A RECURSIVE FORMULA FOR THE JONES POLYNOMIAL OF 2-BRIDGE LINKS AND APPLICATIONS

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Abstract. In this paper, we give a recursive formula for the Jones polynomial of a 2-bridge knot or link with Conway normal form \( C(-2n_1, 2n_2, -2n_3, \ldots, (\pm 1)^r 2n_r) \) in terms of \( n_1, n_2, \ldots, n_r \). As applications, we also give a recursive formula for the Jones polynomial of a 3-periodic link \( L^{(3)} \) with rational quotient

\[ L^{(3)} = C(2, n_1, -2, n_2, \ldots, n_r, (\pm 1)^r 2) \]

for any nonzero integers \( n_1, n_2, \ldots, n_r \) and give a formula for the span of the Jones polynomial of \( L^{(3)} \) in terms of \( n_1, n_2, \ldots, n_r \) with \( n_i \neq \pm 1 \) for all \( i = 1, 2, \ldots, r \).

1. Introduction

The Jones polynomial of an oriented link in \( S^3 \) was first introduced in [4]. Kauffman [8] and Murasugi [15] have used the Jones polynomial in verifying Tait conjecture which states that a reduced alternating diagram has minimal crossing number. Let \( D \) be a connected, prime diagram of an oriented link \( L \). Then the span of the Jones polynomial of \( L \) is less than or equal to the number of crossings of \( D \) and the equality holds if and only if \( D \) is reduced alternating [8, 15, 23].

In 1956, a characterization of 2-bridge knots and links was introduced by Schubert [21]. In [1], Conway introduced another presentation, now called Conway normal form, of 2-bridge knots and links. Several people have studied the Jones polynomials of 2-bridge knots and links [5, 6, 13, 14, 18, 19, 22]. In 1987, Lichorish and Millett [13] gave an algorithm to calculate the Homfly polynomials of 2-bridge knots and links with matrix manipulations. In 2002, Nakabo [19] also presented an explicit formula of the Homfly polynomials of 2-bridge knots and links. Lu and Zhong [14] computed the Kauffman polynomials of 2-bridge knots and links using the Kauffman skein theory and linear algebra techniques. Note that the Jones polynomial can be obtained from the Homfly and Kauffman polynomials by substituting variables.

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On the other hand, Hilden, Lozano, and Montesinos-Amilibia [2] introduced a special kind of Conway normal form of a 2-bridge link with two components and studied the excellent component of the character variety of periodic knots in $S^3$ with rational quotients. In [16], Murasugi described several relationships between the Jones polynomials of a periodic link and its factor link. It is remarkable that the set of periodic links with rational quotients is a special family of periodic links which contains all 2-bridge knots and links, all torus knots and links and some pretzel knots and links, etc. It is known that every 2-bridge knot or link is a 2-periodic link with rational quotient and every 2-periodic link with rational quotient is a 2-bridge knot or link [3]. The second and third authors [10] re-examined Hilden, Lozano, and Montesinos-Amilibia’s presentation to study the Alexander polynomials of 2-bridge links with Conway normal form $C(2, n_1, -2, n_2, \ldots, -2, n_r, (−1)^r 2)$ and $q$-periodic links in $S^3$ with rational quotients $C(2, n_1, -2, n_2, \ldots, n_r, (−1)^r 2)$ in terms of $n_1, n_2, \ldots, n_r$ and its period $q$. Thereafter, some properties for the family of periodic links with rational quotients are studied [3, 9, 10, 11, 12].

In this paper, we first give a recursive formula for the Jones polynomial of a 2-periodic link with rational quotient, which is actually a recursive formula for the Jones polynomial of a 2-bridge knot or link. Generalizing this formula, we also obtain a recursive formula for the Jones polynomial of a 3-periodic link with rational quotient and a formula for the span of the Jones polynomial of this kind of 3-periodic link.

This paper is organized as follows. In Section 2, we review presentations of 2-bridge knots and links and periodic links with rational quotients. Section 3 contains the definition of bracket polynomial and formulas for periodic links with rational quotients. In Section 4, for arbitrary given nonzero integers $n_1, n_2, \ldots, n_r$, we give a recursive formula for the Jones polynomial of a 2-bridge knot or link with Conway normal form $C(−2n_1, 2n_2, −2n_3, \ldots, (−1)^r 2n_r)$ in terms of $n_1, n_2, \ldots, n_r$. In Section 5, we give a recursive formula for the Jones polynomial of a 3-periodic link $L^{(3)}$ with rational quotient $L = C(2, n_1, -2, n_2, \ldots, n_r, (−1)^r 2)$ for arbitrary given nonzero integers $n_1, n_2, \ldots, n_r$ and give a formula for the span of the Jones polynomial of $L^{(3)}$ in terms of $n_1, n_2, \ldots, n_r$ with $n_i \neq ±1$ for all $i = 1, 2, \ldots, r$. The formula for the span gives a lower bound for the minimal crossing number of the 3-periodic link $L^{(3)}$.

2. Periodic links with rational quotients

To each pair $(α, β)$ of two co-prime integers subject to the condition that $β$ is odd and $0 < |β| < α$, Schubert [21] associated an oriented diagram on the 2-sphere $S^2$ of an oriented 2-bridge knot ($α$ odd) or link ($α$ even) $L$ in $S^3$, now called the Schubert normal form of $L$ and denoted by $S(α, β)$, and showed that any (oriented) 2-bridge knots and links in $S^3$ can be represented in this way. Two such pairs of integers $(α, β)$ and $(α’, β’)$ define an equivalent oriented
(resp. unoriented) knot or link if and only if
\[ \alpha = \alpha' \quad \text{and} \quad \beta \equiv \beta' \mod 2\alpha, \]
where \( \beta^{-1} \) denotes the integer with the properties \( 0 < \beta^{-1} < 2\alpha \) and \( \beta \beta^{-1} \equiv 1 \mod 2\alpha. \)

Let \([a_1, a_2, \ldots, a_n]\) denote a continued fraction expansion of \( \alpha/\beta \):
\[ [a_1, a_2, \ldots, a_n] \equiv a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}} = \frac{\alpha}{\beta}. \]

Then \( L = S(\alpha, \beta) \) has also a diagram \( C(a_1, a_2, \ldots, a_n) \), called Conway normal form of \( L \), as shown in Figure 1, depending on whether \( n \) is even or odd [1]. The integral tangles in Figure 1, which are rectangles labeled \( a_i \), are the 2-braids with \( |a_i| \) crossings as shown in Figure 2. It is well known that \( L = S(\alpha, \beta) \) admits a diagram \( C(2b_1, 2b_2, \ldots, 2b_n) \), which is equivalent to \( C(a_1, a_2, \ldots, a_n) \) [7].

\[ \begin{align*}
\text{Figure 1} \\
\text{Figure 2}
\end{align*} \]

It is known [2, 10] that the 2-bridge link \( L = S(\alpha, \beta)(\alpha \text{ even}) \) can also be represented by Conway diagram of the form \( C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r 2) \) as
shown in Figure 3. We choose an orientation of the 2-bridge link $C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r2)$ as shown in Figure 3. Then it is easy to see that the diagram shown in Figure 3 can be deformed to the diagrams in Figure 4 by using Reidemeister moves. Throughout this paper, an oriented 2-bridge link $L$ in $S^3$ represented by the Conway normal form $C(2, n_1, -2, n_2, \ldots, n_r, (-1)^r2)$ is denoted by $L = \overline{C}[[n_1, n_2, \ldots, n_r]]$.

![Figure 3](image1)

![Figure 4](image2)
A link \( L \) in \( S^3 \) is called a \( p \)-periodic link (\( p \geq 2 \) an integer) if there exists an orientation preserving auto-homeomorphism \( h \) of \( S^3 \) such that \( h(L) = L \), \( h \) is of order \( p \) and the set \( \text{Fix}(h) \) of fixed points of \( h \) is a circle disjoint from \( L \). In this case, the link \( L/(h) \cup \text{Fix}(h) \) in the orbit space \( S^3/(h) \cong S^3 \) is called the quotient link of \( L \). Let \( K \) be an oriented link in \( S^3 \) and \( U \) an oriented trivial knot with \( K \cap U = \emptyset \). For any integer \( p \geq 2 \), let \( \phi^p_U : \Sigma^3 \rightarrow S^3 \) be a \( p \)-fold branched cyclic covering branched along \( U \). Then \( \Sigma^3 \) is homeomorphic to the 3-sphere \( S^3 \), and \( (\phi^p_U)^{-1}(K) \) is a \( p \)-periodic link in \( \Sigma^3 \) with \( L = K \cup U \) as its quotient link. We give an orientation to \( (\phi^p_U)^{-1}(K) \) induced by the orientation of \( K \). Note that any periodic knot or link in \( S^3 \) arises in this manner.

**Definition** ([10]). A link \( \tilde{L} \) in \( S^3 \) is called a \( p \)-periodic link with rational quotient if it is a \( p \)-periodic link whose quotient link is a 2-bridge link, or equivalently, if there exists a 2-bridge link \( L = U_1 \cup U_2 \) in \( S^3 \) such that \( \tilde{L} \) is equivalent to the preimage \( (\phi^p_U)^{-1}(U_1) \) of the component \( U_1 \) of \( L \) by a \( p \)-fold cyclic covering \( \phi^p_U : \Sigma^3 \rightarrow S^3 \) branched along the component \( U_2 \) of \( L \).

Note that each component \( U_1 \) and \( U_2 \) of \( L \) is a trivial knot and they can be interchanged each other by an orientation preserving homeomorphism of \( S^3 \) [17]. This implies that \( (\phi^p_{U_2})^{-1}(U_1) \) is equivalent to \( (\phi^p_{U_1})^{-1}(U_2) \). Now let \( L = \overline{C}[[n_1, n_2, \ldots, n_r]] \) be an oriented 2-bridge link as shown Figure 4. Then the diagram \( D^{(p)} \) shown in Figure 5 is a canonical oriented \( p \)-periodic diagram of the oriented \( p \)-periodic link \( (\phi^p_{U_2})^{-1}(U_1) \) with rational quotient \( \overline{C}[[n_1, n_2, \ldots, n_r]] \). In what follows, we shall denote the oriented \( p \)-periodic link \( (\phi^p_{U_2})^{-1}(U_1) \) by \( L^{(p)} \) or \( \overline{C}[[n_1, n_2, \ldots, n_r]]^{(p)} \) for our convenience. Then any \( p \)-periodic link with rational quotient can be represented by \( \overline{C}[[n_1, n_2, \ldots, n_r]]^{(p)} \) for some nonzero integers \( n_1, n_2, \ldots, n_r \) [3, 10].

**Figure 5.** The canonical \( p \)-periodic diagram \( D^{(p)} \) of \( L^{(p)} \)
3. Bracket polynomial of periodic links

The bracket polynomial of an unoriented link diagram $D$, denoted by $\langle D \rangle$, is a Laurent polynomial in a single variable $A$ defined by the following three axioms:

1. If denotes the standard diagram of the unknot, then
   \[ \langle \ \rangle = 1. \]

2. If $\delta = -A^{-2} - A^2$ and $D \cup \mathcal{U}$ denotes the diagram $D$ together with the standard diagram of the unknot, disjoint from $D$, then
   \[ \langle D \cup \mathcal{U} \rangle = \delta \langle D \rangle. \]

3. Suppose that $D_+, D_0$ and $D_\infty$ are the diagrams that are exactly the same except at a neighborhood of one crossing point in which the diagrams differ as shown in Figure 6. Then
   \[ \langle D_+ \rangle = A \langle D_\infty \rangle + A^{-1} \langle D_0 \rangle. \]

![Figure 6](image)

From (3), we also obtain the equation
\[ \langle D_- \rangle = A \langle D_0 \rangle + A^{-1} \langle D_\infty \rangle. \]

It is easy to see that $\langle D \rangle$ is an invariant under Reidemeister moves II and III, but not an invariant under Reidemeister move I. If $\varphi_+, \varphi_-$ and $\varphi_0$ are the diagrams that are exactly the same except at a neighborhood of one crossing point in which the diagrams differ as shown in Figure 7, then we have
\[ \langle \varphi_+ \rangle = (-A)^3 \langle \varphi_0 \rangle, \quad \langle \varphi_- \rangle = (-A)^{-3} \langle \varphi_0 \rangle. \]

![Figure 7](image)
For a link $L$ with its diagram $D$, the Jones polynomial $V_L(t)$ of $L$ is defined as

$$V_L(t) = (-A)^{-3w(D)} \langle D \rangle$$

by setting $A^{-4} = t$ [8].

**Lemma 3.1.** For each integer $n$, let $T(n)$ be a diagram with $n$-half twists and fixed outside as described in Figure 8. Then for any integer $n \geq 1$, we have

$$\langle T(n) \rangle = A^{-n} \langle T(0) \rangle + A^{-n+2} \sum_{i=0}^{n-1} (-A^4)^i \langle T(\infty) \rangle$$

and

$$\langle T(-n) \rangle = A^n \langle T(0) \rangle + A^{n-2} \sum_{i=0}^{n-1} (-A^{-4})^i \langle T(\infty) \rangle.$$  

**Figure 8**

*Proof. First we will prove that the equation (6) holds. If $n = 1$, then $\langle T(1) \rangle = A^{-1} \langle T(0) \rangle + A \langle T(\infty) \rangle$ by (3). For a positive integer $n > 1$, we assume that

$$\langle T(n) \rangle = A^{-n} \langle T(0) \rangle + A^{-n+2} \sum_{i=0}^{n-1} (-A^4)^i \langle T(\infty) \rangle.$$  

Then it follows that

$$\langle T(n+1) \rangle = A^{-1} \langle T(n) \rangle + A(-A)^{3n} \langle T(\infty) \rangle$$

$$= A^{-1} \left( A^{-n} \langle T(0) \rangle + A^{-n+2} \sum_{i=0}^{n-1} (-A^4)^i \langle T(\infty) \rangle \right)$$

$$+ A^{-(n+1)+2} (-A^4)^n \langle T(\infty) \rangle$$

$$= A^{-(n+1)} \langle T(0) \rangle + A^{-(n+1)+2} \sum_{i=0}^{n} (-A^4)^i \langle T(\infty) \rangle.$$  

By a similar argument, we obtain the equation (7). □*
For any nonzero integer \( n \), we define Laurent polynomials \( \alpha_n \) and \( \beta_n \) by

\[
\alpha_n = A^{-n}, \quad \beta_n = \begin{cases} 
A^{-n+2} \sum_{i=0}^{n-1} (-A^4)^i & \text{if } n \geq 1, \\
A^{-n-2} \sum_{i=0}^{-n-1} (-A^{-4})^i & \text{if } n \leq -1.
\end{cases}
\]

Then we have easily:

**Lemma 3.2.** For given nonzero integers \( n \) and \( p \), we have that

\[
\beta_n \delta + p \alpha_n = A^{-n}((-A^4)^n + (p - 1)).
\]

For any nonzero integers \( n_1, n_2, \ldots, n_r \) \((r \geq 1)\) and a positive integer \( p \geq 2 \), let \( L^{(p)} \) be the \( p \)-periodic link in \( S^3 \) with rational quotient \( L = C[[n_1, n_2, \ldots, n_r]] \). We consider the \( p \)-periodic diagram \( D^{(p)} \) of \( L^{(p)} \) as shown in Figure 9. In Figure 9, each \( T_{i,j} \) denotes a 2-tangle with \( n_i \)-half twists as in Figure 3. If \( n_i \) is positive (respectively, negative), each crossing of \( T_{i,j} \) is positive (respectively, negative). Since the writhe, denoted by \( w(D^{(p)}) \), of \( D^{(p)} \) is the sum of crossing signs of crossings in \( D^{(p)} \), we get

\[
w(D^{(p)}) = p \sum_{i=1}^{r} n_i.
\]

**Figure 9**

Put \( T_i = \{T_{i,1}, T_{i,2}, \ldots, T_{i,p}\} \) for each \( i = 1, 2, \ldots, r \). We call a function \( s : T_i \rightarrow \{0, \infty\} \), where 0 denotes \( T(0) \) and \( \infty \) denotes \( T(\infty) \) a weight function of \( T_i \). For each \( i = 1, 2, \ldots, r \), let \( S_i \) denote the set of all weight functions of \( T_i \). For a weight function \( s \in S_r \), let \( D^{(p)}(s) \) be the diagram obtained from \( D^{(p)} \) by replacing each tangle \( T_{r,k} \) in \( T_r \) by a \( s(T_{r,k}) \)-tangle and we denote by \( \phi(s) \) the number of the tangles in \( s^{-1}(0) \). By applying Lemma 3.1 to each tangle \( T_{r,k} \) in \( T_r \), we have:
Proposition 3.3. For given nonzero integers $n_1, n_2, \ldots, n_r$ ($r \geq 1$) and a positive integer $p \geq 2$, let $L^{(p)}$ be the $p$-periodic link with rational quotient $L = \overline{C}[[n_1, n_2, \ldots, n_r]]$ and $D^{(p)}$ its $p$-periodic diagram as shown in Figure 9. Then
\[
\langle D^{(p)} \rangle = \sum_{s \in S_r} o_{n_r}^{\phi(s)} \beta_{n_r}^{\phi(s)} \langle D^{(p)}(s) \rangle.
\]

For an $r$-tuple $(s_1, s_2, \ldots, s_r)$ of weight functions with $s_i \in S_i (i = 1, 2, \ldots, r)$, let $D^{(p)}(s_1, \ldots, s_r)$ be the diagram obtained from $D^{(p)}$ by replacing each tangle $T_{i,j}$ with the $s_i(T_{i,j})$-tangle and we denote by $\phi(s_k)$ the number of tangles in $s_k^{-1}(0)$ for each $k = 1, 2, \ldots, r$. By applying Lemma 3.1 to each tangle $T_{i,j}$ in $D^{(p)}$, we also have:

Proposition 3.4. For given nonzero integers $n_1, n_2, \ldots, n_r$ ($r \geq 1$) and a positive integer $p \geq 2$, let $L^{(p)}$ be the $p$-periodic link with rational quotient $L = \overline{C}[[n_1, n_2, \ldots, n_r]]$ and $D^{(p)}$ its $p$-periodic diagram as shown in Figure 9. Then
\[
\langle D^{(p)} \rangle = \sum_{(s_1, \ldots, s_r) \in S_1 \times \cdots \times S_r} \left( \prod_{k=1}^{r} o_{n_k}^{\phi(s_k)} \beta_{n_k}^{\phi(s_k)} \right) \delta^{D^{(p)}(s_1, \ldots, s_r)-1},
\]
where $\delta = -A^2 - A^{-2}$ and $|D^{(p)}(s_1, \ldots, s_r)|$ is the number of disjoint simple closed curves in $D^{(p)}(s_1, \ldots, s_r)$.

Remark 3.5. Each diagram $D^{(p)}(s_1, \ldots, s_r)$ is a disjoint union of simple closed curves. If we can find a formula for the number of disjoint simple closed curves in each $D^{(p)}(s_1, \ldots, s_r)$ in terms of $n_1, n_2, \ldots, n_r$ and $p$, then the Laurent polynomial $\langle D^{(p)} \rangle$ can be expressed by means of the integers $n_1, n_2, \ldots, n_r$ and $p$. However, it looks very difficult to make such a formula. The authors know of none.

4. Recursive formula for the Jones polynomial of 2-bridge links

It is well known that any 2-bridge knot or link admits a diagram with Conway normal form $C(2a_1, 2a_2, \ldots, 2a_r)$ for some integers $a_1, a_2, \ldots, a_r$ [7]. In [3], Jang, the second and third authors proved that the 2-periodic link $L^{(2)}$ with rational quotient $L = \overline{C}[[n_1, n_2, \ldots, n_k]]$ is a 2-bridge knot or link with Conway normal form $C(-2n_1, 2n_2, -2n_3, \ldots, (-1)^r 2n_r)$. In this section we give a recursive formula for the Jones polynomial of a 2-periodic link with rational quotient and give a formula for the span of the Jones polynomial. Consequently, we get a recursive formula for the Jones polynomial of 2-bridge knot or link with Conway normal form $C(2a_1, 2a_2, \ldots, 2a_r)$ in terms of $a_1, a_2, \ldots, a_r$.

Lemma 4.1. Let $n_1, n_2, \ldots, n_r$ be given nonzero integers. For each $k = 1, 2, \ldots, r$, let $D^{(2)}_k$ be the canonical 2-periodic diagram of the 2-periodic link
with rational quotient $L_k = \overline{C}[[n_1, n_2, \ldots, n_k]]$. Let $D_0^{(2)}$ denote the standard diagram of the unknot. Then we have the following recursive formula:

$$D^{(2)}_0 = 1,$$

$$D^{(2)}_n = \beta^2_n \delta + 2\alpha_n \beta_n + \alpha^2_n \delta,$$

$$D_k^{(2)} = (\beta^2_n \delta + 2\alpha_n \beta_n)\langle D_{k-1}^{(2)} \rangle + \alpha_n^2 A^{6n-1}\langle D_{k-2}^{(2)} \rangle.$$

**Proof.** For a weight function $s \in S_k$, let $D_k^{(2)}(s(T_{k,1}), s(T_{k,2}))$ be the diagram obtained from $D_k^{(2)}$ by replacing each tangle $T_{k,i}$ by an $s(T_{k,i})$-tangle ($i = 1, 2$).

If $k = 1$, then $D_1^{(2)}(0,0)$, $D_1^{(2)}(0, \infty)$, $D_1^{(2)}(\infty,0)$ and $D_1^{(2)}(\infty, \infty)$ consist of simple closed curves without crossings. We observe that $D_1^{(2)}(0,0)$ and $D_1^{(2)}(\infty, \infty)$ have two components and $D_1^{(2)}(0, \infty)$ and $D_1^{(2)}(\infty, 0)$ have one component. By Proposition 3.4, we have

$$D_1^{(2)} = \beta^2_n \delta + 2\alpha_n \beta_n + \alpha^2_n \delta.$$

Now we assume that the recursive formula (9) holds for $n_1, n_2, \ldots, n_{k-1}$ with $k \geq 2$. Then $D_k^{(2)}(0,0)$, $D_k^{(2)}(0, \infty)$, $D_k^{(2)}(\infty,0)$ and $D_k^{(2)}(\infty, \infty)$ are isotopic to the diagrams as shown in Figure 10. Thus $D_k^{(2)}(0, \infty)$ and $D_k^{(2)}(\infty, 0)$ are isotopic to the diagram $D_{k-1}^{(2)}$, and $D_k^{(2)}(\infty, \infty)$ is isotopic to the diagram $D_{k-1}^{(2)} \sqcup D_{k-2}^{(2)}$. Moreover $D_k^{(2)}(0,0)$ is obtained from $D_{k-2}^{(2)}$ by applying the Reidemeister move I. By (2), (5) and Proposition 3.3, we have

$$D_k^{(2)} = \beta^2_k \langle D_k^{(2)}(\infty, \infty) \rangle + \beta_k \alpha_k \langle D_k^{(2)}(\infty, 0) \rangle + \alpha_k \beta_k \langle D_k^{(2)}(0, \infty) \rangle + \alpha_k^2 \langle D_k^{(2)}(0, 0) \rangle$$

$$= (\beta^2_k \delta + 2\alpha_k \beta_k)\langle D_{k-1}^{(2)} \rangle + \alpha_k^2 A^{6(k-1)}\langle D_{k-2}^{(2)} \rangle.$$

This completes the proof. \qed

**Figure 10**

-isotropic to the diagram $D_{k-1}^{(2)}$, and $D_k^{(2)}(\infty, \infty)$ is isotopic to the diagram $D_{k-1}^{(2)} \sqcup D_{k-2}^{(2)}$. Moreover $D_k^{(2)}(0,0)$ is obtained from $D_{k-2}^{(2)}$ by applying the Reidemeister move I. By (2), (5) and Proposition 3.3, we have

$$D_k^{(2)} = \beta^2_k \langle D_k^{(2)}(\infty, \infty) \rangle + \beta_k \alpha_k \langle D_k^{(2)}(\infty, 0) \rangle + \alpha_k \beta_k \langle D_k^{(2)}(0, \infty) \rangle + \alpha_k^2 \langle D_k^{(2)}(0, 0) \rangle$$

$$= (\beta^2_k \delta + 2\alpha_k \beta_k)\langle D_{k-1}^{(2)} \rangle + \alpha_k^2 A^{6(k-1)}\langle D_{k-2}^{(2)} \rangle.$$
For any nonzero integer $n$, let $A_n(t)$ be a Laurent polynomial in $\mathbb{Z}[t^{\pm \frac{1}{2}}]$ defined by

$$A_n(t) = \begin{cases} t^{-\frac{1}{2}} \sum_{i=0}^{n-1} (-t)^{-i} & \text{if } n \geq 1, \\ t^{\frac{1}{2}} \sum_{i=0}^{-n-1} (-t)^i & \text{if } n \leq -1. \end{cases}$$

We note that $\beta_n|_{A=t^{-\frac{1}{2}}} = t^{\frac{1}{2}} A_n(t)$.

**Theorem 4.2.** Let $n_1, n_2, \ldots, n_r$ be given nonzero integers. For each $k = 1, 2, \ldots, r$, let $L_k^{(2)}$ be the 2-periodic link with rational quotient $L_k = C[[n_1, n_2, \ldots, n_k]]$ and let $L_0^{(2)}$ the trivial knot. Let $V_k(t)$ be the Jones polynomial of $L_k^{(2)}$ for each $k = 0, 1, 2, \ldots, r$. Then we have the following recursive formula:

$$V_0(t) = 1,$$

$$V_1(t) = t^{2n_1} \left( A_{2n_1}(t) - t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right),$$

$$V_k(t) = t^{2n_k} A_{2n_k}(t) V_{k-1}(t) + t^{2n_k} V_{k-2}(t).$$

**Proof.** For each $k = 1, 2, \ldots, r$, let $D_k^{(2)}$ be the canonical 2-periodic diagram of the 2-periodic link $L_k^{(2)}$. Then

$$V_k(t) = (-A)^{-3w(D_k^{(2)}) - \langle D_k^{(2)} \rangle} |_{A=t^{-\frac{1}{2}}}.$$ 

For each $k = 1, 2, \ldots, r$, put $f_k(A) = (-A)^{-3w(D_k^{(2)})} \langle D_k^{(2)} \rangle$. Then $V_k(t) = f_k(A)|_{A=t^{-\frac{1}{2}}}$. We note that $w(D_k^{(2)}) = 2 \sum_{i=1}^{k} n_i$ and, by Lemma 3.2, $\beta_n \delta + 2\alpha_n = A^{-n_1}(-A^4)^{n_1} + 1$.

Since $L_0^{(2)}$ is the trivial knot, $V_0(t) = 1$. If $n_1 \geq 1$, then

$$f_1(A) = (-A)^{-6n_1} \left( \beta_{n_1} (\beta_{n_1} \delta + 2\alpha_{n_1}) + \alpha_{n_1}^2 \delta \right)$$

$$= A^{-6n_1} \left( A^{-2n_1+2}(-A^4)^{n_1} + 1 \sum_{i=0}^{n_1-1} (-A^4)^i + A^{-2n_1}(-A^2 - A^{-2}) \right)$$

$$= A^{-8n_1} \left( A^2 \sum_{i=0}^{2n_1-1} (-A^4)^i + (-A^2 - A^{-2}) \right)$$

and hence

$$V_1(t) = t^{2n_1} \left( t^{-\frac{1}{2}} \sum_{i=0}^{2n_1-1} (-t)^{-i} + (-t^{-\frac{1}{2}} - t^{\frac{1}{2}}) \right) = t^{2n_1} \left( A_{2n_1}(t) - t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right).$$
Let and hence
\[ V_1(t) = t^{2n_1} \sum_{i=0}^{2n_1-1} (-t^i + (-t^{-\frac{i}{2}} - t^{\frac{i}{2}})) = t^{2n_1} \left( A_{2n_1}(t) - t^{\frac{i}{2}} - t^{-\frac{i}{2}} \right). \]

For \( k \geq 2 \), from Lemma 4.1, we obtain
\[ f_k(A) = (-A)^{-6n_k}(\beta_{nk}^2 \delta + 2\alpha_{nk}\beta_{nk})f_{k-1}(A) + (-A)^{-6n_k}\alpha_{nk}^2 f_{k-2}(A). \]
Immediately we have \((-A)^{-6n_k}\alpha_{nk}^2|_{A=t^{-\frac{1}{4}}} = t^{2n_k}. \) If \( n_k \geq 1 \), then it follows that
\[ (-A)^{-6n_k}(\beta_{nk}^2 \delta + 2\alpha_{nk}\beta_{nk}) = A^{-8n_k+2} \sum_{i=0}^{2n_k-1} (-A^4)^i \]
and hence
\[ (-A)^{-6n_k}(\beta_{nk}^2 \delta + 2\alpha_{nk}\beta_{nk})|_{A=t^{-\frac{1}{4}}} = t^{2n_k} A_{2n_k}(t). \]
If \( n_k \leq -1 \), then
\[ (-A)^{-6n_k}(\beta_{nk}^2 \delta + 2\alpha_{nk}\beta_{nk}) = A^{-8n_k-2} \sum_{i=0}^{2n_k-1} (-A^4)^i \]
and hence
\[ (-A)^{-6n_k}(\beta_{nk}^2 \delta + 2\alpha_{nk}\beta_{nk})|_{A=t^{-\frac{1}{4}}} = t^{2n_k} A_{2n_k}(t). \]
Therefore we have
\[ V_k(t) = f_k(A) |_{A=t^{-\frac{1}{4}}} \]
\[ = A^{-6n_k}(\beta_{nk}^2 \delta + 2\alpha_{nk}\beta_{nk})|_{A=t^{-\frac{1}{4}}} V_{k-1}(t) + A^{-6n_k}\alpha_{nk}^2|_{A=t^{-\frac{1}{4}}} V_{k-2}(t) \]
\[ = t^{2n_k} A_{2n_k}(t)V_{k-1}(t) + t^{2n_k} V_{k-2}(t). \]
This completes the proof. \( \square \)

**Example 4.3.** Let \( L \) be the 2-bridge knot with Conway normal form \( C(-2, -4) \). It is the mirror image of the knot 5\(_2\) in Rolfsen’s table [20]. By the discussion in the beginning of this section, \( L \) is the 2-periodic knot with rational quotient \( C[[1, 2]]. \) Let \( n_1 = 1 \) and \( n_2 = 2. \) Then we have that \( A_2(t) = t^{-\frac{1}{2}} - t^{\frac{1}{2}} \) and \( A_4(t) = t^{-\frac{3}{2}} - t^{-\frac{1}{2}} + t^{\frac{1}{2}} - t^{\frac{3}{2}}. \) From Theorem 4.2, it follows that \( V_0(t) = 1, \)
\( V_1(t) = -t^{\frac{1}{2}} + t^{\frac{3}{2}} \) and \( V_2(t) = -t^6 + t^5 - t^4 + 2t^3 - t^2 + t. \) Hence the Jones polynomial of \( L \) is
\[ V_L(t) = -t^6 + t^5 - t^4 + 2t^3 - t^2 + t. \]

For the Jones polynomial \( V_L(t) \) of a link \( L, \) we denote the maximum (resp. minimum) degree of \( V_L(t) \) by \( \max \deg V_L(t) \) (resp. \( \min \deg V_L(t) \)). We also denote the span of \( V_L(t) \) by \( \text{span } V_L(t), \) i.e., \( \text{span } V_L(t) = \max \deg V_L(t) - \min \deg V_L(t). \)
Lemma 4.4. For given nonzero integers $n_1, n_2, \ldots, n_r$, let $V_k(t)$ be the Jones polynomial of the 2-periodic link $L^{(2)}_k$ with rational quotient $L = \overrightarrow{C}[n_1, n_2, \ldots, n_k]$ for each $k = 1, 2, \ldots, r$. Put $\epsilon_k = \frac{|n_k|}{n_k} (k = 1, 2, \ldots, r)$. For each $k = 1, 2, \ldots, r$, we have that

\[
\max \deg V_k(t) = \frac{1 - k}{2} + \sum_{i=1}^{k} (n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4} \sum_{j=1}^{k-1} (1 - \epsilon_j)(1 + \epsilon_{j+1})
\]

and

\[
\min \deg V_k(t) = \frac{k - 1}{2} + \sum_{i=1}^{k} (n_i - |n_i|) - \frac{1}{4} \sum_{j=1}^{k-1} (1 + \epsilon_j)(1 - \epsilon_{j+1}).
\]

Proof. Let $n$ be any nonzero integer. From (10), we have that

\[
\max \deg A_{2n}(t) = \frac{1}{2} - |n| - n, \quad \min \deg A_{2n}(t) = \frac{1}{2} - |n| - n.
\]

We will use the recursive formula in Theorem 4.2 and induction on $k$.

If $n_1 \geq 1$, then $\max \deg A_{2n_1}(t) = \frac{1}{2}$ and hence $\max \deg V_1(t) = 2n_1 + \frac{1}{2}$.

If $n_1 \leq -1$, then $\max \deg A_{2n_1}(t) = -\frac{1}{2} - 2n_1$ and hence $\max \deg V_1(t) = 2n_1 - \frac{1}{2} - 2n_1 = -\frac{1}{2}$. Therefore we have

\[
\max \deg V_1(t) = (n_1 + |n_1|) + \frac{\epsilon_1}{2}.
\]

We assume that the formula (14) holds for $k$. 

Case (i): If $n_{k+1} \leq -1$ or $n_k \geq 1$, then

\[
\max \deg A_{2n_{k+1}}(t) V_k(t) = \max \deg A_{2n_{k+1}}(t) + \max \deg V_k(t)
\]

\[
= \frac{1}{2} + |n_{k+1}| - n_{k+1}
\]

\[
+ \frac{1}{2} + \sum_{i=1}^{k} (n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4} \sum_{j=1}^{k-1} (1 - \epsilon_j)(1 + \epsilon_{j+1})
\]

\[
\geq \frac{2 - k}{2} + \sum_{i=1}^{k-1} (n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4} \sum_{j=1}^{k-2} (1 - \epsilon_j)(1 + \epsilon_{j+1})
\]

\[
- 1 + |n_{k+1}| - n_{k+1} + (n_k + |n_k|)
\]

and

\[
\max \deg V_{k-1}(t) = \frac{2 - k}{2} + \sum_{i=1}^{k-1} (n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4} \sum_{j=1}^{k-2} (1 - \epsilon_j)(1 + \epsilon_{j+1}).
\]
Since \(-1 + |n_{k+1}| - n_{k+1} + (n_k + |n_k|) \geq 1,
max \deg \mathcal{A}_{2n_{k+1}}(t)V_k(t) > max \deg V_{k-1}(t).
Thus by (13), we have that
max \deg V_{k+1}(t)
= 2n_{k+1} + max \deg \mathcal{A}_{2n_{k+1}}(t)V_k(t)
= 2n_{k+1} - \frac{1}{2} + |n_{k+1}| - n_{k+1}
+ \frac{1 - k}{2} + \sum_{i=1}^{k}(n_i + |n_i|) + \frac{\epsilon_i}{2} + \frac{1}{4}\sum_{j=1}^{k-1}(1 - \epsilon_j)(1 + \epsilon_{j+1})
= \frac{1 - (k + 1)}{2} + \sum_{i=1}^{k+1}(n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4}\sum_{j=1}^{k}(1 - \epsilon_j)(1 + \epsilon_{j+1}).

Case (ii) : If \(n_{k+1} \geq 1\) and \(n_k \leq -1\), then
max \deg \mathcal{A}_{2n_{k+1}}(t)V_k(t)
= max \deg \mathcal{A}_{2n_{k+1}}(t) + max \deg V_k(t)
= - \frac{1}{2} + \frac{1 - k}{2} + \sum_{i=1}^{k-1}(n_i + |n_i|) + \frac{\epsilon_i}{2} + \frac{1}{4}\sum_{j=1}^{k-2}(1 - \epsilon_j)(1 + \epsilon_{j+1})
and
max \deg V_{k-1}(t) = \frac{2 - k}{2} + \sum_{i=1}^{k-1}(n_i + |n_i|) + \frac{\epsilon_i}{2} + \frac{1}{4}\sum_{j=1}^{k-2}(1 - \epsilon_j)(1 + \epsilon_{j+1}).
Hence max \deg \mathcal{A}_{2n_{k+1}}(t)V_k(t) + 1 = max \deg V_{k-1}(t). Thus we have
max \deg V_{k+1}(t)
= 2n_{k+1} + max \deg V_{k-1}(t)
= 2n_{k+1} + \frac{2 - k}{2} + \sum_{i=1}^{k-1}(n_i + |n_i|) + \frac{\epsilon_i}{2} + \frac{1}{4}\sum_{j=1}^{k-2}(1 - \epsilon_j)(1 + \epsilon_{j+1})
= \frac{1 - (k + 1)}{2} + \sum_{i=1}^{k+1}(n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4}\sum_{j=1}^{k}(1 - \epsilon_j)(1 + \epsilon_{j+1}).

By a similar argument, we also have the formula (15). □

For given nonzero integers \(n_1, n_2, \ldots, n_r\), we define an integer \(\kappa(n_1, n_2, \ldots, n_r)\) (or briefly \(\kappa(n_i; r)\)) as the number of elements in the set \(\{ (n_i, n_{i+1}) \mid n_i n_{i+1} > 0, 1 \leq i \leq r - 1 \}\). For example, \(\kappa(2, 3, 2, -1) = 2, \kappa(1, 2, 3, 4) = 3\) and \(\kappa(-1, 1, -2, 4) = 0\). We note that \(0 \leq \kappa(n_i; r) \leq r - 1\).
Theorem 4.5. For given nonzero integers $n_1, n_2, \ldots, n_r$, let $L^{(2)}$ be the 2-periodic link with rational quotient $L = \overrightarrow{C}[[n_1, n_2, \ldots, n_r]]$. Then the span of the Jones polynomial $V_{L^{(2)}}(t)$ of $L^{(2)}$ is given by

$$\text{span} \ V_{L^{(2)}}(t) = 2 \sum_{i=1}^{r} |n_i| - \kappa(n_i; r).$$

Proof. From Lemma 4.4, we have

$$\text{span} \ V_{L^{(2)}}(t) = \max \deg V_{L^{(2)}}(t) - \min \deg V_{L^{(2)}}(t)$$

$$= \frac{1-r}{2} + \sum_{i=1}^{r} (n_i + |n_i|) + \frac{\epsilon_1}{2} + \frac{1}{4} \sum_{j=1}^{r-1} (1-\epsilon_j)(1+\epsilon_{j+1})$$

$$- \frac{r-1}{2} - \sum_{i=1}^{r} (n_i - |n_i|) - \frac{\epsilon_1}{2} + \frac{1}{4} \sum_{j=1}^{r-1} (1+\epsilon_j)(1-\epsilon_{j+1})$$

$$= (1-r) + 2 \sum_{i=1}^{r} |n_i| + \frac{1}{4} \sum_{j=1}^{r-1} [(1-\epsilon_j)(1+\epsilon_{j+1}) + (1+\epsilon_j)(1-\epsilon_{j+1})]$$

$$= 2 \sum_{i=1}^{r} |n_i| - \left\{ (r-1) - \sum_{j=1}^{r-1} \frac{1-\epsilon_j\epsilon_{j+1}}{2} \right\}.$$  

Because $(1-\epsilon_j\epsilon_{j+1})/2$ is 1 if $n_j$ and $n_{j+1}$ have different signs and 0 otherwise, $\sum_{j=1}^{r-1} (1-\epsilon_j\epsilon_{j+1})/2$ counts the number of pairs $(n_j, n_{j+1})$ with $n_jn_{j+1} < 0$. Therefore,

$$\kappa(n_i; r) = (r-1) - \sum_{j=1}^{r-1} \frac{1-\epsilon_j\epsilon_{j+1}}{2},$$

hence we have

$$\text{span} \ V_{L^{(2)}}(t) = 2 \sum_{i=1}^{r} |n_i| - \kappa(n_i; r).$$

This completes the proof. \( \square \)

Corollary 4.6. For given nonzero integers $a_1, a_2, \ldots, a_r$, let $L$ be the 2-bridge knot or link with Conway normal form $C(2a_1, 2a_2, \ldots, 2a_r)$. Then the crossing number of $L$ is given by

$$c(L) = 2 \sum_{i=1}^{r} |a_i| - \kappa(-a_1, a_2, -a_3, \ldots, (-1)^r a_r).$$

Proof. From [3, Theorem 2.1], $L$ is a 2-periodic link with rational quotient $\overrightarrow{C}([-a_1, a_2, \ldots, (-1)^r a_r])$ (for more detail, see Remark 4.7 (2)). By Theorem 4.5, we obtain that

$$\text{span} \ V_L(t) = 2 \sum_{i=1}^{r} |a_i| - \kappa(-a_1, a_2, -a_3, \ldots, (-1)^r a_r).$$
Since any 2-bridge link is alternating, the span of its Jones polynomial is equal to its crossing number. Hence the crossing number of \( L \) is given by
\[
c(L) = 2 \sum_{i=1}^{r} |a_i| - \kappa(-a_1, a_2, -a_3, \ldots, (-1)^r a_r).
\]
This completes the proof. \( \square \)

**Remark 4.7.** (1) It should be noticed that the result in Corollary 4.6 is not new. It is well known that every 2-bridge knot or link \( L \) has the standard Conway normal form \( C(b_1, b_2, \ldots, b_n) \) such that all \( b_1, b_2, \ldots, b_n \) are either positive or negative and \( C(b_1, b_2, \ldots, b_n) \) is a reduced alternating diagram for \( L \). Hence \( c(L) = |b_1| + |b_2| + \cdots + |b_n| \). It is also known that \( L \) admits a Conway normal form \( C(2a_1, 2a_2, \ldots, 2a_r) \) for some nonzero integers \( a_1, a_2, \ldots, a_r \) [7], adopted in Corollary 4.6. The authors do not know whether the formula (17) of Corollary 4.6 can be directly derived from the standard Conway normal form \( C(b_1, b_2, \ldots, b_n) \) or not.

(2) Let \( L \) be a link of two components and let \( L_1 \) be the same link as \( L \) but with the opposite orientation on only one component of \( L \). Note that \( L \) and \( L_1 \) are may be different. But the crossing numbers of \( L \) and \( L_1 \) are the same. Since every 2-bridge link is invertible, there are at most two oriented 2-bridge links with the same unoriented diagram. Without loss of generality, in the proof of Corollary 4.6, we can consider that \( L \) is a 2-periodic link with rational quotient \( \overline{C}([-a_1, a_2, \ldots, (-1)^r a_r]) \).

5. **Recursive formula for the Jones polynomial of 3-periodic links**

In this section, we give a recursive formula for the Jones polynomial of a 3-periodic link with rational quotient. We also calculate the span of the Jones polynomial under certain conditions.

**Lemma 5.1.** Let \( n_1, n_2, \ldots, n_r \) be given nonzero integers. For each \( k = 1, 2, \ldots, r \), let \( D_k^{(3)} \) be the canonical 3-periodic diagram of the 3-periodic link with rational quotient \( L_k = \overline{C}([n_1, n_2, \ldots, n_k]) \). Let \( D_0^{(2)} \) denote the standard diagram of the unknot. Then we have the following recursive formula:

\[
\begin{align*}
\langle D_0^{(3)} \rangle &= 1, \\
\gamma_1 &= \delta, \\
\gamma_2 &= (3a_{n_1} a_{n_1}^3 + (3a_{n_1} a_{n_1}^2 + a_{n_1}^3) \delta + 3a_{n_1} \delta^2), \\
\gamma_k &= (-A)^{3n_k-1} \left( a_{n_{k-1}}^3 \gamma_{n_{k-1}} + (a_{n_{k-1}}^2 \delta + 2a_{n_{k-1}} a_{n_k}) D_k^{(3)}(D_{k-2}^{(3)}) \right), \\
\langle D_0^{(3)} \rangle &= (3a_{n_k}^3 \delta + 3a_{n_k} a_{n_k}^2 D_k^{(3)}(D_{k-1}^{(3)})) + 3a_{n_k}^2 a_{n_k} \gamma_k \\
+ (-A)^{3n_k-1} a_{n_k}^3 D_k^{(3)}(D_{k-2}^{(3)}) & \quad k = 2, 3, \ldots, r.
\end{align*}
\]
Proof. For a weight function \( s \in S_k \), let \( D_{k,i}^{(3)}(s(T_{k,1}), s(T_{k,2}), s(T_{k,3})) \) be the diagram obtained from \( D_k^{(3)} \) by replacing each tangle \( T_{k,i} \) by a \( s(T_{k,i}) \)-tangle \((i = 1, 2, 3)\).

If \( k = 1 \), then \( D_1^{(3)}(0,\infty,\infty) \), \( D_1^{(3)}(\infty,0,\infty) \) and \( D_1^{(3)}(\infty,\infty,0) \) consist of a simple closed curve. We also observe that \( D_1^{(3)}(0,\infty,\infty) \) and \( D_1^{(3)}(\infty,\infty,\infty) \) consist of two simple closed curves, and \( D_1^{(3)}(0,0,0) \) consists of three simple closed curves. By Proposition 3.4, we have

\[
\langle D_1^{(3)} \rangle = 3\alpha_{n_1}\beta_{n_1}^2 + (3\alpha_{n_1}\beta_{n_1} + \beta_{n_1}^3)\delta + \alpha_{n_1}^3\delta^2.
\]

Now we assume that the recursive formula (18) holds for \( n_1, n_2, \ldots, n_{k-1} \) with \( k \geq 2 \). Then \( D_k^{(3)}(\infty,\infty,\infty), D_k^{(3)}(0,\infty,\infty), D_k^{(3)}(\infty,0,\infty), D_k^{(3)}(\infty,\infty,0), D_k^{(3)}(0,0,\infty), D_k^{(3)}(0,\infty,0), D_k^{(3)}(\infty,0,0) \) and \( D_k^{(3)}(0,0,0) \) are isotopic to the diagrams as shown in Figure 11. Thus \( D_k^{(3)}(0,\infty,\infty), D_k^{(3)}(\infty,0,\infty) \) and \( D_k^{(3)}(\infty,\infty,0) \) are isotopic to the diagram \( D_{k-1}^{(3)} \cup \emptyset \). Moreover \( D_k^{(3)}(0,0,0) \) is obtained from \( D_{k-2}^{(3)} \) by applying the Reidemeister move I. Since \( D_k^{(3)} \) is a periodic diagram, \( D_k^{(3)}(0,\infty,\infty), D_k^{(3)}(0,0,\infty) \) and \( D_k^{(3)}(\infty,0,0) \) are isotopic to each other. Let \( D_k^{(3)}(\infty,0,0) \) be the diagram in Figure 12(a) and (b), respectively. They are obtained from \( D_{k-1}^{(3)}(\infty,0,0) \) and \( D_{k-2}^{(3)}(\infty,0,0) \) respectively by applying the Reidemeister move I. For each \( k = 1, 2, \ldots, r \), we define a Laurent polynomials \( \gamma_k \) by

\[
\gamma_k = \langle D_k^{(3)}(\infty,0,0) \rangle.
\]

Since \( D_k^{(3)}(\infty,0,0) \) is of two components, \( \gamma_1 = \delta \). By applying Lemma 3.1 to \( T_{k-1,1} \) and \( T_{k-1,2} \) in \( D_k^{(3)}(\infty,0,0) \), we get

\[
\gamma_k = \alpha_{n_k-1}^2 \langle D_k^{(3)}(\infty,0,0) \rangle + (\alpha_{n_k-1} + 2\alpha_{n_k-1}\beta_{n_k-1}\delta) \langle D_k^{(3)}(\infty,0,0) \rangle
\]

\[
= (-A)^{n_k-1} \left( \alpha_{n_k-1}^2 \gamma_{n_k-1} + (\alpha_{n_k-1}^2 + 2\alpha_{n_k-1}\beta_{n_k-1}) \langle D_{k-2}^{(3)}(\infty,0,0) \rangle \right).
\]

Hence we have

\[
\langle D_k^{(3)} \rangle = \beta_{n_k}^3 \langle D_k^{(3)}(\infty,\infty,\infty) \rangle + \beta_{n_k}^2 \alpha_{n_k} \langle D_k^{(3)}(\infty,\infty,0) \rangle + \beta_{n_k} \alpha_{n_k} \langle D_k^{(3)}(\infty,0,\infty) \rangle + \alpha_{n_k} \langle D_k^{(3)}(\infty,0,0) \rangle
\]

\[
+ \beta_{n_k}^3 \alpha_{n_k} \langle D_k^{(3)}(0,\infty,\infty) \rangle + \beta_{n_k}^2 \alpha_{n_k} \langle D_k^{(3)}(0,\infty,0) \rangle + \beta_{n_k} \alpha_{n_k} \langle D_k^{(3)}(0,0,\infty) \rangle + \alpha_{n_k} \langle D_k^{(3)}(0,0,0) \rangle
\]

\[
= (\beta_{n_k}^3 \delta + 3\alpha_{n_k} \beta_{n_k}^2) \langle D_{k+1}^{(3)} \rangle + 3\alpha_{n_k} \beta_{n_k} \gamma_k + (-A)^{n_k-1} \alpha_{n_k}^3 \langle D_{k-2}^{(3)} \rangle.
\]

This completes the proof. \( \square \)
For any nonzero integer \( n \), let \( B_n(t) \) be a Laurent polynomial in \( \mathbb{Z}[t^{\pm \frac{1}{2}}] \) defined by

\[
B_n(t) = \begin{cases} 
  t^{-1}((-t)^{-n} + 2) \left( \sum_{i=0}^{n-1} (-t)^{-i} \right)^2 & \text{if } n \geq 1, \\
  t((-t)^{-n} + 2) \left( \sum_{i=0}^{-n-1} (-t)^{i} \right)^2 & \text{if } n \leq -1.
\end{cases}
\]

We note that \( B_n(t) = ((-t)^{-n} + 2)A_n(t)^2 \).
Theorem 5.2. Let \( n_1, n_2, \ldots, n_r \) be given nonzero integers. Let \( L_k^{(3)} \) be 3-periodic link with rational quotient \( L_k = \overline{C}[[n_1, n_2, \ldots, n_k]] \) and let \( L_k^{(3)} \) the trivial knot. Let \( V_k(t) \) be the Jones polynomial of \( L_k^{(3)} \) for each \( k = 0, 1, 2, \ldots, r \). Then we have the following recursive formula:

\[
V_0(t) = 1,
\]

\[
V_1(t) = (-t)^{3n_1} (B_{n_1}(t) + 3(-t)^{-n_1} + t^{-1} - 1 + t),
\]

\[
V_k(t) = (-t)^{3n_k} (B_{n_k}(t)V_{k-1}(t) + 3A_{nk}(t)\lambda_k(t) + V_{k-2}(t)),
\]

where

\[
\lambda_1(t) = -t^{-\frac{1}{2}} - t^{\frac{1}{2}},
\]

\[
\lambda_k(t) = \left(2^{n_k-1} \lambda_{k-1}(t) + \mathcal{A}_{2nk-1}(t)V_{k-2}(t)\right).
\]

Proof. For each \( k = 1, 2, \ldots, r \), let \( D_k^{(3)} \) be the canonical 3-periodic diagram of the 3-periodic link \( L_k^{(3)} \) with rational quotient \( L = \overline{C}[[n_1, n_2, \ldots, n_k]]. \) Then

\[
V_k(t) = (-A)^{-3w(D_k^{(3)})} \langle D_k^{(3)} \rangle_{A=\frac{1}{t}^{-\frac{1}{4}}},
\]

For each \( k = 1, 2, \ldots, r \), put \( f_k(A) = (-A)^{-3w(D_k^{(3)})}\langle D_k^{(3)} \rangle. \) Then \( V_k(t) = f_k(A) \big|_{A=\frac{1}{t}^{-\frac{1}{4}}} \). We note that \( w(D_k^{(3)}) = 3 \sum_{i=1}^{k} n_i \) and that, by Lemma 3.2, \( \beta_{nk} + 3\alpha_{nk} = A^{-n}((-A^4)^{n_1} + 2). \)

Since \( L_0^{(3)} \) is the trivial knot, \( V_0(t) = 1. \) If \( n_1 \geq 1, \) then

\[
f_1(A) = (-A)^{-9n_1}(\beta_{n_1}^2(3\alpha_{n_1} + \beta_{n_1}) + \alpha_{n_1}^2\delta(3\beta_{n_1} + \alpha_{n_1}))
\]

\[= (-1)^{n_1} A^{-12n_1+4} \left( \sum_{i=0}^{n_1-1} (-A^4)^i \right)^2 \left((-A^4)^{n_1} + 2\right)
\]

\[+ (-1)^{n_1} A^{-12n_1+4}(-1 - A^{-4}) \left(3 \sum_{i=0}^{n_1-1} (-A^4)^i + (-1 - A^{-4})\right)\]
and hence
\[ V_1(t) = (-1)^{n_1} t^{3n_1-1} \left( \sum_{i=0}^{n_1-1} (-t)^i \right)^2 ((-t)^{-n_1} + 2) \]
\[ + (-1)^{n_1} t^{3n_1-1} (1 - t) \left( \sum_{i=0}^{n_1-1} (-t)^i + (-1 - t) \right) \]
\[ = (-1)^{3n_1} \left( B_{n_1} (t) + 3(-t)^{-n_1} + t^{-1} - 1 + t \right). \]

If \( n_1 \leq -1 \), then
\[ f_1(A) = (-1)^{n_1} A^{-12n_1-4} \left( \sum_{i=0}^{n_1-1} (-A^{-4})^i \right)^2 ((-A^4)^{n_1} + 2) \]
\[ + (-1)^{n_1} A^{-12n_1+4} (-1 - A^{-4}) \left( 3A^{-4} \sum_{i=0}^{n_1-1} (-A^{-4})^i + (-1 - A^{-4}) \right) \]
and hence
\[ V_1(t) = (-1)^{n_1} t^{3n_1+1} \left( \sum_{i=0}^{n_1-1} (-t)^i \right)^2 ((-t)^{-n_1} + 2) \]
\[ + (-1)^{n_1} t^{3n_1-1} (1 - t) \left( 3t \sum_{i=0}^{n_1-1} (-t)^i + (-1 - t) \right) \]
\[ = (-1)^{3n_1} \left( B_{n_1} (t) + 3(-t)^{-n_1} + t^{-1} - 1 + t \right). \]

Put \( \lambda_1(t) = \delta \mid_{A=t^{-\frac{1}{4}}} \) and \( \lambda_k(t) = (-A)^{-\frac{9}{2}} \sum_{i=1}^{n_1} \gamma_{k-1} \mid_{A=t^{-\frac{1}{4}}} \) for each \( k = 2, 3, \ldots, r \). Then \( \lambda_1(t) = -t^{-\frac{1}{2}} - t^\frac{1}{2} \) and
\[ \lambda_k(t) = (-A)^{-\frac{9}{2}} \sum_{i=1}^{n_1} (-A)^{3n_k-1} \alpha_{n_k-1}^2 \gamma_{n_k-1} \mid_{A=t^{-\frac{1}{4}}} \]
\[ + (-A)^{-\frac{9}{2}} \sum_{i=1}^{n_1} (-A)^{3n_k-1} \beta_{n_k-1} \beta_{n_k-1} \delta + 2\alpha_{n_k-1} \gamma_{n_k-1} \]
\[ = A^{-8n_k-1} (-A)^{-\frac{9}{2}} \sum_{i=1}^{n_1} \gamma_{n_k-1} \mid_{A=t^{-\frac{1}{4}}} \]
\[ + A^{-7n_k-1} ((-A^{4n_k-1} + 1) \beta_{n_k-1} (-A)^{-\frac{9}{2}} \sum_{i=1}^{n_1} \gamma_{n_k-1} \mid_{A=t^{-\frac{1}{4}}} \]
\[ = t^{2n_k} \lambda_{k-1}(t) + t^{2n_k} ((-t)^{-n_k-1} + 1) A_{n_k-1} (t) V_{k-2}(t) \]
\[ = t^{2n_k} \lambda_{k-1}(t) + t^{2n_k} A_{2n_k-1} (t) V_{k-2}(t). \]
For \( k \geq 2 \), by Lemma 5.1, we obtain

\[
f_k(A) = (-A)^{-9} \sum_{i=1}^{k} (\beta_{n_k}^3 \delta + 3\alpha_{n_k} \beta_{n_k}^2)(D_{k-1}^{(3)})
\]
\[+ (-A)^{-9} \sum_{i=1}^{k} 3\alpha_{n_k}^2 \beta_{n_k} \gamma_k (+ (-A)^{-9} \sum_{i=1}^{k} (-A)^{g_{n_k-1}} \alpha_{n_k}^3)(D_{k-2}^{(3)})
\]
\[= (-A)^{-g_{n_k}}(\beta_{n_k}^3 \delta + 3\alpha_{n_k} \beta_{n_k}^2)f_{k-1}(A)
\]
\[+ (-A)^{-9} \sum_{i=1}^{k} 3\alpha_{n_k}^2 \beta_{n_k} \gamma_k (+ (-A)^{-9} \alpha_{n_k}^3 f_{k-2}(A).)
\]

Note that \((-A)^{-g_{n_k}} \alpha_{n_k}^3|_{A=t^{-\frac{1}{2}}} = (-1)^{n_k} \alpha_{n_k}^3 \beta_{n_k} \gamma_k |_{A=t^{-\frac{1}{2}}} = (-1)^{n_k} \alpha_{n_k}^3 \beta_{n_k} \gamma_k (t). \]

If \( n_k \geq 1 \), then by Lemma 3.2 we have

\[(-A)^{-g_{n_k}}(\beta_{n_k}^3 \delta + 3\alpha_{n_k} \beta_{n_k}^2) = (-1)^{n_k} A^{-12n_k - 4}(-A^4)^{n_k} + 2 \left( \sum_{i=0}^{n_k-1} (-A^4)^{i} \right)^2
\]

and hence

\[(-A)^{-g_{n_k}}(\beta_{n_k}^3 \delta + 3\alpha_{n_k} \beta_{n_k}^2) |_{A=t^{-\frac{1}{2}}} = (-1)^{n_k} t^{3n_k - 1}((-t)^{-n_k + 2}) \left( \sum_{i=0}^{n_k-1} (-t)^{i} \right)^2
\]
\[= (-t)^{3n_k} B_{n_k}(t).
\]

If \( n_k \leq -1 \), then by Lemma 3.2 we also have

\[(-A)^{-g_{n_k}}(\beta_{n_k}^3 \delta + 3\alpha_{n_k} \beta_{n_k}^2) = (-1)^{n_k} A^{-12n_k - 4}((-A^4)^{n_k} + 2) \left( \sum_{i=0}^{n_k-1} (-A^4)^{i} \right)^2
\]

and hence

\[(-A)^{-g_{n_k}}(\beta_{n_k}^3 \delta + 3\alpha_{n_k} \beta_{n_k}^2) |_{A=t^{-\frac{1}{2}}} = (-1)^{n_k} t^{3n_k + 1}((-t)^{-n_k + 2}) \left( \sum_{i=0}^{n_k-1} (-t)^{i} \right)^2
\]
\[= (-t)^{3n_k} B_{n_k}(t).
\]

Therefore we have

\[V_k(t) = f_k(A)|_{A=t^{-\frac{1}{2}}}
\]
\[= (-A)^{-g_{n_k}}(\beta_{n_k}^3 \delta + 3\alpha_{n_k} \beta_{n_k}^2)f_{k-1}(A)|_{A=t^{-\frac{1}{2}}}
\]
\[+ (-A)^{-9} \sum_{i=1}^{k} 3\alpha_{n_k}^2 \beta_{n_k} \gamma_k |_{A=t^{-\frac{1}{2}}} + (-A)^{-9} \alpha_{n_k}^3 f_{k-2}(A)|_{A=t^{-\frac{1}{2}}}
\]
\[= (-t)^{3n_k} B_{n_k}(t)V_{k-1}(t) + 3(-t)^{3n_k} A_{n_k}(t) \lambda_k(t) + (-t)^{3n_k} V_{k-2}(t).
\]

This completes the proof. □
Example 5.3. Let $L$ be the 3-periodic knot with rational quotient $\frac{C}{[[1, -1, 1]]}$. Then $L$ is the knot $9_{40}$ in Rolfsen’s table [20]. Let $n_1 = 1$, $n_2 = -1$ and $n_3 = 1$. Then we have that $A_1(t) = t^{-\frac{3}{2}}$, $A_{-1}(t) = t^{\frac{3}{2}}$, $A_2(t) = t^{-\frac{3}{2}} - t^{-\frac{5}{2}}$, $A_{-2}(t) = t^{\frac{3}{2}} - t^{\frac{5}{2}}$, $B_1(t) = 2t^{-1} - t^{-2}$ and $B_{-1}(t) = 2t - t^2$. From Theorem 5.2, we get that $V_0(t) = 1$, $V_1(t) = -t^4 + t^3 + t$, $\lambda_2 = -t^{\frac{3}{2}} - t^{\frac{5}{2}}$, $V_2(t) = -t^3 + 3t^{\frac{3}{2}} - 2t + 4 - 2t^{-1} + 3t^{-2} - t^{-3}$, $\lambda_3 = t^{\frac{3}{2}} - 2t^{\frac{5}{2}} + t^{\frac{7}{2}} - 2t^\frac{5}{2} + 8t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ and $V_3(t) = t^7 - 4t^6 + 8t^5 - 11t^4 + 13t^3 - 13t^2 + 11t - 8 + 5t^{-1} - t^{-2}$. Hence the Jones polynomial of $L$ is

$$V_L(t) = t^7 - 4t^6 + 8t^5 - 11t^4 + 13t^3 - 13t^2 + 11t - 8 + 5t^{-1} - t^{-2}.$$ 

Let $n$ be any nonzero integer. From (19), we note that

(25) $\max \deg B_n(t) = -1 + \frac{3}{2}(|n| - n), \quad \min \deg B_n(t) = 1 - \frac{3}{2}(|n| + n).$

From (10), we also note that

(26) $\max \deg A_n(t) = -\frac{1}{2} + \frac{1}{2}(|n| - n), \quad \min \deg A_n(t) = 1 - \frac{1}{2}(|n| + n).$

Lemma 5.4. For given nonzero integers $n_1, n_2, \ldots, n_r$, let $V_k(t)$ be the Jones polynomial of the 3-periodic link $L_k^{(3)}$ with rational quotient $L_k = C^{\infty}([[n_1, n_2, \ldots, n_k]])$. Suppose that $n_i \neq 1$ for all $i = 1, 2, \ldots, r$. In the recursive formula in Theorem 5.2, we have the following properties for each $k = 2, 3, \ldots, r$:

1. If $n_k > 1$ and $n_{k-1} \leq -1$, then $\max \deg B_{n_k}(t)V_{k-1}(t) < \max \deg V_{k-2}(t)$ and $\max \deg A_{n_k}(t)\lambda_k(t) < \max \deg V_{k-2}(t)$.

2. If $n_k \leq -1$ or $n_{k-1} > 1$, then $\max \deg V_{k-2}(t) < \max \deg B_{n_k}(t)V_{k-1}(t)$ and $\max \deg A_{n_k}(t)\lambda_k(t) < \max \deg B_{n_k}(t)V_{k-1}(t)$.

Proof. We will use induction on $k$. If $n_1 > 1$, then $\max \deg B_{n_1}(t) = -1$. From (21), we have $\max \deg V_1(t) = 3n_1 + 1$. If $n_1 \leq -1$, then $\max \deg B_{n_1}(t) = -1 - 3n_1$. From (21), we get $\max \deg V_1(t) = -1$. Therefore we have

$$\max \deg V_1(t) = \frac{3}{2}(|n_1| + n_1) + \epsilon_1.$$ 

From (20) and (24), we note that $\max \deg \lambda_2(t) = (|n_1| + n_1) + \frac{1}{2}\epsilon_1$ and $\max \deg V_0(t) = 0$. If $n_2 > 1$ and $n_1 \leq -1$, then $\max \deg B_{n_2}(t)V_1(t) = -2$ and $\max \deg A_{n_2}(t)\lambda_2(t) = -1$. Hence we have that if $n_2 > 1$ and $n_1 \leq -1$, then

$$\max \deg B_{n_2}(t)V_1(t) < \max \deg V_0(t)$$

and

$$\max \deg A_{n_2}(t)\lambda_2(t) < \max \deg V_0(t).$$

If $n_2 \leq -1$, then

$$\max \deg B_{n_2}(t)V_1(t) = -1 - 3n_2 + \frac{3}{2}(|n_1| + n_1) + \epsilon_1$$

and

$$\max \deg A_{n_2}(t)\lambda_2(t) = -\frac{1}{2} - n_2 + (|n_1| + n_1) + \frac{1}{2}\epsilon_1.$$
If $n_1 > 1$, then
\[
\max \deg B_{n_2}(t)V_1(t) = \frac{3}{2}|n_2| - n_2 + 3n_1
\]
and
\[
\max \deg A_{n_2}(t)\lambda_2(t) = \frac{1}{2}(|n_2| - n_2) + 2n_1.
\]
Hence we have that if $n_2 \leq -1$ or $n_1 > 1$, then
\[
\max \deg V_0(t) < \max \deg B_{n_2}(t)V_1(t)
\]
and
\[
\max \deg A_{n_2}(t)\lambda_2(t) < \max \deg B_{n_2}(t)V_1(t).
\]

Now we assume that the statements hold for $\leq k$. From now on we will prove that the statements hold for $k + 1$.

**Case (i):** Suppose that $n_{k+1} > 1$ and $n_k \leq -1$. By the induction hypothesis and (22), we have
\[
\max \deg V_k(t) = 3n_k + \max \deg B_{n_k}(t)V_{k-1}(t).
\]
Hence we have
\[
\max \deg B_{n_{k+1}}(t)V_k(t) = -1 + \max \deg V_k(t)
\]
\[
= -1 + 3n_k + \max \deg B_{n_k}(t)V_{k-1}(t)
\]
\[
= -1 + 3n_k + (-1 - 3n_k) + \max \deg V_{k-1}(t)
\]
\[
= -2 + \max \deg V_{k-1}(t)
\]
\[
< \max \deg V_{k-1}(t).
\]

From (24), it is true that either
\[
\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg \lambda_k(t)
\]
or
\[
\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg A_{2n_k}(t)V_{k-1}(t).
\]
If $\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg \lambda_k(t)$, then, by the induction hypothesis, we get
\[
\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg \lambda_k(t)
\]
\[
< 2n_k - \max \deg A_{n_k}(t) + \max \deg B_{n_k}(t)V_{k-1}(t)
\]
\[
= -\frac{1}{2} + \max \deg V_{k-1}(t).
\]

If $\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg A_{2n_k}(t)V_{k-1}(t)$, then we have
\[
\max \deg \lambda_{k+1}(t) \leq -\frac{1}{2} + \max \deg V_{k-1}(t).
\]

Hence we know
\[
\max \deg A_{n_{k+1}}(t)\lambda_{k+1}(t) = -\frac{1}{2} + \max \deg \lambda_{k+1}(t)
\]
\[
\leq -1 + \max \deg V_{k-1}(t)
\]
\[
< \max \deg V_{k-1}(t).
\]

By (27) and (28), it follows that the statement (1) holds.
Case (ii) : Suppose that \( n_{k+1} \leq -1 \) or \( n_k > 1 \). By the induction hypothesis and (22), we have that

\[
\max \deg V_k(t) \geq 3n_k + \max \deg B_{n_k}(t) + \max \deg V_{k-1}(t)
\]

(29)

\[
= -1 + \frac{3}{2}(|n_k| + n_k) + \max \deg V_{k-1}(t)
\]

and

\[
\max \deg \lambda_k(t) \leq \max \deg V_k(t) - 3n_k - \max \deg A_{n_k}(t)
\]

(30)

\[
= \max \deg V_k(t) - 3n_k + \frac{1}{2} - \frac{1}{2}(|n_k| - n_k).
\]

From (29), we get

\[
\max \deg B_{n_k+1}(t)V_k(t)
\]

\[
= -1 + \frac{3}{2}(|n_k+1| - n_k+1) + \max \deg V_k(t)
\]

\[
\geq -1 + \frac{3}{2}(|n_k+1| - n_k+1) - 1 + \frac{3}{2}(|n_k| + n_k) + \max \deg V_{k-1}(t)
\]

\[
= -2 + \frac{3}{2}(|n_k+1| - n_k+1) + \frac{3}{2}(|n_k| + n_k) + \max \deg V_{k-1}(t)
\]

(31)

\[
> \max \deg V_{k-1}(t).
\]

From (24), we know that either

\[
\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg \lambda_k(t)
\]

or

\[
\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg A_{2n_k}(t) + \max \deg V_{k-1}(t).
\]

If \( \max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg \lambda_k(t) \), then from (30), we calculate

\[
\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg \lambda_k(t)
\]

\[
\leq 2n_k + \max \deg V_k(t) - 3n_k + \frac{1}{2} - \frac{1}{2}(|n_k| - n_k)
\]

\[
= \max \deg V_k(t) + \frac{1}{2} - \frac{1}{2}(|n_k| + n_k)
\]

\[
= \max \deg B_{n_k+1}(t)V_k(t) - \max \deg A_{n_k+1}(t)
\]

\[
+ \left(1 - \frac{1}{2}(|n_k| + n_k) - (|n_{k+1}| - n_{k+1})\right).
\]
If \( \max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg \mathcal{A}_{2n_k}(t) + \max \deg \mathcal{V}_{k-1}(t) \), then from (29), we have
\[
\max \deg \lambda_{k+1}(t) \leq 2n_k + \max \deg \mathcal{A}_{2n_k}(t) + \max \deg \mathcal{V}_{k-1}(t)
\]
\[
\leq 2n_k - \frac{1}{2}(|n_k| - n_k) + 1 - \frac{3}{2}(|n_k| + n_k) + \max \deg \mathcal{V}_k(t)
\]
\[
= \frac{1}{2} - \frac{1}{2}(|n_k| + n_k) + \max \deg \mathcal{V}_k(t)
\]
\[
= \max \deg \mathcal{B}_{n_{k+1}}(t) \mathcal{V}_k(t) - \max \deg \mathcal{A}_{n_{k+1}}(t)
\]
\[
+ \left( 1 - \frac{1}{2}(|n_k| + n_k) - (|n_{k+1}| - n_{k+1}) \right).
\]
Since \( n_{k+1} \leq -1 \) or \( n_k > 1 \), we get that \( 1 - \frac{1}{2}(|n_k| + n_k) - (|n_{k+1}| - n_{k+1}) < 0 \) and hence
(32) \( \max \deg \mathcal{A}_{n_{k+1}}(t) \lambda_{k+1}(t) < \max \deg \mathcal{B}_{n_{k+1}}(t) \mathcal{V}_k(t) \).

By (31) and (32), it follows that the statement (2) holds. This completes the proof.

**Lemma 5.5.** For given nonzero integers \( n_1, n_2, \ldots, n_r \), let \( \mathcal{V}_k(t) \) be the Jones polynomial of the 3-periodic link \( L_k^{(3)} \) with rational quotient \( L_k = \mathbb{C}[[[n_1, n_2, \ldots, n_k]]] \). Suppose that \( n_i \neq -1 \) for all \( i = 1, 2, \ldots, r \). In the recursive formula in Theorem 5.2, we have the following properties for each \( k = 2, 3, \ldots, r \):

1. If \( n_k < -1 \) and \( n_{k-1} \geq 1 \), then \( \min \deg \mathcal{B}_{n_k}(t) \mathcal{V}_{k-1}(t) > \min \deg \mathcal{V}_{k-2}(t) \) and \( \min \deg \mathcal{A}_{n_k}(t) \lambda_k(t) > \min \deg \mathcal{V}_{k-2}(t) \).
2. If \( n_k \geq 1 \) or \( n_{k-1} < -1 \), then \( \min \deg \mathcal{V}_{k-2}(t) > \min \deg \mathcal{B}_{n_k}(t) \mathcal{V}_{k-1}(t) \) and \( \min \deg \mathcal{A}_{n_k}(t) \lambda_k(t) > \min \deg \mathcal{B}_{n_k}(t) \mathcal{V}_{k-1}(t) \).

**Proof.** Let \( m_i = -n_i \) for all \( i = 1, 2, \ldots, r \) and let \( \bar{V}_k(t) \) be the Jones polynomial of the 3-periodic link \( \bar{L}_k^{(3)} \) with rational quotient \( \bar{L}_k = \mathbb{C}[[m_1, m_2, \ldots, m_k]] \). Let \( \bar{\lambda}_k(t) \) be the Laurent polynomial recursively defined by
\[
\bar{\lambda}_1(t) = -t^{-\frac{1}{2}} - t^\frac{1}{2}, \quad \bar{\lambda}_k(t) = t^{2m_{k-1}}(\bar{\lambda}_{k-1}(t) + \mathcal{A}_{2m_{k-1}}(t) \bar{V}_{k-2}(t)).
\]
By (25) and (26),
\[
\max \deg \mathcal{B}_{m_k}(t) = -\min \deg \mathcal{B}_{m_k}(t), \quad \max \deg \mathcal{A}_{m_k}(t) = -\min \deg \mathcal{A}_{n_k}(t).
\]
We observe that \( \bar{L}_k^{(3)} \) is the mirror image of \( L_k^{(3)} \) and hence \( \bar{V}_k(t) = V_k(t^{-1}) \). Therefore
\[
\max \deg \bar{V}_k(t) = -\min \deg \mathcal{V}_k(t), \quad \max \deg \bar{\lambda}_k(t) = -\min \deg \lambda_k(t).
\]
By Lemma 5.4,

1. If \( m_k > 1 \) and \( m_{k-1} \leq -1 \), then \( \max \deg \mathcal{B}_{m_k}(t) \bar{V}_{k-1}(t) < \max \deg \bar{V}_{k-2}(t) \) and \( \max \deg \mathcal{A}_{m_k}(t) \bar{\lambda}_k(t) < \max \deg \bar{V}_{k-2}(t) \).
2. If \( m_k \leq -1 \) or \( m_{k-1} > 1 \), then \( \max \deg \bar{V}_{k-2}(t) < \max \deg \mathcal{B}_{m_k}(t) \bar{V}_{k-1}(t) \) and \( \max \deg \mathcal{A}_{m_k}(t) \bar{\lambda}_k(t) < \max \deg \mathcal{B}_{m_k}(t) \bar{V}_{k-1}(t) \).
This completes the proof. \hfill \Box

**Theorem 5.6.** For given nonzero integers \(n_1, n_2, \ldots, n_r\), let \(V_k(t)\) be the Jones polynomial of the 3-periodic link \(L_k^{(i)}\) with rational quotient \(L_k = C[n_1, n_2, \ldots, n_k]\). Suppose that \(n_i \neq \pm 1\) for all \(i = 1, 2, \ldots, r\). Then

\[
\max \deg V_k(t) = (1 - k) + \frac{3}{2} \sum_{i=1}^{k} (n_i + |n_i|) + \epsilon_1 + \frac{1}{2} \sum_{j=1}^{k-1} (1 - \epsilon_j)(1 + \epsilon_{j+1})
\]

and

\[
\min \deg V_k(t) = (k - 1) + \frac{3}{2} \sum_{i=1}^{k} (n_i - |n_i|) + \epsilon_1 - \frac{1}{2} \sum_{j=1}^{k-1} (1 + \epsilon_j)(1 - \epsilon_{j+1}).
\]

**Proof.** In the proof of Lemma 5.4, we have

\[
\max \deg V_1(t) = \frac{3}{2} (|n_1| + n_1) + \epsilon_1.
\]

If \(n_2 > 1\) and \(n_1 < -1\), then by (1) in Lemma 5.4 we obtain

\[
\max \deg V_2(t) = 3n_2 + \max \deg V_0(t) = \frac{3}{2} (n_2 + |n_2|).
\]

If \(n_2 < -1\) or \(n_1 > 1\), then by (2) in Lemma 5.4 we get

\[
\max \deg V_2(t) = 3n_2 + \max \deg B_{n_2}(t)V_1(t)
\]

\[
= 3n_2 + \left( -1 + \frac{3}{2} (|n_2| - n_2) + \frac{3}{2} (|n_1| + n_1) + \epsilon_1 \right)
\]

\[
= -1 + \frac{3}{2} (|n_2| + n_2) + \frac{3}{2} (|n_1| + n_1) + \epsilon_1.
\]

Therefore we have

\[
\max \deg V_2(t) = -1 + \frac{3}{2} \sum_{i=1}^{2} (n_i + |n_i|) + \epsilon_1 + \frac{1}{2} (1 - \epsilon_1)(1 + \epsilon_2).
\]

If \(n_{k+1} > 1\) and \(n_k < -1\), then by (1) in Lemma 5.4 we get

\[
\max \deg V_{k+1}(t) = 3n_{k+1} + \max \deg V_{k-1}(t)
\]

\[
= 3n_{k+1} + (2 - k) + \frac{3}{2} \sum_{i=1}^{k-1} (n_i + |n_i|) + \epsilon_1
\]

\[
+ \frac{1}{2} \sum_{j=1}^{k-2} (1 - \epsilon_j)(1 + \epsilon_{j+1}).
\]
If \( n_{k+1} < -1 \) or \( n_k > 1 \), then by (2) in Lemma 5.4 we calculate
\[
\max \deg V_{k+1}(t) = 3n_{k+1} + \max \deg B_{n_{k+1}}(t)V_k(t)
\]
\[
= -1 + \frac{3}{2}(n_{k+1} + n_{k+1}) + (1 - k)
\]
\[
+ \frac{3}{2} \sum_{i=1}^{k} (n_i + |n_i|) + \epsilon_1 + \frac{1}{2} \sum_{j=1}^{k-1} (1 - \epsilon_j)(1 + \epsilon_{j+1}).
\]
Therefore we have
\[
\max \deg V_{k+1}(t) = -1 + \frac{3}{2} \sum_{i=1}^{k+1} (n_i + |n_i|) + \epsilon_1 + \frac{1}{2} \sum_{j=1}^{k} (1 - \epsilon_j)(1 + \epsilon_{j+1}).
\]

By a similar argument, we also have
\[
\min \deg V_{k+1}(t) = k + \frac{3}{2} \sum_{i=1}^{k+1} (n_i - |n_i|) + \epsilon_1 - \frac{1}{2} \sum_{j=1}^{k} (1 + \epsilon_j)(1 - \epsilon_{j+1}).
\]
This completes the proof. □

**Theorem 5.7.** For given nonzero integers \( n_1, n_2, \ldots, n_r \), let \( L^{(3)} \) be 3-periodic link with rational quotient \( L = \overline{C}[[n_1, n_2, \ldots, n_r]] \). If \( |n_i| \geq 2 \) for all \( i = 1, 2, \ldots, r \), then the span of the Jones polynomial \( V_{L^{(3)}}(t) \) of \( L^{(3)} \) is given by
\[
\text{span } V_{L^{(3)}}(t) = 3 \sum_{i=1}^{r} |n_i| - 2\kappa(n_i; r).
\]

**Proof.** From Theorem 5.6, we have
\[
\text{span } V_{L^{(3)}}(t) = \max \deg V_r(t) - \min \deg V_r(t)
\]
\[
= (1 - r) + \frac{3}{2} \sum_{i=1}^{r} (n_i + |n_i|) + \epsilon_1 + \frac{1}{2} \sum_{j=1}^{r-1} (1 - \epsilon_j)(1 + \epsilon_{j+1})
\]
\[
- (r - 1) + \frac{3}{2} \sum_{i=1}^{r} (n_i - |n_i|) - \epsilon_1 + \frac{1}{2} \sum_{j=1}^{r-1} (1 + \epsilon_j)(1 - \epsilon_{j+1})
\]
\[
= 3 \sum_{i=1}^{r} |n_i| - 2\kappa(n_i; r)
\]

because
\[
\kappa(n_i; r) = r - 1 - \frac{1}{4} \sum_{j=1}^{r-1} [(1 - \epsilon_j)(1 + \epsilon_{j+1}) + (1 + \epsilon_j)(1 - \epsilon_{j+1})]
\]
as we have seen in the proof of Theorem 4.5. This completes the proof. □
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