzero $2 \times 1$ vector $\nu_0$ such that $B_1 \nu_0 = -\lambda_1 \nu_0 = 0, B_2 \nu_0 = (\lambda_2 0 \lambda_3) \nu_0 = 0$, this reduces to either of the following two equalities:

$$
\begin{align*}
\lambda_1 &= 0, \\
\lambda_2 &= 0,
\end{align*}
$$

or

$$
\begin{align*}
\lambda_1 &= 0, \\
\lambda_3 &= 0.
\end{align*}
$$

□

![Figure 1. Wavelet and scaling function with second accuracy](image)

By equation (4.20), to obtain second accuracy and the minimal length of the filters, we should choose $N = 1$, that is, $\cos \gamma_1 = -\frac{1}{2}$. In fact, in this case, $\psi_2, \phi_2$ are the Daubechies’s wavelet function db2 and the corresponding scaling function. The graph of the above functions are plotted in Figure 1.

On the other hand, to obtain third accuracy and minimal length, we have to choose $N = 2$. Here the equations (4.20) and (4.21) have four solutions:

$$
\begin{align*}
\gamma_1 &= \pi + \arcsin 0.5374, & \gamma_2 &= -\arcsin 0.9392; \\
\gamma_1 &= \pi + \arcsin 0.9756, & \gamma_2 &= \pi - \arcsin 0.9600; \\
\gamma_1 &= \pi - \arcsin 0.5374, & \gamma_2 &= -\arcsin 0.9392; \\
\gamma_1 &= \pi - \arcsin 0.9756, & \gamma_2 &= \pi + \arcsin 0.9600.
\end{align*}
$$

Take the first solution, and we present the graph of the above functions in Figure 2.
Figure 2. Wavelet and scaling function with third accuracy

References


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