ON QUASI-STABLE EXCHANGE IDEALS

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Abstract. We introduce, in this article, the quasi-stable exchange ideal for associative rings. If \( I \) is a quasi-stable exchange ideal of a ring \( R \), then so is \( M_n(I) \) as an ideal of \( M_n(R) \). As an application, we prove that every square regular matrix over quasi-stable exchange ideal admits a diagonal reduction by quasi invertible matrices. Examples of such ideals are given as well.

1. Introduction

Following Ara (cf. [1]), an ideal \( I \) of a ring \( R \) is an exchange ideal provided that for every \( x \in I \) there exist an idempotent \( e \in I \) and elements \( r, s \in I \) such that \( e = xe = x + s - xs \). Clearly, an ideal \( I \) of a ring \( R \) is an exchange ideal if and only if for any \( x \in I \), there exists an idempotent \( e \in xR \) such that \( 1 - e \in (1 - x)R \). Exchange ideal plays a key role in the direct sum decomposition theory of exchange rings. Many authors have studied such ideals, e.g., [1] and [12].

So as to investigate directly infinite rings, we introduce a new class of exchange ideals, i.e., quasi-stable exchange ideals of a ring \( R \). If \( I \) is a quasi-stable exchange ideal of a ring \( R \), we will show that \( M_n(I) \) is a quasi-stable exchange ideal of \( M_n(R) \). As is well known, every square matrix over a unit-regular ring admits a diagonal reduction. Ara et. al. extended this result and proved that every square regular matrix over a separative exchange ring admits a diagonal reduction by invertible matrices (cf. [2]). It is interesting to investigate diagonal reduction of matrices over an ideal of a ring \( R \) even though there exist some square matrices over \( R \) which can not be reduced. As an application, we prove that every square regular matrix over quasi-stable exchange ideal admits a diagonal reduction by quasi invertible matrices. These also give nontrivial generalizations of [4, Theorem 16] and [6, Theorem 11].

Throughout, all rings are associative with identity, all ideals are two-sided ideals and all modules are right unitary modules. We use \( M_n(R) \) to denote

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the ring of $n \times n$ matrices over $R$ with identity $I_n$. $GL_n(R)$ denotes the $n$-dimensional general linear group of $R$. Set $GL_n(I) = GL_n(R) \cap (I_n + M_n(I))$. An element $x \in R$ is regular provided that $x = xyx$ for a $y \in R$. $\Gamma(I)$ stands for the set of all products of a left invertible element and a right invertible element in $1 + I$, i.e., $\{w \in R | \exists s, t \in 1 + I \text{ such that } su = 1, vt = 1\}$.

2. Equivalent characterizations

**Definition 2.1.** Let $I$ be an ideal of a ring $R$. We say that $I$ is a right quasi-stable ideal if $aR + bR = R$ with $a \in I, b \in R$ implies that there exists $y \in R$ such that $a + by \in \Gamma(I)$. We say that $I$ is a left quasi-stable ideal if $Ra + Rb = R$ with $a \in I, b \in R$ implies that there exists $z \in R$ such that $a + zb \in \Gamma(I)$. An ideal $I$ of a ring $R$ is a quasi-stable ideal in case it is both right and left quasi-stable ideal.

Let $J(R)$ be the Jacobson radical of rings $R$. If $ax + b = 1$ with $a \in J(R), x, b \in R$, then $b \in U(R)$. Hence, $a + b \cdot b^{-1} = 1 + J(R) \in \Gamma(J(R))$. Thus, $J(R)$ is a right quasi-stable exchange ideal. The purpose of this section is to investigate several equivalent characterizations of right quasi-stable ideals. The left quasi-stable ideals have analogous results.

**Theorem 2.2.** Let $I$ be an exchange ideal of a ring $R$. Then the following are equivalent:

1. $I$ is a right quasi-stable ideal.
2. Every element in $I$ is a product of an idempotent in $I$ and an element in $\Gamma(I)$.

**Proof.** (1)⇒(2) Given any $x \in I$, there exists $y \in I$ such that $x = xyx$. Since $xy + (1 - xy) = 1$ with $x \in I$, we have $z \in R$ such that $x + (1 - xy)z = w \in \Gamma(I)$. So $x = xyx = xy(x + (1 - xy)z) = ew$, where $e = xy \in I$ is an idempotent.

(2)⇒(1) Suppose that $ax + b = 1$ with $a \in I, x, b \in R$. Then $b \in 1 + I$. Since $I$ is an exchange ideal of $R$, by [1, Lemma 1.1], we have an idempotent $e = bs$ and $1 - e = (1 - b)t$ for some $s, t \in R$. Hence $axt + e = 1$, and then $(1 - e)axt + e = 1$. So $(1 - e)a \in I$ is regular. Thus we have an idempotent $f \in I$ and a $w \in \Gamma(I)$ such that $(1 - e)a = fw$. So $f + e = 1$, and then $f + e(1 - f)w = f + e - f = (1 - f)w$. As a result, $a + bs((1 - f)w - a) = (1 + f(1 - f))^{-1}w \in \Gamma(I)$. Therefore $I$ is a right quasi-stable ideal. □

**Corollary 2.3.** Let $I$ be an exchange ideal of a ring $R$. Then the following are equivalent:

1. $I$ is a right quasi-stable ideal.
2. Whenever $ax + b = 1$ with $a, x \in I, b \in 1 + I$, there exists $y \in R$ such that $a + by \in \Gamma(I)$. 


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Proof. (1)⇒(2) is trivial.

(2)⇒(1) Let $x \in I$ be regular. Then we have $y \in I$ such that $x = xyz$. Since $xy + (1 - xy) = 1$ with $x, y \in I$, $1 - xy \in 1 + I$, by hypothesis, there exists $z \in R$ such that $x + (1 - xy)z = w \in \Gamma(I)$. Thus, $x = xyz = xy(x + (1 - xy)z) = cw$, where $c = xy \in I$ is an idempotent. According to Theorem 2.2, we obtain the result. □

Recall that an ideal $I$ of a ring $R$ has stable range one provided that $aR + bR = R$ with $a \in I, b \in R$ implies that there exists $y \in R$ such that $a + by \in GL_1(R)$. We recall a simple known result.

Lemma 2.4. Given $ax + b = 1, a, x, b \in R$, then the following hold:

(1) If $u(a + by) = 1$, then $(x + (1 - xy)ab)(a + y(1 - xa)) = 1$. If $(a + by)u = 1$, then $(a + y(1 - xa))(x + (1 - xy)ab) = 1$.

(2) If $(x + zb)v = 1$, then $(x + (1 - xa)z)(a + bv(1 - za)) = 1$. If $v(x + zb) = 1$, then $(a + bv(1 - za))(x + (1 - xa)z) = 1$.

Proof. Straightforward. □

Proposition 2.5. Let $I$ be an exchange ideal of a ring $R$. If $I$ has stable range one, then $I$ is a right quasi-stable ideal

Proof. Assume that $ax + b = 1$ with $a, x \in I, b \in 1 + I$. Then $(a + (1 - a)b)(x + b) + (1 - a)b(1 - (x + b)) = 1$, where $a + (1 - a)b \in 1 + I$. Since $I$ has stable range one, we have $y \in R$ such that $(a + (1 - a)b) + (1 - a)b (1 - (x + b))y \in GL_1(I)$. That is, $a + (1 - a)b(1 + (1 - (x + b)y)) \in GL_1(I)$. As $a(x + b) + (1 - a)b = 1$, we can find $z \in R$ such that $x + b + z(1 - a)b \in GL_1(I)$, i.e., $x + (1 + z(1 - a))b \in GL_1(I)$. By using Lemma 2.4 again, we have $t \in R$ such that $a + bt \in GL_1(I)$. Therefore $I$ is a right quasi-stable ideal, as desired. □

It follows from Lemma 2.4 that stable range one for ideals is right and left symmetric. Recall that a ring $R$ is perfect in case $R/J(R)$ is a division ring and idempotents lift modulo $J(R)$. Consequently, every ideal of a perfect ring is quasi-stable.

Proposition 2.6. Let $I$ be an exchange ideal of a ring $R$. Then the following are equivalent:

(1) $I$ is a right quasi-stable ideal.

(2) For any regular $a, b \in I$, $aR = bR$ implies that there exists $w \in \Gamma(I)$ such that $a = bw$.

Proof. (1)⇒(2) Suppose that $aR = bR$ with regular $a, b \in I$. Then we have $x, y \in R$ such that $ax = b$ and $a = by$. Assume that $b = bb'$. Replacing $b'by$ with $y$, we may assume that $y \in I$. From $yx + (1 - yx) = 1$, we have $z \in R$ such that $y + (1 - yx)z = w \in \Gamma(I)$. Hence $a = by = b(y + (1 - yx)z) = bw$, as required.
(2) ⇒ (1) For any regular \( x \in I \), there exists an idempotent \( e \in I \) such that \( xR = eR \). So \( x = ew \) for some \( w \in \Gamma(I) \). Therefore \( I \) is a right quasi-stable ideal by Theorem 2.2.

3. Extensions of matrices

A natural problem asks whether quasi-stable exchange ideal of a ring is invariant under matrix extension. In this section, we give this problem an affirmative answer. In the sequel, we say that the pair \((a, b)\) is an \( I \)-unimodular row in case \( ax + by = 1 \) for some \( x \in I, y \in R \). The \( I \)-unimodular row \((a, b)\) is called \( I \)-reducible if there exists \( z \in R \) such that \( a + bz \in \Gamma(I) \).

**Lemma 3.1.** Let \((a, b)\) be a \( I \)-unimodular row in a ring \( R \). Let \( u, v \in GL_1(I) \) and \( c \in R \). Then \((vau + vbc, vb)\) is also \( I \)-unimodular row. Furthermore, \((a, b)\) is \( I \)-reducible if and only if so is \((vau + vbc, vb)\).

**Proof.** Since \((a, b)\) is an \( I \)-unimodular row in a ring \( R \), we have \( x \in I, y \in R \) such that \( ax + by = 1 \). Hence \((vau + vbc)(u^{-1}xv^{-1}) + vb(y - cu^{-1}x)v^{-1} = 1 \). Clearly, \( u^{-1}xv^{-1} \in I \). So \((vau + vbc, vb)\) is an \( I \)-unimodular row. Assume that \((a, b)\) is \( I \)-reducible. We then have \( y \in R \) such that \( a + by \in \Gamma(I) \). Choose \( z = yu - c \). Then we see that \((vau + vbc) + (vb)z = v(a + by)u \in \Gamma(I)\); hence, \((au + vbc, vb)\) is \( I \)-reducible. Conversely, assume that there exists \( z \in R \) such that \( vau + vbc + vbz \in \Gamma(I) \). Then \( v(a + b(c + z)u^{-1})u \in \Gamma(I) \). As \( u, v \in GL_1(I) \), \( a + b(c + z)u^{-1} \in \Gamma(I) \). Therefore \((a, b)\) is \( I \)-reducible. \(\Box\)

**Theorem 3.2.** Let \( I \) be a right quasi-stable exchange ideal of a ring \( R \). Then \( M_n(I) \) is a right quasi-stable exchange ideal of \( M_n(R) \) for all \( n \in \mathbb{N} \).

**Proof.** By [1, Theorem 1.4], \( M_n(I) \) is an exchange ideal of \( M_n(R) \). We now induct on \( n \). Assume inductively that the result holds for \( n \). It will suffice to show that the result holds for \( n + 1 \). Suppose that

\[
(*) \quad \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1(n+1)} \\
  a_{21} & a_{22} & \cdots & a_{2(n+1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)(n+1)}
\end{pmatrix}
\begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1(n+1)} \\
  b_{21} & b_{22} & \cdots & b_{2(n+1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{(n+1)1} & b_{(n+1)2} & \cdots & b_{(n+1)(n+1)}
\end{pmatrix}
= I_{n+1}
\]

in \( M_{n+1}(R) \), where

\[
\begin{pmatrix}
  a_{11} & \cdots & a_{1(n+1)} \\
  a_{21} & \cdots & a_{2(n+1)} \\
  \vdots & \ddots & \vdots \\
  a_{(n+1)1} & \cdots & a_{(n+1)(n+1)}
\end{pmatrix},
\begin{pmatrix}
  b_{11} & \cdots & b_{1(n+1)} \\
  b_{21} & \cdots & b_{2(n+1)} \\
  \vdots & \ddots & \vdots \\
  b_{(n+1)1} & \cdots & b_{(n+1)(n+1)}
\end{pmatrix} \in M_{n+1}(I).
\]
Then $a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1(n+1)}b_{(n+1)1} + c_{11} = 1$ with $a_{11} \in I$. As $I$ is a quasi-stable exchange ideal of $R$, we have $z_1 \in R$ such that $a_{11} + (a_{12}b_{21} + \cdots + a_{1n}b_{n1} + c_{11})z_1 \in \Gamma(I)$. Since

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
b_{21}z_1 & 1 & 0 & \cdots & 0 \\
b_{31}z_1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{(n+1)1}z_1 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

by virtue of Lemma 3.1, $(\ast)$ is $M_{n+1}(I)$-reducible if and only if this is so for the $M_{n+1}(I)$-unimodular row with elements

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1(n+1)} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2(n+1)} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3(n+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \cdots & a_{(n+1)(n+1)}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
b_{21}z_1 & 1 & 0 & \cdots & 0 \\
b_{31}z_1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{(n+1)1}z_1 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\
c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\
c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)}
\end{pmatrix}
\begin{pmatrix}
z_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

So we assume that $a_{11} \in \Gamma(I)$. From $c_{21}, \ldots, c_{(n+1)1} \in I$, we have $a_{ij} \in I$ (either $i \neq 1$ or $j \neq 1$) in $(\ast)$. Write $a_{11} = uv$, $su = 1, vt = 1, s, t \in 1 + I$. Then $sa_{11}t = 1$, and so

$$
\begin{pmatrix}
s & 0 & 0 & \cdots & 0 \\
1 - a_{11}ts & a_{11}t & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$
\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1(n+1)} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2(n+1)} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3(n+1)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \cdots & a_{(n+1)(n+1)}
\end{pmatrix}
\times
\begin{pmatrix}
  t & 1 - ts_{a_{11}} & 0 & \cdots & 0 \\
  0 & s_{a_{11}} & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\times
\begin{pmatrix}
  1 & d_{12} & d_{13} & \cdots & d_{1(n+1)} \\
  d_{21} & d_{22} & d_{23} & \cdots & d_{2(n+1)} \\
  d_{31} & d_{32} & d_{33} & \cdots & d_{3(n+1)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  d_{(n+1)1} & d_{(n+1)2} & d_{(n+1)3} & \cdots & d_{(n+1)(n+1)}
\end{pmatrix},
\]

where
\[
\begin{pmatrix}
  s & 0 & 0 & \cdots & 0 \\
  1 - a_{11}ts & a_{11}t & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1
\end{pmatrix}
= \begin{pmatrix}
  1 - a_{11}ts & 0 & \cdots & 0 \\
  0 & s & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1
\end{pmatrix}^{-1},
\]

\[
\begin{pmatrix}
  t & 1 - ts_{a_{11}} & 0 & \cdots & 0 \\
  0 & s_{a_{11}} & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1
\end{pmatrix}
= \begin{pmatrix}
  s_{a_{11}} & 0 & 0 & \cdots & 0 \\
  1 - ts_{a_{11}} & t & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1
\end{pmatrix}^{-1}
\in GL_{n+1}(I).
\]

Thus (\(\ast\)) is \(M_{n+1}(I)\)-reducible if and only if this is so for the \(M_{n+1}(I)\)-unimodular row with elements
\[
\begin{pmatrix}
  1 & d_{12} & d_{13} & \cdots & d_{1(n+1)} \\
  d_{21} & d_{22} & d_{23} & \cdots & d_{2(n+1)} \\
  d_{31} & d_{32} & * & \cdots & * \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  d_{(n+1)1} & d_{(n+1)2} & * & \cdots & d_{(n+1)(n+1)}
\end{pmatrix},
\]

\[
\begin{pmatrix}
  s & 0 & 0 & \cdots & 0 \\
  1 - a_{11}ts & a_{11}t & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\times
\begin{pmatrix}
  c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\
  c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\
  c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)}
\end{pmatrix}.
\]
In (\(\ast\)), we may assume that \(d_{ij} \in I\) (either \(3 \leq i \leq n + 1\) or \(3 \leq j \leq n + 1\)) and \(d_{12} = sa_{11}(1 - tsa_{11}) + sa_{12}a_{11}.d_{21} = (1 - a_{11}ts)a_{11}t + a_{11}ta_{21}t, d_{22} = ((1 - a_{11}ts)a_{11} + a_{11}ta_{22})(1 - tsa_{11}) + ((1 - a_{11}ts)a_{12} + a_{11}ta_{22})sa_{11} \in I\). By Lemma 3.1 again, (\(\ast\)) is \(M_{n+1}(I)\)-reducible if and only if this is so for the \(M_{n+1}(I)\)-unimodular row with elements

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & * & * & \cdots & * \\
& & & & \\
& & & & \\
0 & * & * & \cdots & * \\
\end{pmatrix}
\]

So we may assume that \(a_{11} = 1, a_{11} = 0 = a_{11} (2 \leq i \leq n + 1)\) in (\(\ast\)). Furthermore, we may assume that (\(\ast\)) is in the following form:

\[
\begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & D
\end{pmatrix}
\begin{pmatrix}
\epsilon_{11} & E_{12} \\
E_{21} & E_{22}
\end{pmatrix}
\begin{pmatrix}
c_{11} & C_{12} \\
0_{n \times 1} & Z_{2}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & \mathbf{I}_n
\end{pmatrix}.
\]

\(D \in M_n(I)\) and \(\begin{pmatrix}
\epsilon_{11} & E_{12} \\
E_{21} & E_{22}
\end{pmatrix} \in M_{n+1}(I)\). This infers that \(DE_{22} + C_{22} = I_n\). By the induction hypothesis, \(M_n(I)\) is a quasi-stable exchange ideal of \(M_n(R)\). So we can find \(Z_2 \in M_n(R)\) such that \(D + C_{22}Z_2 \in \Gamma(M_n(I))\). Thus, we pass to the \(M_{n+1}(I)\)-unimodular row with elements

\[
\begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & D
\end{pmatrix}
\begin{pmatrix}
c_{11} & C_{12} \\
0_{n \times 1} & Z_2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & \mathbf{I}_n
\end{pmatrix}.
\]

In addition, we have \(C_{12} \in M_{1 \times n}(I)\). It suffices to prove that \(M_{n+1}(I)\)-unimodular row with elements

\[
\begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & C_{12}Z_2
\end{pmatrix}
= \begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & U
\end{pmatrix}
\begin{pmatrix}
1 & C_{12}Z_2 \\
0_{n \times 1} & V
\end{pmatrix}.
\]

is \(M_{n+1}(I)\)-reducible. Write \(D + C_{22}Z_2 = UV, SU = I_n, VT = I_n, S, T \in I_n + M_n(I)\). Thus,

\[
\begin{pmatrix}
1 & C_{12}Z_2 \\
0_{n \times 1} & D + C_{22}Z_2
\end{pmatrix} = \begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & U
\end{pmatrix}
\begin{pmatrix}
1 & C_{12}Z_2 \\
0_{n \times 1} & V
\end{pmatrix},
\]

\[
\begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & S
\end{pmatrix}
\begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & U
\end{pmatrix} = I_{n+1},
\]

\[
\begin{pmatrix}
1 & C_{12}Z_2 \\
0_{n \times 1} & V
\end{pmatrix}
\begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & T
\end{pmatrix}
= \begin{pmatrix}
1 & -C_{12}Z_2T \\
0_{n \times 1} & I_2
\end{pmatrix} = I_{n+1},
\]

\[
\begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & S
\end{pmatrix}
\begin{pmatrix}
1 & 0_{1 \times n} \\
0_{n \times 1} & T
\end{pmatrix}
= \begin{pmatrix}
1 & -C_{12}Z_2T \\
0_{n \times 1} & I_2
\end{pmatrix} \in I_{n+1} + M_{n+1}(I).
This implies that \( \begin{pmatrix} 1 & \frac{C_{12}Z_2}{D + C_{22}Z_2} \\ 0 & D + C_{22}Z_2 \end{pmatrix} \in \Gamma(M_{n+1}(I)), \) as required. □

**Corollary 3.3.** Let \( I \) be a right quasi-stable exchange ideal of a ring \( R \). Then every regular \( n \times n \) matrix over \( I \) is a product of an idempotent \( n \times n \) matrix over \( I \) and an matrix in \( \Gamma(M_n(I)) \).

**Proof.** Since \( I \) is a right quasi-stable exchange ideal of \( R \), by Theorem 3.2, \( M_n(I) \) is a right quasi-stable exchange ideal of \( M_n(R) \). Therefore we complete the proof from Theorem 2.2. □

Let \( FP(I) \) denote the set of finitely generated projective right \( R \)-module \( P \) such that \( P = PI \).

**Lemma 3.4.** Let \( I \) be an exchange ideal of a ring \( R \). If \( P \in FP(I) \). Then there exist idempotents \( e_1, \ldots, e_n \in I \) such that \( P \cong e_1R \oplus \cdots \oplus e_nR \).

**Proof.** See [1, Proposition 1.5]. □

**Lemma 3.5.** Let \( I \) be a quasi-stable exchange ideal of a ring \( R \). For any regular \( a, b \in I \), \( aR \cong bR \) implies that \( a = w_1bw_2 \) for some \( w_1, w_2 \in \Gamma(I) \).

**Proof.** Suppose that \( \psi : aR \cong bR \). Then one easily checks that \( Ra = R\psi(a) \) and \( \psi(a)R = bR \). As \( a \in I \), we have \( \psi(a) \in Ra \subseteq I \). Since \( I \) is a right quasi-stable ideal, it follows by Proposition 2.6 that there exists \( w_2 \in \Gamma(I) \) such that \( bw_2 = \psi(a) \). Likewise, we have \( w_1 \in \Gamma(I) \) such that \( a = w_1\psi(a) \). Therefore \( a = w_1bw_2 \), where \( w_1, w_2 \in \Gamma(I) \). □

We use \( A^T \) to denote the transpose of the matrix \( A \). We now derive the main result of this article.

**Theorem 3.6.** Let \( I \) be a quasi-stable exchange ideal of a ring \( R \). Then every square regular matrix over \( I \) admits a diagonal reduction by quasi invertible matrices.

**Proof.** Given any regular \( A \in M_n(I) \), we have an idempotent matrix \( E \in M_n(I) \) such that \( AR^{n \times 1} = E^{n \times 1}R^{n \times 1} \), where \( R^{n \times 1} = \{(x_1, \ldots, x_n)^T \mid x_1, \ldots, x_n \in R\} \). Clearly, \( ER^{n \times 1} \subseteq FP(I) \). By Lemma 3.4, there exist idempotents \( e_1, \ldots, e_n \in I \) such that \( ER^{n \times 1} \cong e_1R \oplus \cdots \oplus e_nR \cong \text{diag}(e_1, \ldots, e_n)R^{n \times 1} \) as right \( R \)-modules. Set \( R^{1 \times n} = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in R\} \). Then \( AR^{n \times 1} \otimes_R R^{1 \times n} \cong \text{diag}(e_1, \ldots, e_n)R^{n \times 1} \otimes_R R^{1 \times n} \). So \( AM_n(R) \cong \text{diag}(e_1, \ldots, e_n)M_n(R) \). Therefore the result follows. □

Let \( I \) be an ideal of a ring \( R \). We use \( TM_n(R) \) to denote the ring of all \( n \times n \) lower triangular matrices over \( R \) and \( TM_n(I) \) to denote the ideal of all \( n \times n \) lower triangular matrices over \( I \).

**Lemma 3.7.** Let \( I \) be an ideal of a ring \( R \), and let \( n \in \mathbb{N} \). If \( u_{ij} \in \Gamma(I) \) \( (1 \leq i \leq n, \ j \leq n \) and \( u_{ij} = 0 \) \( (i < j, 1 \leq i, j \leq n) \). Then \( (u_{ij})_{n \times n} \in \Gamma(TM_n(I)) \).
Proof. Straightforward.

**Proposition 3.8.** Let $I$ be a right quasi-stable exchange ideal of a ring $R$, and let $n \in \mathbb{N}$. Then $TM_n(I)$ is a right quasi-stable exchange ideal of $TM_n(R)$.

**Proof.** Obviously, $TM_n(I)$ is an exchange ideal of $TM_n(R)$. Given 

\[
\begin{pmatrix}
  a_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  * & \cdots & a_n
\end{pmatrix}
\begin{pmatrix}
  x_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  \vdots & \cdots & x_n
\end{pmatrix}
+ 
\begin{pmatrix}
  b_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  \vdots & \cdots & b_n
\end{pmatrix}
= I_n
\]

with 

\[
\begin{pmatrix}
  a_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  * & \cdots & a_n
\end{pmatrix}
, 
\begin{pmatrix}
  x_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  \vdots & \cdots & x_n
\end{pmatrix}
\in TM_n(I),
\]

then for each $i$ ($1 \leq i \leq n$) we get $a_{ii}x_{ii} + b_{ii} = 1$ with $a_{ii} \in I, x_{ii}, b_{ii} \in R$. As $I$ is a right quasi-stable ideal, we can find $y_i \in R$ such that $a_{ii} + b_{ii}y_i \in \Gamma(I)$. Clearly, $b_{ii} \in 1 + I$ and $a_{ij}, b_{ij} \in I$ ($j < i, 1 \leq i, j \leq n$). By virtue of Lemma 3.7, we get 

\[
\begin{pmatrix}
  a_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  * & \cdots & a_n
\end{pmatrix}
+ 
\begin{pmatrix}
  b_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  \vdots & \cdots & b_n
\end{pmatrix}
\begin{pmatrix}
  y_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  \vdots & \cdots & y_n
\end{pmatrix}
= 
\begin{pmatrix}
  a_{11} + b_{11}y_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  * & \cdots & a_{nn} + b_{nn}y_n
\end{pmatrix}
\in \Gamma(TM_n(I)),
\]

as required. □

4. Examples

The aim of this section is to construct several examples of quasi-stable ideals. A natural problem asks that if right quasi-stable ideal is right and left symmetric. So far, we can not answer this question. Now we establish an interesting properties of such ideals, which is an extension of [4, Lemma 14].

**Proposition 4.1.** Let $I$ be a right quasi-stable ideal of a ring $R$. Then for any regular $x \in I$, there exist an idempotent $e \in R$, a right invertible $u \in 1 + I$, a left invertible $v \in 1 + I$ such that $x = euv$.

**Proof.** Assume that $A = (a_{ij}) \in GL_2(R) \cap \left( \frac{1+t}{1+t} \frac{1+i}{1+i} \right)$, where $a_{12} \in \Gamma(I)$. Write $a_{12} = uv, su = 1, vt = 1, s, t \in 1 + I$. Then $sa_{12}t = 1$. Clearly, we have 

\[
\begin{pmatrix}
  s & 0 \\
  1 - a_{12}ts & a_{12}t
\end{pmatrix}
= 
\begin{pmatrix}
  a_{12}t & 1 - a_{12}ts \\
  0 & s
\end{pmatrix}^{-1},
\]

\[
\begin{pmatrix}
  sa_{12} & 0 \\
  1 - tsa_{12} & t
\end{pmatrix}
= 
\begin{pmatrix}
  t & 1 - ts a_{12} \\
  0 & sa_{12}
\end{pmatrix}^{-1} \in GL_2(I).
\]
So we get
\[
\begin{pmatrix}
  s & 0 \\
 1 - a_{12}ts & a_{12}t
\end{pmatrix}
\begin{pmatrix}
  sa_{12} & 0 \\
 1 - tsa_{12} & t
\end{pmatrix}
= \begin{pmatrix}
  * & 1 \\
  * & * 
\end{pmatrix}
\in GL_2(R) \cap \begin{pmatrix}
  1 + I & 1 + I \\
  I & 1 + I
\end{pmatrix}.
\]

We infer that
\[
A = \begin{pmatrix}
  s & 0 \\
 1 - a_{12}ts & a_{12}t
\end{pmatrix}^{-1}
\begin{pmatrix}
  * & 1 \\
  * & *
\end{pmatrix}
\begin{pmatrix}
  sa_{12} & 0 \\
 1 - tsa_{12} & t
\end{pmatrix}^{-1}.
\]

Therefore
\[
A^{-1} = \begin{pmatrix}
  sa_{12} & 0 \\
 1 - tsa_{12} & t
\end{pmatrix}
\begin{pmatrix}
  * & 1 \\
  * & *
\end{pmatrix}
\begin{pmatrix}
  s & 0 \\
 1 - a_{12}ts & a_{12}t
\end{pmatrix}^{-1}.
\]

From \((1 \ 0) \in GL_2(R) \cap \begin{pmatrix}
  1 + I & 1 + I \\
  I & 1 + I
\end{pmatrix}\), we can find \(u \in GL_1(I)\) such that
\[
\begin{pmatrix}
  1 & 0 \\
  * & 1
\end{pmatrix}
\begin{pmatrix}
  * & 1 \\
  * & *
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  * & 1
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 \\
  u & 0
\end{pmatrix}
= \begin{pmatrix}
  0 & u^{-1} \\
  1 & 0
\end{pmatrix},
\]
where \((1 \ 0) \in GL_2(R) \cap \begin{pmatrix}
  1 + I & 0 \\
  -1 - I & 1
\end{pmatrix}\). Thus
\[
\begin{pmatrix}
  * & 1 \\
  * & *
\end{pmatrix}^{-1}
= \begin{pmatrix}
  1 & 0 \\
  * & 1
\end{pmatrix}
\begin{pmatrix}
  0 & u^{-1} \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  * & 1
\end{pmatrix}
= \begin{pmatrix}
  * & u^{-1} \\
  * & *
\end{pmatrix}.
\]

So we deduce that
\[
A^{-1} = \begin{pmatrix}
  sa_{12} & 0 \\
 1 - tsa_{12} & t
\end{pmatrix}
\begin{pmatrix}
  * & u^{-1} \\
  * & *
\end{pmatrix}
\begin{pmatrix}
  s & 0 \\
 1 - a_{12}ts & a_{12}t
\end{pmatrix}^{-1}.
\]

As \(u \in 1 + I\), we have \(u^{-1} \in 1 + I\). Set \(w = sa_{12}u^{-1}a_{12}t\). As \(sa_{12}t = 1\), we see that \(sa_{12} \in 1 + I\) is right invertible and \(u^{-1}a_{12}t \in 1 + I\) is left invertible.

Assume that \(B = (b_{ij}) \in GL_2(I)\). Write \(B^{-1} = (c_{ij})\). Then \(B^{-1} \in GL_2(I)\); hence, \(c_{12}R + c_{11}R = R\) with \(c_{12} \in I\). As \(I\) is a right quasi-stable ideal, we can find \(y \in R\) such that \(c_{12} + c_{11}y \in \Gamma(I)\). Obviously, \(y \in 1 + I\), and so
\[
B^{-1} \begin{pmatrix}
  1 & y \\
  0 & 1
\end{pmatrix} = \begin{pmatrix}
  * & c_{12} + c_{11}y \\
  * & *
\end{pmatrix}
\in GL_2(R) \cap \begin{pmatrix}
  1 + I & 1 + I \\
  I & 1 + I
\end{pmatrix}.
\]

By the consideration above, we can find some \(w_1 \in 1 + I\) such that
\[
\begin{pmatrix}
  1 & -y \\
  0 & 1
\end{pmatrix} B = (B^{-1} \begin{pmatrix}
  1 & y \\
  0 & 1
\end{pmatrix})^{-1} = \begin{pmatrix}
  * & c_{12} + c_{11}y \\
  * & *
\end{pmatrix}^{-1} = \begin{pmatrix}
  * & w_1 \\
  * & *
\end{pmatrix},
\]
where \(w_1\) is the product of a right invertible element and a left invertible element \(v \in 1 + I\).

Given \(ax + b = 1\) with \(a, x \in I, b \in R\), then \(\begin{pmatrix}
  1 & x \\
  -a & b
\end{pmatrix} = (1 - xa - ax)^{-1} \in GL_2(I)\).

By the proceeding discussion, we can find \(z \in R\) such that \((\frac{1}{b} \ 1) (\frac{-1}{-a} \ b) = \)
Let \( w_2 \in 1 + I \) be the product of a right invertible element and a left invertible element \( v \in 1 + I \). Therefore \( x + zb = w_2 \).

For any regular \( x \in I \), it follows from \( xy + (1 - xy) = 1 \) that \( w := x + (1 - xy)z \in 1 + I \) is the product of a right invertible element and a left invertible element \( v \in 1 + I \). Set \( e := xy \in I \). Then \( x = xy(x + (1 - xy)z) = ew \), where \( e = e^2 \in I \) is an idempotent. Therefore we complete the proof. \( \Box \)

Recall that an ideal \( I \) of a ring \( R \) is regular provided that for any \( x \in I \) there exists \( y \in I \) such that \( x = xyx \). We say that a ring \( R \) is right quasi-stable in case it is a right quasi-stable ideal as itself.

**Proposition 4.2.** Let \( I \) be a regular ideal of a ring \( R \). If \( eRe \) is a right quasi-stable ring for all idempotents \( e \in I \), then \( I \) is a right quasi-stable exchange ideal of \( R \).

**Proof.** By [1, Example], \( I \) is an exchange ideal. Given \( ax + b = 1 \) with \( a \in I, x, b \in R \), then \( a = aa'a \) for \( a' \in R \). Set \( c = a'ax \). Then \( ac + b = 1 \) with \( a, c \in I, b \in 1 + I \). As \( a, c, 1 - b \in I \). In view of [7, Lemma 3.2], there exists an idempotent \( e \in I \) such that \( a, x, 1 - b \in eRe \). Hence, \( (1 - b)(1 - e) = 0 \), and so \( b(1 - e) = 1 - e \). In addition, \( (1 - b)e = 1 - b \); hence, \( b = be + 1 - e \). Thus, \( ax + be = e \). This implies that \( be \in eRe \), and so \( ebe = be \). Since \( ax + ebe = c \), by hypothesis, we can find some \( u, v, s, t \in eRe \) such that \( a + ebev = ew, su = e, vt = e \) for some \( y \in R \). Thus, \( a + ebev + 1 - e = (u + 1 - e)(v + 1 - e) \), and so \( a + b(eve + 1 - e) = (u + 1 - e)(v + 1 - e) \), where \( (s + 1 - e)(u + 1 - e) \). Therefore \( I \) is a right quasi-stable ideal of \( R \), as desired. \( \Box \)

**Corollary 4.3.** Let \( I \) be a regular ideal of a ring \( R \). If \( aR + bR = R \) with \( a \in 1 + I, b \in R \) implies that there exists \( y \in R \) such that \( a + by \in R \) is right or left invertible, then \( I \) is a quasi-stable exchange ideal of \( R \).

**Proof.** Let \( e \in I \) be an idempotent. In view of [5, Lemma 4.1], \( eRe \) is one-sided unit-regular. For any \( x \in eRe \), by [3, Theorem 4], there exist an idempotent \( f \in eRe \) and a right or left \( u \in eRe \) such that \( x = eu \). This implies that \( eRe \) is a right quasi-stable ring from Theorem 2.2. According to Proposition 4.2, \( I \) is a right quasi-stable exchange ideal. By the symmetry of one-sided unit-regularity, we establish the result. \( \Box \)

Recall that an ideal \( I \) of a regular ring \( R \) satisfies the comparability axiom provided that for any \( x, y \in I \), either \( xR \lesssim yR \) or \( yR \lesssim xR \) (cf. [10]). Let \( I \) be an ideal of a regular ring \( R \). If \( I \) satisfies the comparability axiom, we note that \( aR + bR = R \) with \( a \in 1 + I, b \in R \) implies that \( a + by \in R \) is right or left invertible for any \( y \in R \).

**Corollary 4.4.** Let \( I \) be a regular ideal of a ring \( R \). If \( I \) satisfies the comparability axiom, then \( I \) is quasi-stable.
Clearly, it is directly proved that \( aR + bR = R \) with \( a \in 1 + I, b \in R \) implies that \( a + by \in R \) is right or left invertible. Therefore we complete the proof by Corollary 4.3.

By [8, Corollary 9.15], every regular, right self-injective ring satisfies general comparability. We now extend this result to right injective ideals of regular rings.

**Proposition 4.5.** Let \( I \) be a regular ideal of a ring \( R \). If \( I \) is an injective right \( R \)-module, then \( I \) is a quasi-stable ideal of \( R \).

**Proof.** Since \( I \) is regular, \( I \) is an exchange ideal. As \( I \) is injective, there exists a splitting exact sequence \( 0 \to I \to R \to R/I \to 0 \). Thus, we have a right \( R \)-module \( C \cong R/I \) such that \( R = I \oplus C \). Thus, \( I = eR \) for some idempotent \( e \in I \). Let \( f \in I \) be an idempotent. Then we have an inclusion \( i : eR \hookrightarrow eR \).

Construct a \( R \)-morphism \( \varphi : eR \to fR \) given by \( \varphi(ER) = fer \) for any \( r \in R \). It is easy to verify that \( \varphi i = 1_E \). This implies that the exact sequence \( 0 \to fR \to eR \to eR/fR \to 0 \) splits. Thus, we have a right \( R \)-module \( D \cong eR/fR \) such that \( eR = fR \oplus D \). Since \( eR \) is injective, so is \( fR \).

For any \( m \in Z(fR) \), there exists some \( z \in R \) such that \( m = zmz \). Hence, \( r(m) = (1 - zm)R \). As \( r(m) \cap zmR = 0 \), we get \( zmR = 0 \); hence, \( m = mzR = 0 \). That is, \( Z(fR) = 0 \), i.e., \( fR \) is nonsingular. In view of [8, Corollary 1.23], \( fR \cong \text{End}_R(fR) \) is a regular, right self-injective ring. According to [8, Corollary 9.15], \( eRe \) satisfies general comparability.

Let \( x \in fRf \), we can find an idempotent \( g \in fRf \) and a related unit \( w \in fRf \) such that \( x = gw \). As \( w \in fRf \) is a related unit, there exists an idempotent \( g \in fRf \) such that \( gw \in g(fRf) \) is right invertible and \( (f - g)w \in (f - g)(fRf) \) is left invertible. Thus, \( w = ((f - g)w + g)(gw + f - g) \).

According to Proposition 4.2, \( I \) is a right quasi-stable ideal. Analogously, we show that \( I \) is a left quasi-stable ideal. Therefore \( I \) is quasi-stable, as desired.

Let \( R \) be a regular ring, and let \( a \in R \). If \( RaR \) is injective, it follows from Proposition 4.5 and Theorem 2.2 that \( a \) is the product of an idempotent, a left invertible element and a right invertible element.

**Example 4.6.** Let \( R \) be regular, and let

\[ I = \{ x \in R \mid xR \text{ is injective} \}. \]

Then \( I \) is a quasi-stable ideal.

**Proof.** It is directly proved that \( I \) is an ideal of \( R \). For any \( a \in I \), there exists an idempotent \( e \in I \) such that \( a \in eRe \) from [6, Lemma 3.2]. As \( eR \) is injective, it follows from [8, corollary 1.23] that \( eRe \) is a regular, right self-injective ring. Thus, it satisfies related comparability. Hence, there exists an idempotent \( f \in eRe \) and a related unit \( w \in eRe \) such that \( a = eu \). This implies that \( a = e(u + 1 - e) \), where \( e \in I \) is an idempotent and \( u + 1 - e \in \Gamma(I) \).

According to Theorem 2.2, \( I \) is a right quasi-stable ideal. Similarly, we show that \( I \) is a left quasi-injective ideal, as asserted.
5. Directly finite ideals

We say that an ideal $I$ of a ring $R$ is directly finite provided that for any $a, b \in I$, $(1 + a)(1 + b) = 1$ implies that $(1 + b)(1 + a) = 1$. An ideal $I$ of a ring $R$ is said to be of bounded index provided that there exists some $n \in \mathbb{N}$ such that $x^n = 0$ for any nilpotent element $x \in I$. Let $R$ be a regular ring, and let $I = \{ x \in R \mid \text{End}_R(xR) \text{ is of bounded index} \}$. Then $I$ is a directly finite, quasi-stable exchange ideal.

**Lemma 5.1.** Let $I$ be a directly finite, right quasi-stable exchange ideal of a ring $R$. Suppose that $AX + B = I_n$ with $A, X \in M_n(I), B \in M_n(R)$. Then

1. There exists some $Y \in M_n(R)$ such that $A + BY \in GL_n(I)$.
2. There exists some $Z \in M_n(R)$ such that $X + ZB \in GL_n(I)$.

**Proof.** (1) Since $I$ is directly finite, one easily checks that $\Gamma(I) = GL_1(I)$. By iteration of the process of Theorem 3.2 and replacing the elements in $\Gamma(I)$ by invertible elements in $1 + I$, we can find some $Y \in M_n(R)$ such that $A + BY \in GL_n(I)$.

(2) By (1), there is $Y \in M_n(R)$ such that $A + BY \in GL_n(I)$. In view of Lemma 2.4, one directly verifies that $(X + (X_n - XY)(A + BY)^{-1}B)^{-1} = A + Y(I_n - XA)$. Check $Z = (X_n - XY)(A + BY)^{-1}$. Then $X + ZB \in GL_n(I)$, as asserted.

**Theorem 5.2.** Let $I$ be a directly finite, right quasi-stable exchange ideal of a ring $R$. Then for any regular $A \in M_n(I)$ there exist $U, V \in GL_n(I)$ such that $UAV = \text{diag}(e_1, \ldots, e_n)$ for some idempotents $e_1, \ldots, e_n \in I$.

**Proof.** Given any regular matrix $A \in M_n(I)$, there exists $E = E^2 \in M_n(I)$ such that $AM_n(R) = EM_n(R)$. Similarly to Theorem 3.4, we have idempotents $e_1, \ldots, e_n \in I$ such that $\varphi : AM_n(R) \cong \text{diag}(e_1, \ldots, e_n)M_n(R)$, where $M_n(R)A = M_n(R)\varphi(A), \varphi(A)M_n(R) = \text{diag}(e_1, \ldots, e_n)M_n(R)$. One directly verifies that there exist some $X, Y \in M_n(I)$ such that $XA = \varphi(A)$ and $A = Y\varphi(A)$. Since $XY + (I_n - YX) = I_n$, it follows by Lemma 5.1 that there exists some $Z \in M_n(R)$ such that $U := X + Z(I_n - YX) \in GL_n(I)$. Hence $UA = (X + Z(I_n - YX))A = AX = \varphi(A)$. Likewise, we can find some $V \in GL_n(I)$ such that $\varphi(A)V = \text{diag}(e_1, \ldots, e_n)$. Therefore $UAV = \text{diag}(e_1, \ldots, e_n)$, as asserted.

Let $I$ be an ideal of a ring $R$. Set $B(I) = \{ e \in I \mid e = e^2 \text{ and } ex = xe \text{ for any } x \in I \}$. We say that $I$ is an abelian ideal in case every idempotent in $I$ is in $B(I)$. For example, every semicommutative ideal of a ring is an abelian ideal.

**Corollary 5.3.** Let $I$ be an abelian exchange ideal of a ring $R$. Then for any regular $A \in M_n(I)$ there exist $U, V \in GL_n(I)$ such that $UAV = \text{diag}(e_1, \ldots, e_n)$ for some idempotents $e_1, \ldots, e_n \in I$. 
Proof. For any regular \( x \in I \), we have \( y \in I \) such that \( x = xyx \) and \( y = yxy \). Since \( I \) is an abelian exchange ideal of \( R \), we have \( x = x^2y = yx^2 \), and then \( x = xy(1 + x - xy) \). Set \( e = xy \) and \( u = 1 + x - xy \). Then \( e = e^2 \in I \) and \( u(1 + y - xy) = 1 \). Hence \( u \in \Gamma(I) \). Thus \( I \) is a right quasi-stable ideal by Theorem 2.2.

Suppose that \( (1 - x)(1 - y) = 1 \) with \( x, y \in I \); hence, \( (1 - y)(1 - x) \in 1 + I \) is an idempotent. Since \( I \) is an abelian ideal of \( R \), \( (1 - (1 - y)(1 - x))x = x(1 - (1 - y)(1 - x)) \). Furthermore, we get \( (1 - y)(1 - x)(1 - x) = (1 - x)(1 - y)(1 - x) \). So \( (1 - y)(1 - x) = (1 - y)(1 - x)(1 - y)d = (1 - x)(1 - y)(1 - y) = 1 \). That is, \( I \) is a directly finite ideal. Therefore we complete the proof from Theorem 5.2.

Recall that an ideal \( I \) of a ring \( R \) is periodic provided that for any \( x \in I \), there exists \( n(x) \in \mathbb{N} \) such that \( x = x^{n(x)+1} \).

**Corollary 5.4.** Let \( I \) be a periodic ideal of a ring \( R \). Then for any regular \( A \in M_n(I) \) there exist \( U, V \in \text{GL}_n(I) \) such that \( UAV = \text{diag}(e_1, \ldots, e_n) \) for some idempotents \( e_1, \ldots, e_n \in I \).

**Proof.** For any idempotent \( e \in I \) and any idempotent \( x \in I \), we have \( (ex - exe)^2 = 0 \). So we deduce that \( ex = exe \) because \( I \) is a periodic ideal. Likewise, we have \( xe = exe \); hence, \( ex = xe \). This means that \( I \) is an abelian ideal of \( R \). On the other hand, \( I \) is a strongly \( n \)-regular ideal; hence, it is an exchange ideal. So the proof is true by Corollary 5.3.

**Example 5.5.** Let \( V \) be a countably generated infinite-dimensional vector space over a division \( D \), and let \( (a_{ij}) \in M_n(\text{End}_D(V)) \) with all \( \dim_D(a_{ij}V) < \infty \). Then there exist \( U, V \in \text{GL}_n(\text{End}_D(V)) \) such that \( UAV = \text{diag}(e_1, \ldots, e_n) \) for some idempotents \( e_1, \ldots, e_n \in \text{End}_D(V) \).

**Proof.** Let \( I = \{ x \in \text{End}_D(V) \mid \dim_D(xV) < \infty \} \). Obviously, \( I \) is an ideal of \( \text{End}_D(V) \). For any idempotent \( e \in I \), \( eRe \) is unit-regular; hence, \( I \) has stable range one. This implies that \( I \) is directly finite. In view of Proposition 2.5, \( I \) is a quasi-stable ideal. According to Theorem 5.2, the result follows.

Let \( R \) be an exchange ring, and let \( (a_{ij}) \in M_n(R) \). If each \( R_{a_{ij}} \) has stable range one, analogously, we conclude that there exist \( (u_{ij}), (v_{ij}) \in \text{GL}_n(R) \) such that \( (u_{ij})(a_{ij})(v_{ij}) = \text{diag}(e_1, \ldots, e_n) \) for some idempotents \( e_1, \ldots, e_n \in R \). In addition, \( R_{u_{ij}}R, R_{v_{ij}}R \) \( (i \neq j) \), \( R(1 - u_{ij})R \) and \( R(1 - v_{ij})R \) all have stable range one.

**References**


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